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Optimal Fiscal Policy in an Emerging Economy with Credit Constraints: Theory and Application for Poland

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Abstract

In this paper we develop an open-economy endogenous growth model to examine the influence of fiscal policy on the economy in the long run. We allow for public deficit and 5 types of taxes. One of the novel features is separate treatment of interest rates on public and private debt, both of which are linear functions of appropriate debt-to-GDP ratios. Two extreme situations are analyzed: a model of "decentralized economy", where economic agents do not take into account any externalities, and a model of "benevolent social planner". We derive the rules of optimal fiscal policy that induce economic agents to internalize all externalities. Theoretical results are illustrated with an empirical analysis for Poland. The optimal values of several fiscal policy instruments for Poland are calculated.

Keywords: optimal fiscal policy, imperfect capital mobility, credit constraints, production externalities

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1 Introduction

The literature on fiscal policy in the Ramsey-type growth theory of an open economy is growing continuously. Early examples include Nielsen and Sorensen (1991), Rebelo (1992) and Razin and Yuen (1994), (1996), who study the dynamic effects of various forms of capital income taxation under perfect capital mobility, and Asea and Turnovsky (1998), who analyze the effects of capital income taxation on economic growth in a stochastic model of a small open economy.

One of the most significant contributions is a monograph by Turnovsky (2009), based on his earlier extensive research. It contains a range of models of optimal fiscal policy in a small open economy (SOE) under perfect or imperfect mobility of capital. He examines productive government expenditures and 3 types of taxes: taxes on consumption, production and foreign debt of the private sector. Important qualitative differences between closed economy and SOE are exposed. For example, the capital income tax ceases to have any effect on the long-run growth rate of the economy. The equilibrium growth rate is independent of almost all fiscal instruments, including public expenditures. The only tool of fiscal policy that is not neutral is the tax rate on foreign interest income.

Another important contribution is Fisher (2010), who investigates fiscal policy shocks in a SOE growth model, where domestic capital accumulation, subject to installation costs, is the engine of economic growth. He examines the short- and long-run dynamics of the model by considering several fiscal instruments: government expenditure, capital tax, a tax on international financial assets, and consumption tax. He shows that a permanent fiscal expansion leads to a temporary increase in growth, whereas a rise in consumption tax results in a temporary decrease in growth. There are also numerous related theoretical and empirical papers within the closed economy setting, e.g., Judd (1999), who studies the welfare-maximizing tax structure in a two-sector model of endogenous growth with human capital, Lee and Gordon (2005), who empirically explore how taxes affect growth rates across countries, and Konopczyński (2014a), who investigates the relationship between economic growth in Poland and four types of taxes and human capital investment.

Although these publications provided valuable insight into the long-run consequences of fiscal policy in an open economy, they suffer from one severe simplification: the assumption of permanently balanced government budget (zero deficit and debt in each period of time), or – at best – limited role of fiscal policy. This assumption is in fact typical for open economy growth models. Amazingly, even some of the latest research papers allow the private sector to borrow abroad, but at the same time prohibit any active fiscal policy of the government; see e.g., Assibey-Yeboah and Mohsin (2014). (An exception is Konopczyński (2014b), who investigates the implications of the size and structure of budget deficit in an open economy under perfect capital mobility.) This approach is probably inherited from closed economy endogenous growth models, where usually Ricardian equivalence holds, and hence budget deficit does not matter for the long-run welfare and the rate of growth. (One important exception is presented by Turnovsky (2002) – see Proposition 2 therein.) With the proportional tax on capital, Ricardian equivalence does not hold any more. However, "an increase in the tax on capital reduces the growth rate of capital, but leaves the growth rate of consumption unaffected". This paper demonstrates that, in the SOE context, disregarding government deficit and public debt is unjustified.

Furthermore, there is another important assumption that permeates the open-economy growth theory: perfect capital mobility. In our view, it is a serious flaw of the existing theory. The sovereign debt crisis in the eurozone proves that even within the common currency area individual countries face different interest rates. It could have been misunderstood, as the interest rates on public debt (bond yields) in the eurozone were almost identical until late 2009. However, from that moment on, it became painfully obvious that the higher the level of debt, the higher the risk of insolvency, which translates into higher interest rates. This is true not only for developing countries – which has been recognized since the publication of Bardhan (1967) – but also in so-called emerging economies and developed world.

Several researchers have incorporated this observation into the open-economy endogenous growth theory in several ways. The earliest papers assumed that the risk premium depends upon the level of debt; e.g., Obstfeld (1982), Bhandari, Haque, and Turnovsky (1990), and Fisher (1995). However, if the economy in equilibrium is growing at certain constant rate, this assumption is inappropriate. Therefore, researchers more often assume that the risk premium is a function of the level of debt in relation to either output, or domestic capital, or some other measure of wealth. This approach was first mentioned by Bhandari, Haque, and Turnovsky (1990), and later adopted by van der Ploeg (1996), Turnovsky (1997), Fisher and Terrell (2000), Schmitt-Grohe and Uribe (2003), Chatterjee and Turnovsky (2007), Assibey-Yeboah and Mohsin (2012, 2014).

Quite amazingly, though, all these papers grant fiscal policy a very passive role, if any. Generally, they assume that the government maintains a balanced budget in each period, and thus has zero debt, or – at best – they consider an aggregate debt of the entire country. Obstfeld (1982), Schmitt-Grohe and Uribe (2003), and Assibey-Yeboah and Mohsin (2014) abstract completely from fiscal policy, i.e., these models have no room for public sector. Chatterjee and Turnovsky (2007) assume that the government behaves passively: it runs "continuously balanced budget", and hence there is no public debt, which stands in sharp contrast to the private sector that can borrow abroad. Bhandari, Haque, and Turnovsky (1990) and van der Ploeg (1996) allow the government to run deficit financed by borrowing at home or abroad, but they effectively merge public and private foreign debts by assuming a uniform interest rate for both sectors. Similarly, Fisher (1995), and Fisher and Terrell (2000), using their own words, "abstract from government policy, there is no distinction between private sector and sovereign' debt. Consequently, the implications of the differential 'risk' characteristics of private and government debt are not addressed in this paper".

In this paper we give the government an active role by assuming (more in par with reality) that government expenditures may exceed revenues (in the growing economy even permanently), whereas public deficit is financed by domestic and foreign debt, entirely independently of the private sector's foreign debt.

One important novel feature of our model – to the best of our knowledge, new in the literature – is the fact that the cost of borrowing to the private sector and the cost of borrowing to the public sector are described by two independent equations, which reflects empirical evidence that (on average) the cost of borrowing by governments is much lower than by corporations and other private entities. It allows for a significant generalization of the existing normative conclusions from the theory of open economy endogenous growth. Hopefully, it also provides certain theoretical insight into the process of economic growth in heavily indebted countries that face significant credit constraints.

To be more precise, our model is a significant modification and generalization of existing models, especially those presented by Turnovsky (2009). The government can (even indefinitely) run deficit financed by public debt composed of domestic and foreign debt. There are 5 types of taxes: on wages, capital income, consumption, interest paid by private sector to foreign lenders and interest on government bonds held by domestic investors. Productive capital depreciates (which is also often neglected in the literature) at an exogenous rate. We introduce public consumption as a substitute to private consumption. Following Acemoglu (2008), we apply a slightly modified utility function, where the rate of discount depends on the rate of growth of population.

Identically to Turnovsky (2009), we analyze two distinct situations: the model of the benevolent social planner, and the model of atomized representative agents (or "decentralized economy"). Put simply, representative agents are so small, that they behave just like standard, textbook firm in perfectly competitive market: it is fair for them to assume that their individual decisions have negligible impact on market prices, as well as several other aggregate numbers, including all characteristics of the public sector. In other words, there are some externalities of individual decisions, that are not incorporated into decentralized, individual decision-making. To be more precise, there are 4 types of externalities in our model (all except for the last one are explicitly or implicitly present in the models of Turnovsky, 2009). First, firms do not realize positive externalities of investment in capital related to learning-by-doing and spillover-effects, for reasons described by Romer (1986) and van der Ploeg (1996), but most importantly the underdeveloped (inefficient) patent market, which is certainly true in case of emerging economies and developing countries. Second, individual agents have negligible influence on all prices (of output and factors of production), so they treat all prices as exogenous. Third, they neglect negative externalities associated with the aggregate level of private foreign debt, i.e., they assume that their individual decisions regarding borrowing from abroad have negligible impact on the interest rate, whereas in fact their aggregate decisions do influence the risk premium, which translates into the cost of borrowing. Fourth, individual agents do not take into account negative externalities associated with an increasing ratio of public debt to GDP. Strictly speaking, they take the interest rate on government bonds as an exogenous value, on which their individual decisions have no impact. However, in fact, their aggregate consumption and investment decisions impact on the revenues and expenditures of the public sector, and therefore public debt, which translates into the interest rate on government bonds.

The paper is organized as follows. Section 2 sets out the model. In sections 3 and 4, we solve the optimal control problem for the decentralized economy and the benevolent social planner, i.e., we derive the second-best and the first-best solution. Section 5 presents the sensitivity of these two types of economies to fiscal policy. In section 6 we solve the problem of full and partial replication of the first-best equilibrium. In section 7 we compare our model with Turnovsky's models and point to some important differences in implications.

The second part of the paper contains an empirical analysis for Poland. Section 8 summarizes the calibration of the model for Poland. In section 9, we present the baseline scenario, where all parameters preserve their recent values long into the future. Next we present the solution of the problem of replication for the Polish economy. Finally, in section 11, we search for the global optimum: the set of values of policy instruments that maximizes the welfare of the nation. Section 12 summarizes the main theoretical and empirical results.

2 The model

2.1 The interest rates

Let Z be the net foreign debt of the private sector. The real interest rate on private foreign debt is a linear function of the debt-to-GDP ratio:

$$r_Z = r_Z(Z/Y) = r_Z(z/y) = \varepsilon_Z + p_Z z/y, \tag{1}$$

where ε_Z represents the base interest rate for the private sector, z denotes private foreign debt, and $p_Z > 0$ is the risk premium parameter. Throughout the paper capital letters denote real values of variables in domestic currency (e.g., D), while lowercase letters denote real values per capita, e.g., $d = D/L$, where L is the supply of labor. An analogous equation applies to the real interest rate on public debt:

$$r_D = r_D(D/Y) = r_D(d/y) = \varepsilon_D + p_D d/y, \tag{2}$$

where ε_D is the base interest rate for the public sector, d represents total public debt per capita, and $p_D > 0$ is the public debt risk premium parameter. Note that unlike virtually all existing literature, we apply separate base interest rates as well as separate risk premiums for private and public sector. This reflects an undeniable empirical fact mentioned in the Introduction: throughout the world the cost of borrowing by

governments is (on average) much lower than by corporations and households. For example, in Poland the real 10-year Treasury bond yields in the period 2001–2012 were on average equal to 3.29%, whereas in the same period the average real cost of foreign lending to the private sector was on average equal to 5.99% (for details see Table 5 below). There are plenty of reasons: different level of (objective as well as perceived) risk, huge variance in the quality of collateral (in case of the government the "collateral" has an ultimate form of an implicit guarantee of the power to "print money"), scale effects, huge differences in intermediation costs, and etc.
Note that this assumption may result in a seemingly awkward situation: it could happen (and in fact, in our simulations it does happen) that the private sector is lending to the government at some (low) interest rate, and at the same time borrowing from abroad at a much higher interest rate. Many would ask: why would a household (or a firm) be willing to do that? Shouldn't these interest rates be equal (at least in equilibrium), reflecting some "no arbitrage" condition? The answer is NO, precisely for the reasons mentioned above.
It is worth noting that linear relationships assumed in Eqs. (1) and (2) are supported by bulk of empirical research; see, e.g., Chinn, Frankel (2003), Kinoshita (2006), Laubach (2009), Poghosyan (2013). Nonetheless, some researchers detect nonlinear effects; see, e.g., Faini (2006) and Ardagna (2007). However, nonlinear effects are statistically significant only at a very high level of public debt, exceeding 100% of GDP. Poland, which will be the subject of our interest in the empirical part of this paper, is currently way below this level. Thus, nonlinear effects will not be taken into account.

2.2 The technology and the markets for factors of production

The output of a representative firm is described by the Cobb-Douglas production function:

$$Y_i = F\left(K_i, L_i\right) = aK_i^{\alpha}\left(EL_i\right)^{\beta} \text{ with } \alpha+\beta=1,\ \alpha,\beta>0,\ a>0, \tag{3}$$

where K_i denotes the stock of physical capital, L_i represents raw labor, and E is the labor-augmenting technology index. Obviously, the aggregate output of the whole economy is: $Y = aK^{\alpha}(EL)^{\beta}$, where K is the aggregate stock of capital and L is the supply of labor in the country, which is assumed to grow exponentially: $L = L_0e^{nt}$. Dividing both sides by L yields the per capita production function: $y = Y/L = ak^{\alpha}(E)^{\beta}$. We assume positive externalities related to learning-by-doing and spillover-effects that are reflected in the labor-augmenting technology index E, which is proportional to the capital per worker ratio, i.e., $E = xK/L$, where $x = \mathit{const.} > 0$. (These ideas were first introduced by Arrow (1962) and Lucas (1988); an overview of the literature related to these externalities is provided by Barro and Sala-i-Martin (2004).) Thus, the per capita production function can be written as $y = Ak$, where $A = ax^{\beta} = \mathit{const} > 0$. Similarly, the aggregate output function can be written as $Y = aK^{\alpha}(EL)^{\beta} = AK^{\alpha}(K)^{\beta} = AK$. Firms are maximizing profits in perfectly competitive markets, which in particular implies that the marginal product of capital is equal to the real rental rate:

$$\forall t \quad \partial Y/\partial K = \alpha a K^{\alpha-1}(EL)^{\beta} = \alpha Y/K = \alpha A = w_K. \tag{4}$$

The accumulation of capital is described in a standard way (in per capita terms):

$$\dot{k} = i - (n+\delta)k, \tag{5}$$

where $\delta > 0$ is the depreciation rate. Investment requires quadratic adjustment costs, and hence in order to attain net investment equal to I, a firm needs expenditures equal to:

$$\Phi(I, K) = I\left(1 + \frac{\chi}{2}\frac{I}{K}\right), \quad \text{with } \chi > 0. \tag{6}$$

Note that the concept of adjustment costs is attributed to Hayashi (1982).

2.3 Consumer preferences

The preferences of the representative household are expressed by the following intertemporal utility function:

$$U = \int_0^{\infty} \frac{1}{\gamma}\left(cg_C^{\kappa}\right)^{\gamma} e^{-(\rho-n)t} dt, \rho > 0, \rho > n, \tag{7}$$

where c denotes private consumption and g_C is public consumption. The elasticity of substitution between both types of consumption is expressed by $\kappa > 0$. A fraction $1/(1-\gamma)$ is equal to the intertemporal elasticity of substitution. We assume that $\gamma < 0$, which is justified on the basis of empirical research; see e.g., Turnovsky (2009), p. 177. The effective rate of discount equal to $\rho - n$ is adopted from Acemoglu (2008), p. 310. It reflects the assumption that a household derives utility from its own consumption and also from the consumption of its descendants (children, grandchildren, etc.), the number of which is growing at an annual rate n. We assume that $\rho > n$. Otherwise, the integral in (7) would not be convergent.

It follows that the higher the rate of population growth (n) in a country, the smaller the effective rate of discount, because the number of children, grandchildren, etc., who will be consuming in the future is greater. Intuitively – the more children (per family), the more we value future consumption. In our opinion this is a realistic assumption – parents of 3 or more kids plan their lifetime spending flow differently (leaving more for the future and deriving satisfaction from bequests that their children will get) than parents of one child, not to mention a couple without children, or singles. In our view, this assumption brings the standard Ramsey-type optimization somewhat closer to the more realistic overlapping generations approach.

2.4 The public sector (the government)

The total tax revenue of the government in real terms is:

$$T = \tau_L wL + \tau_K w_K K + \tau_C C + \tau_Z r_Z Z + \tau_D r_D D_D. \quad (8)$$

where τ_L, τ_K, τ_C, τ_Z, τ_D are the average tax rates on wages, capital income, consumption, interest paid by private sector to foreign lenders, and interest on government bonds held by domestic investors, respectively. The deficit of the public sector is the difference between total government spending and tax revenue, i.e. in real terms: $J = G + r_D D - T$, where G is the government spending and D represents the total public debt. We assume that the budget deficit is a fixed percentage of GDP, i.e., $J = \xi Y$, where $\xi = const > 0$ is a decision parameter. Therefore, the budgetary rule can be written as:

$$G = T - r_D D + \xi Y. \quad (9)$$

The deficit is financed by government bonds, which raises the public debt according to the equation: $\dot{D} = \xi Y$. A certain portion (ω) of bonds is sold to foreign investors, and the remainder is purchased by domestic lenders, i.e.

$$\dot{D}_F = \omega \xi Y, \quad (10)$$

$$\dot{D}_D = (1 - \omega)\xi Y. \quad (11)$$

where D_D and D_F represent domestic and foreign debt of the government, respectively. The government spending consists of two components:

$$G = G_C + G_T = \sigma_C C + G_T, \quad 0 < \sigma_C < 1, \quad (12)$$

where G_C is the public consumption (by assumption proportional to private consumption), and G_T represents cash transfers to the private sector.

2.5 The private sector

The private sector receives income in the form of remuneration of labor and capital, the interest on domestic public debt, and cash transfers from the government. It must, however, pay interest to foreign creditors. The private sector's real disposable income after taxes is defined as:

$$Y_d = (1 - \tau_L)\, wL + (1 - \tau_K)\, w_K K + (1 - \tau_D)\, r_D D_D - (1 + \tau_Z) r_Z Z + G_T. \quad (13)$$

This income is spent on consumption and investment, as well as purchases of government bonds. Any difference is covered by (net) borrowing from abroad. Therefore, the instantaneous budget constraint in real terms is expressed as follows: $Y_d = C(1 + \tau_C) + \Phi(I, K) + \dot{D}_D - \dot{Z}$. Substituting Eq. (11), and rearranging yields:

$\dot{Z} = C(1+\tau_C) + \Phi(I,K) + (1-\omega)\xi Y - Y_d$. Using Eqs. (6) and (13), this budget constraint can be transformed into the per capita form:

$$\begin{aligned} \dot{z} &= c\left(1+\tau_C\right) + i\left(1+\frac{\chi}{2}\frac{i}{k}\right) + \left[(1+\tau_Z)r_Z - n\right]z - (1-\tau_L)w + \\ &\quad - \left(1-\tau_K\right)w_K k - \left(1-\tau_D\right)r_D d_D - g_T + (1-\omega)\xi y. \end{aligned} \tag{14}$$

It is worth to emphasize that the representative agent treats all prices and fiscal variables as exogenous to its private decisions, as his individual influence on the market is negligible. Thus, when making decisions, he pays attention to the budget constraint (14) treating w, w_K, g_T, g_C and d_D as constants.

To the contrary, the benevolent social planner has full information about the economy, including all externalities, aggregate effects and fiscal rules. Therefore, even though his optimal control problem must formally incorporate the same budget constraint (14), it can be transformed into a simpler form that can be derived by applying all information about the economy. From Eqs. (9) and (12), it follows that $g_T = t + \xi y - r_D d - g_C$. Substituting it into Eq. (14) yields:

$$\begin{aligned} \dot{z} &= c(1+\tau_C) + i\left(1+\frac{\chi}{2}\frac{i}{k}\right) + \left[(1+\tau_Z)r_Z - n\right]z - (1-\tau_L)w + \\ &\quad -(1-\tau_K)w_K k - (1-\tau_D)r_D d_D - t - \xi y + r_D d + g_C + (1-\omega)\xi y. \end{aligned} \tag{15}$$

From Eq. (8), it follows that $t = \tau_L w + \tau_K w_K k + \tau_Z r_Z z + \tau_D r_D d_D + \tau_C c$, and hence Eq. (15) can be reduced to: $\dot{z} = c + i\left(1+\frac{\chi}{2}\frac{i}{k}\right) + \left[r_Z - n\right]z - w - w_K k + r_D(d - d_D) + g_C - \omega\xi y$. Obviously, $w + w_K k = y$, and $d - d_D = d_F$. Therefore, the budget constraint of the social planner reduces to the following form:

$$\dot{z} = c + i\left(1+\frac{\chi}{2}\frac{i}{k}\right) + \left(r_Z - n\right)z - \left(1+\omega\xi\right)y + r_D d_F + g_C. \tag{16}$$

3 The representative agent (the decentralized economy)

The private sector chooses its flows of consumption and investment so as to maximize the level of utility expressed by Eq. (7), subject to the budget constraint (14). Note that d_D is not a decision variable faced by private households: at each moment of time it is the government who decides about the level of public debt (both domestic and foreign) in accordance with the rules (10)–(11). To this end, the private sector in our model behaves passively, accepting any decisions of the government, and buying bonds that the government supplies. Waiving this assumption would significantly add to the complexity of the model, and would render comparisons with existing models far more difficult.

The initial values of variables (endowments) are given by z_0, $k_0 > 0$, $d_0 \geq 0$, $d_{F0} \geq 0$, $d_{D0} \geq 0$ with $d_{F0} + d_{D0} = d_0$. The following variables are treated by an individual decision-maker as exogenous: w, w_K, g_T, g_C, d_D, d_F, r_Z, r_D. The current value hamiltonian is:

$$H_c = \frac{1}{\gamma}\left(c\, g_C^{\kappa}\right)^{\gamma} + \lambda_1' \cdot \left[c(1+\tau_C) + i\left(1 + \frac{\chi}{2}\frac{i}{k}\right) + \left[(1+\tau_Z)r_Z - n\right] z - (1-\tau_L)w + \right.$$
$$\left. -(1-\tau_K)w_K k - (1-\tau_D)r_D^N d_D - g_T + (1-\omega)\xi y\right] + \lambda_2 \cdot \Big[i - (n+\delta)\, k\Big].$$

The solution of this optimization problem (details in the appendix) can be expressed as the following system of differential equations:

$$\begin{bmatrix} \dot{\underline{c}} \\ \dot{\underline{z}} \\ \dot{q} \\ \dot{\underline{d}}_F \\ \dot{\underline{d}}_D \end{bmatrix} = \begin{bmatrix} f^1(\underline{c}, \underline{z}, q, \underline{d}_F, \underline{d}_D) \\ f^2(\underline{c}, \underline{z}, q, \underline{d}_F, \underline{d}_D) \\ f^3(\underline{c}, \underline{z}, q, \underline{d}_F, \underline{d}_D) \\ f^4(\underline{c}, \underline{z}, q, \underline{d}_F, \underline{d}_D) \\ f^5(\underline{c}, \underline{z}, q, \underline{d}_F, \underline{d}_D) \end{bmatrix}, \tag{17}$$

where q represents the market price of capital in relation to the market price of foreign bonds (see Appendix for details). Hereafter, throughout the paper, bars over variables denote their steady-state values. In order to find the steady state one must solve the system of 5 equations: $f^i(\underline{c}, \underline{z}, q, \underline{d}_F, \underline{d}_D) = 0$ $(i = 1, \ldots, 5)$, taking into account that $\bar{r}_Z = \varepsilon_Z + p_Z\bar{\underline{z}}$ and $\bar{\varphi} = (\bar{q}-1)/\chi - (n+\delta)$. First, notice that $f^1 = 0$ immediately yields:

$$\bar{\varphi} = \frac{(1+\tau_Z)\bar{r}_Z - \rho}{A_1}, \tag{18}$$

which is the steady-state rate of growth of all per capita variables: $\hat{\bar{y}} = \hat{\bar{k}} = \hat{\bar{c}} = \hat{\bar{z}} = \bar{\varphi}$. From $f^3 = 0$, it follows that $(1+\tau_Z)\bar{r}_Z = \frac{(1-\tau_K)\alpha A - (1-\omega)\xi A}{\bar{q}} + \frac{(\bar{q}-1)^2}{2\chi\bar{q}} - \delta$. Using this in Eq. (18), and rearranging yields a quadratic equation in $\bar{q}$:

$$a_1 \cdot \bar{q}^2 + a_2 \cdot \bar{q} + a_3 = 0, \tag{19}$$

where $a_1 = 1 - 2\gamma(1+\kappa) > 0$, $a_2 = 2\left[\chi\rho + \gamma(1+\kappa) - \chi n + \chi\gamma(1+\kappa)(n+\delta)\right]$, $a_3 = -\left(1 + 2\chi A\left[(1-\tau_K)\alpha - (1-\omega)\xi\right]\right)$. Notice that $a_1 > 0$, as $\gamma < 0$. We shall assume that $a_3 < 0$, otherwise the deficit would have to be extremely high. (More formal justification for this assumption is presented in the appendix.) The signs of a_1 and a_3 imply that Eq. (19) has 2 real roots: one positive and one negative. The negative one is rejected for the sake of economic interpretation of q. Thus the only viable solution to Eq. (19) is:

$$\bar{q} = \left(\sqrt{\Delta} - a_2\right)/2a_1 > 0, \quad \Delta = a_2^2 - 4a_1a_3 > 0. \tag{20}$$

From Eq. (A.7), it follows that the steady-state rate of growth of the economy is $\bar{\varphi} = (\bar{q}-1)/\chi - (n+\delta)$. The remaining unknown steady-state values can be obtained easily. Eq. (18) yields:

$$\bar{r}_Z = (A_1\bar{\varphi} + \rho)/(1+\tau_Z). \tag{21}$$

From $\bar{r}_Z = \varepsilon_Z + p_Z\bar{\underline{z}}$ and $f^4 = 0$ with $f^5 = 0$, we obtain the steady-state debt-to-GDP ratios:

$$\bar{\underline{z}} = (\bar{r}_Z - \varepsilon_Z)/p_Z, \tag{22}$$

$$\bar{\underline{d}}_F = \frac{\omega\xi}{n+\bar{\varphi}}, \tag{23}$$

$$\bar{\underline{d}}_D = \frac{(1-\omega)\xi}{n+\bar{\varphi}}. \tag{24}$$

Finally, from $f^2 = 0$ one can derive the steady-state consumption-to-GDP ratio:

$$\bar{\underline{c}} = \left(1 + \omega\xi - \frac{\bar{q}^2 - 1}{2A\chi} - (\bar{r}_Z - n - \bar{\varphi})\,\bar{\underline{z}} - \bar{r}_D\bar{\underline{d}}_F\right) \Big/ (1+\sigma_C). \tag{25}$$

In the appendix we prove that the transversality conditions (A.ef) are satisfied if, and only if,

$$(1+\tau_Z)\bar{r}_Z > \bar{\varphi} \; + n, \tag{26}$$

which means that the tax-adjusted interest rate on the private sector's foreign debt must be higher than the GDP growth rate along the balanced growth path. Using Eq. (21), condition (26) can be rewritten as:

$$\rho > n + \gamma(1+\kappa)\bar{\varphi}, \tag{27}$$

which has a very simple interpretation: the rate of discount must simply be sufficiently high.

Proposition 1. *(details in the appendix). The decentralized equilibrium (the balanced growth path) has the form of the stable saddle path. The linear approximation of the model yields the following solution (trajectories):*

$$\begin{bmatrix} \underline{c} & \underline{z} & q & \underline{d}_F & \underline{d}_D \end{bmatrix}^T = \begin{bmatrix} \bar{\underline{c}} & \bar{\underline{z}} & \bar{q} & \bar{\underline{d}}_F & \bar{\underline{d}}_D \end{bmatrix}^T + \sum_{i=1}^{3} s_i e^{r_i t} v^i, \tag{28}$$

where r_i ($i = 1,2,3$) are the negative eigenvalues, and v^i are the corresponding eigenvectors of the Jacobian matrix of Eq. (17) *calculated in the equilibrium. The unknown constants s_i can be obtained by plugging the initial values of debt indicators into Eq.* (28)*, which results in the following system of 3 equations in 3 unknowns:*

$$\underline{z}_0 = \bar{\underline{z}} + \sum_{i=1}^{3} s_i v_2^i, \underline{d}_{F0} = \bar{\underline{d}}_F + \sum_{i=1}^{3} s_i v_4^i, \underline{d}_{D0} = \bar{\underline{d}}_D + \sum_{i=1}^{3} s_i v_5^i.$$

Knowing the eigenvalues, eigenvectors and constants s_i, the remaining two equations of the system (28) *yield the initial values of* $\underline{c}$ *and* q*:* $\underline{c}_0 = \bar{\underline{c}} + \sum_{i=1}^{3} s_i v_1^i$, $q_0 = \bar{q} + \sum_{i=1}^{3} s_i v_3^i$.

Proposition 2. *(proof in the appendix). Welfare in the decentralized economy is equal to*

$$\Omega \approx \tfrac{1}{\gamma}\sigma_C{}^{\kappa\gamma} c_0{}^{\gamma(1+\kappa)} \cdot \cdot \int_0^\infty e^{\left[\frac{[(1+\tau_Z)\bar{r}_Z-\rho]t}{A_1} + \frac{(1+\tau_Z)}{A_1} p_Z \sum_{i=1}^{3} \frac{s_i v_2^i}{r_i}\left(e^{r_i t}-1\right)\right]\gamma(1+\kappa)} e^{-(\rho-n)t} dt. \tag{29}$$

The integral in the above formula is convergent, which follows immediately from the transversality condition (26) *together with the fact that* $Re(r_i) < 0$ *(*$i = 1,2,3$*) and* $\gamma < 0$*. Unfortunately, as we have demonstrated, analytical formulas for* r_i *do not exist. It follows that the examination of the impact of each parameter on welfare is only possible by numerical methods. We take care of this in the empirical part of the paper, having calibrated the model for the economy of Poland.*

4 The benevolent social planner

The social planner maximizes the utility expressed by Eq. (7), subject to the budget constraint (16). He can freely decide on the flow of public consumption, and hence g_C is now an additional control variable. Unlike in the decentralized economy, both interest rates are not exogenous for the decision maker. In particular, in accordance with Eq. (2), the interest rate on public debt is a function of total public debt: $d = d_D + d_F$. Therefore, the optimal control problem needs 2 additional state variables, d_D and d_F, together with the appropriate equations of motion, which can be derived easily from Eqs. (A.15) and (A.16), and have the following form: $\dot{d}_F = \omega\xi y - n d_F$, $\dot{d}_D = (1-\omega)\xi y - n d_D$. Therefore, the current value hamiltonian is:

$$\begin{aligned} H_c &= \frac{1}{\gamma}\left(c\, g_C^{\kappa}\right)^{\gamma} + \lambda_1' \cdot \left[c + i\left(1 + \frac{\chi}{2}\frac{i}{k}\right) + (r_Z - n)z - (1+\omega\xi)\, y + r_D d_F + g_C\right] + \\ &\quad + \lambda_2 \cdot [i - (n+\delta)k] + \lambda_3 \cdot \left[\omega\xi y - n d_F\right] + \lambda_4 \cdot \left[(1-\omega)\xi y - n d_D\right]. \end{aligned}$$

The solution of social planner's optimization problem (details in the appendix) can be expressed as the system of differential equations similar in structure to (17), but with different $f^i(\underline{c}, \underline{z}, q, \underline{d}_F, \underline{d}_D) = 0$ $(i = 1,2,3)$ functions.

Proposition 3. *(proof in the appendix). Finding the first-best steady-state equilibrium (the balanced growth path) requires solving the following fifth-order polynomial equation*

$$w_5\bar{\varphi}^5 + w_4\bar{\varphi}^4 + w_3\bar{\varphi}^3 + w_2\bar{\varphi}^2 + w_1\bar{\varphi} + w_0 = 0, \tag{30}$$

where the coefficients w_i are very complicated nonlinear functions of parameters. (Explicit formulae for these coefficients are presented in the appendix.)

A fifth-order polynomial equation can have up to 5 real roots. Therefore, there is a potential problem of nonuniqueness and nonexistence of a balanced growth equilibrium. Due to the complexity of Eq. (30), these problems cannot be eliminated by any simple algebraic assumptions. The only way to cast some light on these issues is by numerical methods. We have performed numerous (virtually thousands) simulations for this model, calibrated on the basis of statistical data on the Polish economy, widely varying decision parameters, as well as numerous exogenous parameters. Most parameters were varied within intervals ranging more than ±50% from their baseline values (see below) – wide range of simulations is presented in chapter 5 of Konopczyński (2015). In all cases, without any exception, Eq. (30) always turned out 1 positive real root, 2 negative real roots (not acceptable, as it would shrink the economy to zero), and 2 complex conjugate roots (not feasible for obvious reasons). Therefore, henceforth we assume that Eq. (30) has a unique viable (real and positive) solution $\bar{\varphi}^*$, which we call the balanced growth rate (the BGR). Henceforth, the optimal solution obtained by the benevolent social planner is denoted by stars, to distinguish it clearly from the second best solution obtained by the representative agent in the decentralized economy.

Obviously, knowing $\bar{\varphi}^*$ allows a straightforward derivation of all other steady-state values. In the appendix we prove that the transversality conditions (F.h–k) are satisfied if, and only if:

$$\rho > n + \gamma(1+\kappa)\bar{\varphi}^*, \tag{31}$$

which is identical with Eq. (27) for the decentralized economy.

Proposition 4. *(details in the appendix). The first-best equilibrium (the balanced growth path) has the form of the stable saddle path. The linear approximation of the model yields the following solution (trajectories):*

$$\begin{bmatrix} \underline{c}^* & \underline{z}^* & q^* & \underline{d}_F^* & \underline{d}_D^* \end{bmatrix}^T = \begin{bmatrix} \bar{\underline{c}}^* & \bar{\underline{z}}^* & \bar{q}^* & \bar{\underline{d}}_F^* & \bar{\underline{d}}_D^* \end{bmatrix}^T + \sum_{i=1}^{3} s_i e^{r_i t} v^i, \tag{32}$$

where r_i $(i = 1, \ldots, 3)$ are the negative eigenvalues, and v^i are the corresponding eigenvectors of the respective Jacobian matrix M^*. The unknown constants s_i can be obtained by plugging the initial values of debt indicators into Eq. (32)*, which results in the following system of 3 equations in 3 unknowns:*

$$\underline{z}_0 = \bar{\underline{z}}^* + \sum_{i=1}^{3} s_i v_2^i, \underline{d}_{F0} = \bar{\underline{d}}_F^* + \sum_{i=1}^{3} s_i v_4^i, \underline{d}_{D0} = \bar{\underline{d}}_D^* + \sum_{i=1}^{3} s_i v_5^i.$$

Knowing the eigenvalues, eigenvectors and constants s_i the remaining two equations of Eq. (32) *determine the initial values of $\underline{c}$ and q: $\underline{c}_0 = \bar{\underline{c}}^* + \sum_{i=1}^{3} s_i v_1^i$, $\underline{q}_0 = \bar{q}^* + \sum_{i=1}^{3} s_i v_3^i$.*

Proposition 5. *(proof in the appendix). Welfare delivered by the benevolent social planner is equal to:*

$$\begin{aligned}\Omega^* &= \frac{1}{\gamma}\kappa^{\kappa\gamma}{c_0}^{\gamma(1+\kappa)} \cdot \\ &\cdot \int_0^\infty e^{\left[\frac{(\bar{r}_Z^* + p_Z \underline{\bar{z}}^* - \rho)t}{A_1} + \frac{2p_Z}{A_1}\sum_{i=1}^{3}\frac{s_i v_2^i}{r_i}\left(e^{r_i t} - 1\right)\right]\gamma(1+\kappa)} e^{-(\rho-n)t} dt.\end{aligned} \tag{33}$$

The integral in this formula converges, which follows from the transversality condition (31) *together with the fact that* $Re(r_i) < 0$ *(*$i = 1, 2, 3$*) and* $\gamma < 0$*.*

5 Fiscal policy and the long-run equilibrium

5.1 The income and consumption taxes

All the income and consumption tax rates are fully neutral for the benevolent social planner, i.e., they influence neither the steady state, nor the transitory dynamics, nor the level of welfare delivered by the social planner, Ω^*. (It follows directly from a simple observation that none of the tax rates τ_i show up in the formulae describing the balanced growth path and the linear approximation of the trajectories converging towards this path.) All taxes are therefore fully neutral in the first-best equilibrium. In the decentralized economy three tax rates remain fully neutral as well: taxes on labor, consumption and domestic bonds. To the contrary, taxes on capital income and interest on private foreign debt do affect the economy. The derivatives of the steady state with respect to these tax rates together with their signs are reported in table 1. Importantly, the welfare Ω in the decentralized economy depends on these two tax rates, but investigating these relationships is only possible by numerical methods.

5.2 The budget deficit and its financing

All the remaining parameters of fiscal policy also influence both economies. In particular, the trajectories of virtually all variables depend on the size of public deficit, and the structure of public debt (the share of foreign creditors). Consequently, these parameters do impact welfare in both economies. Unfortunately, an analytical examination of these relationships is not possible in case of the first-best solution, because an analytical solution to the model of the benevolent social planner does not exist. To draw any conclusions numerical methods are necessary.

To the contrary, in the decentralized economy, this type of analysis is simple. Table 2 reports the derivatives (and their signs) of the long-run equilibrium (the steady state) with respect to ξ and ω.

The higher the rate of budget deficit ξ, the lower the investment-to-GDP rate $\underline{\bar{i}}$ and the long-run rate of growth $\bar{\varphi}$. It follows directly from Eqs. (23) and (24) that higher deficit (as a percentage of GDP) together with lower growth rate of GDP have

Table 1: How the steady state in the decentralized economy is influenced by taxation

(.)	$\partial(.)/\partial\tau_K$	sign	$\partial(.)/\partial\tau_Z$	sign
$\bar{\underline{i}}$	$-\frac{2\alpha}{\sqrt{\Delta}}$	$-$	0	0
$\bar{\varphi}$	$-\frac{2\alpha A}{\sqrt{\Delta}}$	$-$	0	0
$\bar{r}_Z$	$-\frac{A_1}{(1+\tau_Z)}\frac{2\alpha A}{\sqrt{\Delta}}$	$-$	$-\frac{\bar{r}_Z}{1+\tau_Z}$	$-$
$\bar{\underline{z}}$	$-\frac{A_1}{p_Z(1+\tau_Z)}\frac{2\alpha A}{\sqrt{\Delta}}$	$-$	$-\frac{\bar{r}_Z}{p_Z(1+\tau_Z)}$	$-$
$\bar{r}_D$	$\frac{2\alpha A\xi p_D}{(n+\bar{\varphi})^2\sqrt{\Delta}}$	$+$	0	0
$\bar{\underline{d}}_F$	$\frac{2\alpha A\xi\omega}{(n+\bar{\varphi})^2\sqrt{\Delta}}$	$+$	0	0
$\bar{\underline{d}}_D$	$\frac{2\alpha A\xi(1-\omega)}{(n+\bar{\varphi})^2\sqrt{\Delta}}$	$+$	0	0
$\bar{\underline{c}}$	*complex formula*	?	$\frac{\bar{r}_Z(p_Z\bar{\underline{z}}+\bar{r}_Z-n-\bar{\varphi})}{p_Z(1+\tau_Z)(1+\sigma_C)}$	?
Ω	*analytical formula does not exist*	?	*analytical formula does not exist*	?

negative impact on the equilibrium levels of both indicators of public debt: $\bar{\underline{d}}_F$ and $\bar{\underline{d}}_D$. As the result, the interest on public debt also rises. The situation in the private sector is different: lower rate of investment $\bar{\underline{i}}$, which is in part financed with foreign loans, implies a reduction in private foreign debt $\bar{\underline{z}}$. As the result, the interest rate $\bar{r}_Z$ on this debt falls.
There is also a relationship between ξ and private consumption, as follows:

$$\begin{aligned} \frac{\partial\bar{\underline{c}}}{\partial\xi} \quad = \quad & \frac{1}{1+\sigma_C}\Bigg[\omega+\frac{2(1-\omega)}{\sqrt{\Delta}}\left(\bar{q}-A\bar{\underline{z}}+\frac{AA_1(\bar{r}_Z+p_Z\bar{\underline{z}}-n-\bar{\varphi})}{p_Z(1+\tau_Z)}\right)+ \\ & -\left(n+\bar{\varphi}-\xi\frac{\partial\bar{\varphi}}{\partial\xi}\right)\frac{p_D\bar{\underline{d}}_F+\bar{r}_D\omega}{(n+\bar{\varphi})^2}\Bigg]. \end{aligned} \tag{34}$$

Given all the assumptions taken so far, this derivative can be negative or positive (or, in the special case, equal to zero), depending on the values of the individual parameters of the model. Let us look more closely at Eq. (34). Note that the sign (and value) of this derivative depends, among other things, on financial parameters: ε_D and p_D. Moreover, these parameters influence solely one part of Eq. (34), i.e. the following expression $\left(n+\bar{\varphi}-\xi\frac{\partial\bar{\varphi}}{\partial\xi}\right)(p_D\bar{\underline{d}}_F+\bar{r}_D\omega)/(n+\bar{\varphi})^2$. Note that this expression is always positive, and it increases with both ε_D and p_D. It implies that, with sufficiently high values of ε_D and p_D, the derivative $\partial\bar{\underline{c}}/\partial\xi<0$. The intuition behind this result is simple: “world financial crisis”, or “world crisis of confidence” (manifesting itself with a high value of the average ‘world’ interest rate on sovereign

Table 2: How the steady state in the decentralized economy depends on the size of government deficit and the share of foreign creditors in public debt

(.)	$\partial(.)/\partial\xi$	sign	$\partial(.)/\partial\omega$	sign
$\bar{q}$	$-\frac{2\chi A(1-\omega)}{\sqrt{\Delta}}$	$-$	$\frac{2\chi A\xi}{\sqrt{\Delta}}$	$+$
$\underline{\bar{i}}$	$-\frac{2(1-\omega)}{\sqrt{\Delta}}$	$-$	$\frac{2\xi}{\sqrt{\Delta}}$	$+$
$\bar{\varphi}$	$-\frac{2A(1-\omega)}{\sqrt{\Delta}}$	$-$	$\frac{2A\xi}{\sqrt{\Delta}}$	$+$
$\bar{r}_Z$	$-\frac{2AA_1(1-\omega)}{(1+\tau_Z)\sqrt{\Delta}}$	$-$	$\frac{2AA_1\xi}{(1+\tau_Z)\sqrt{\Delta}}$	$+$
$\underline{\bar{z}}$	$-\frac{2AA_1(1-\omega)}{p_Z(1+\tau_Z)\sqrt{\Delta}}$	$-$	$\frac{2AA_1\xi}{p_Z(1+\tau_Z)\sqrt{\Delta}}$	$+$
$\bar{r}_D$	$\frac{p_D}{(n+\bar{\varphi})^2}\left(n+\bar{\varphi}-\xi\frac{\partial\bar{\varphi}}{\partial\xi}\right)$	$+$	$\frac{-2p_D A\xi^2}{\sqrt{\Delta}(n+\bar{\varphi})^2}$	$-$
$\underline{\bar{d}}_F$	$\frac{\omega}{(n+\bar{\varphi})^2}\left(n+\bar{\varphi}-\xi\frac{\partial\bar{\varphi}}{\partial\xi}\right)$	$+$	$\underline{\bar{d}}\left(1-\frac{2A\xi\omega}{\sqrt{\Delta}(n+\bar{\varphi})}\right)$	?
$\underline{\bar{d}}_D$	$\frac{1-\omega}{(n+\bar{\varphi})^2}\left(n+\bar{\varphi}-\xi\frac{\partial\bar{\varphi}}{\partial\xi}\right)$	$+$	$-\underline{\bar{d}}\left(1+\frac{2A\xi(1-\omega)}{\sqrt{\Delta}(n+\bar{\varphi})}\right)$	$-$
$\underline{\bar{d}}$	$\frac{1}{(n+\bar{\varphi})^2}\left(n+\bar{\varphi}-\xi\frac{\partial\bar{\varphi}}{\partial\xi}\right)$	$+$	$-\underline{\bar{d}}\frac{2A\xi}{\sqrt{\Delta}(n+\bar{\varphi})}$	$-$
$\underline{\bar{c}}$	Eq. (34)	?	Eq. (35)	?
Ω	*analytical formula does not exist*	?	*analytical formula does not exist*	?

debt ε_D) and/or a "country-specific crisis of confidence" (manifesting itself with a high value of the country-specific risk premium p_D), increasing the budget deficit results in a decrease in the long-run share of private consumption in GDP. Recall that public consumption is proportional to private, therefore public consumption (as a share of GDP) also falls. As higher budget deficit lowers both shares of consumption in GDP, and, on the other hand, it also reduces the growth rate of GDP, thus it unambiguously reduces the value of the achieved welfare Ω (trajectories of private and public consumption per capita shift down). It means that, with sufficiently high cost of borrowing by the government, from the point of view of welfare it's worth to minimize the budget deficit. This proposition is intuitively understandable: if the lender demands high interest rate, the government should reduce public debt to zero, allowing the decentralized economy to reach maximum welfare.

To the contrary, with sufficiently low values of ε_D and p_D, the derivative $\partial\underline{\bar{c}}/\partial\xi$ may well be positive (though not necessarily, because the values of individual components in Eq. (34) also depend on other parameters of the model). If it is indeed positive, then the situation is different: increasing the budget deficit raises both shares of consumption in GDP. However, we know that it also reduces the long-run growth rate of the GDP, so even if $\partial\underline{\bar{c}}/\partial\xi > 0$, we do not know whether the trajectories of private and public consumption per capita shift up or down. Thus, we cannot be sure whether higher budget deficit is beneficial from the point of welfare, or otherwise.

To cast some more light on this issue, consider the extreme case of free credit for the government ($\varepsilon_D = p_D = 0$). Obviously, in that case it's worth to borrow as much as possible, i.e. to maximize the deficit (as long as both parameters remain equal to zero). Unlimited, free capital from abroad would effectively remove the "budget constraint" of the nation as a whole, and – at least in theory – allow to reach infinitely high welfare. This is, of course, purely hypothetical scenario – in reality, as public debt rises, so does the interest rate.

To summarize, our model recommends minimizing the budget deficit, if the cost of borrowing by the government is high enough (the "crisis of confidence"). In addition, it is quite possible that even under conditions of cheap credit the government should do exactly the same. Only in case of very low cost of borrowing it could be beneficial (for the nation), if the government would borrow as much as possible.

As a final point, note that it appears that the influence of public deficit ξ on the level of welfare achieved by the decentralized economy (Ω) could be investigated analytically: we do have an analytical formula (29) for Ω, so maybe we could calculate the derivative $\partial\Omega/\partial\xi < 0$? Unfortunately, this is not possible, for a simple reason: Ω depends on the eigenvalues and eigenvectors of the Jacobian matrix M, which in turn depends on ξ. Thus Ω is indeed a function of ξ. Nevertheless, this function does not have an analytical form, because the eigenvalues of the matrix M are the roots of the polynomial of order 5, so there are no analytical formulas for them. Thus the derivative $\partial\Omega/\partial\xi$ does not have an analytical form: it can only be calculated numerically after substituting certain values for all parameters of the model.

Let us turn our attention to the share of foreign lenders in public debt, ω. The higher the value of ω, the higher the investment-to-GDP rate $\underline{\bar{i}}$ and the long-run rate of growth $\bar{\varphi}$. The intuition behind this result is simple: at a given level of $\xi > 0$, the more the government borrows from abroad, the less financing it requires from domestic lenders, so more resources remain in the hands of the private sector, which are later partly invested in productive capital. (Borrowing more from abroad effectively shifts the national budget constraint up.) A higher rate of investment coupled with faster economic growth affect private foreign debt and the related interest rate $\bar{r}_Z$: the equilibrium levels of both go up. This might seem a little surprising, but it follows directly from Eqs. (21) and (22).

Obviously, an increase in ω also affects the long-run equilibrium levels of all indicators of public debt: $\underline{\bar{d}}_D$ (domestic), $\underline{\bar{d}}_F$ (foreign) and $\underline{\bar{d}}$ (total). The relationship between $\underline{\bar{d}}_D$ and ω is unambiguous: for any given level of deficit $\xi > 0$, an increase in ω together with an induced increase in the long-run rate of growth $\bar{\varphi}$ decreases the numerator and raises the denominator in Eq. (24). Thus $\partial\underline{\bar{d}}_D/\partial\omega < 0$, which means that the higher the share of foreign lenders in public debt, the lower the equilibrium level of domestic public debt (as a share of GDP). It would seem that in the case of foreign

public debt it must be the opposite. However, the situation is more complicated. An increase in the value of the parameter ω raises the numerator of the fraction in Eq. (23). However, the denominator of this fraction also raises, due to a higher value of the rate of growth $\bar{\varphi}$. Without extra assumptions it is not known, which of these two effects will prevail. It can easily be verified analytically that the adopted assumptions do not prejudge the sign of the derivative $\partial\bar{d}_F/\partial\omega$. It can be negative or positive – it depends on the values of many parameters characterizing the economy. (Strictly, on all those that occur in the formula for the derivative of $\partial\bar{d}_F/\partial\omega$ given in Table 2.) Interestingly, despite this ambiguity, the derivative $\partial\bar{d}/\partial\omega$ is unambiguously negative. This means that the higher the share of foreigners in public debt, the lower is the equilibrium level of the total public debt relative to GDP. Consequently, the lower is the interest rate $\bar{r}_D$.
There is also a complex relationship between ω and private consumption, as follows:

$$\begin{aligned}\frac{\partial\underline{\bar{c}}}{\partial\omega} &= \frac{1}{1+\sigma_C}\left[\xi - \frac{2\xi}{\sqrt{\Delta}}\left(\bar{q} - A\underline{\bar{z}} + \frac{AA_1\left(\bar{r}_Z + p_Z\underline{\bar{z}} - n - \bar{\varphi}\right)}{p_Z(1+\tau_Z)}\right) + \right.\\ &\left. + \frac{2A\underline{\bar{d}}\bar{d}_F}{\sqrt{\Delta}}\left(p_D\underline{\bar{d}} + \bar{r}_D\right) - \bar{r}_D\underline{\bar{d}}\right].\end{aligned} \tag{35}$$

This derivative can be negative or positive (or, in the special case, equal to zero), depending on the values of the individual parameters of the model. The sign (and value) of this derivative depends, among other things, on financial parameters: ε_D and p_D. Thus it is possible to make a deeper inquiry – very similar to what we did above with Eq. (34). Without going into mathematical details, it is easy to show that with sufficiently high values of ε_D and p_D, the derivative $\partial\underline{\bar{c}}/\partial\omega < 0$. Thus, if borrowing abroad (by the government) becomes expensive enough, increasing the share of foreigners in public debt results in a decrease in the long-run share of private consumption in GDP. Since private consumption is proportional to public consumption, the latter (as a share of GDP) also falls. Apart from these detrimental effects, we also have a favorable effect: recall that an increase in ω unambiguously raises the rate of growth $\bar{\varphi}$. Thus (unlike in the case of public deficit ξ), we cannot be sure whether it implicates lower welfare Ω. Generally, we cannot draw any definitive conclusions regarding the optimal level of ω, even in case of "very high cost of foreign credit". In order to draw clear-cut conclusions the model must be calibrated and simulated. Without numerical analysis, we can only put forward the following rather trivial observation. In case of free credit for the government ($\varepsilon_D = p_D = 0$), it's worth to borrow abroad as much as possible, i.e. to maximize the share of foreign lenders in public debt. (As we noted above, free capital from abroad would effectively remove the "budget constraint" of the nation, and allow to reach infinitely high welfare.) Therefore, if ε_D and p_D remain sufficiently low, the government should maximize the share of foreign lenders (which entails $\omega = 1$).
As a final point, note that the relationship between ω and the level of welfare Ω cannot

be investigated analytically, for identical reasons as in the case of ξ: the derivative $\partial\Omega/\partial\omega$ does not have an analytical form: it can only be calculated numerically.

6 The replication – the optimal fiscal policy

The optimal control problem of the benevolent social planner incorporates all the externalities and rules governing the economy, which are not taken into account by the representative agent. This fact alone implies that the decentralized economy cannot outperform the social planner in terms of welfare, i.e., $\Omega \leq \Omega^*$. However, the social planner may induce individual economic agents to internalize all the externalities by proper adjustment of fiscal policy, which is known in the literature as the problem of replication of the first-best solution.

Proposition 6. *(proof in the appendix). The decentralized economy replicates the first-best solution if, for every $t \geq 0$, the following conditions hold:*

$$\sigma_C^{opt} = \kappa, \tag{36}$$

$$\tau_Z^{opt}(t) = \frac{r_Z^*(t) - \varepsilon_Z}{r_Z^*(t)}, \tag{37}$$

and the capital income tax rate appropriately evolves over time. However, it is impossible to derive an analytical formula for the optimal trajectory $\tau_K^{opt}(t)$– not only for the whole trajectory, but even for a single value of this parameter in a selected moment of time t. Individual (momentary) values of the trajectory $\tau_K^{opt}(t)$ can only be calculated numerically, for a given set of values of all the model parameters.

Note that other tax rates (τ_D, τ_C, τ_L) do not affect the replication, because (as we exposed in section 5) they are neutral for both types of the economy. It is interesting and less obvious, however, that two other parameters of fiscal policy (ξ, ω) also do not affect the replication, which of course does not necessarily mean that they do not affect the level of welfare in the first-best equilibrium (we investigate this issue in section 11).

Proposition 6 is a prescription for the internalization of all the externalities listed in Introduction. However, in a practical perspective, a real-world implementation of this recipe is hard to imagine, as the two tax rates involved would have to be continuously adjusted over time (in practice, very often, perhaps once per year). For this reason, it is worth to solve a simpler problem of partial replication (of the steady state only). Obviously, full replication implies partial replication, but not otherwise.

Proposition 7. *(proof in the appendix). If the decentralized economy replicates the first-best steady state, then the following (necessary) conditions hold together:*

$$\sigma_C = \sigma_C^{rep} = \kappa, \tag{38}$$

$$\tau_Z = \tau_Z^{rep} = \frac{\bar{r}_Z^* - \varepsilon_Z}{\bar{r}_Z^*}, \tag{39}$$

and the tax rate on capital, τ_K, is constant over time and equal to τ_K^{rep}, the value of which can only be calculated numerically. (An explicit formula for this value does not exist.)

7 The comparison with the existing models

As mentioned in the Introduction, our model is an extension of the existing models. Therefore, it is possible to reduce our model to the ones discussed in the literature. In particular, by setting certain parameters to zero and equating the tax rate on wages with the tax rate on capital income our model becomes (almost) identical with one of the models presented in chapter 4 of Turnovsky (2009). Just one minor difference remains: we apply a linear form of the interest rate function (1), whereas Turnovsky applies a more general form: some unspecified, but increasing and differentiable function (4.1). Hereafter, all references to Turnovsky's model in this section refer to Turnovsky (2009), chapter 4. Let us now turn to details.

First, we need to make the two interest rates, $r_Z$and r_D, identical. So instead of two separate Eqs. (1) and (2) we apply:

$$r_D = r_Z = r_Z(Z/Y) = r_Z(z/y) = \varepsilon_Z + p_Z z/y, \tag{1a}$$

which happens to be a special case of Turnovsky's assumption (4.1). This assumption means that the private sector and the public sector face the same base interest rate and the same risk premium. (Note that instead of shares in GDP Turnovsky applies ratios to capital. However, with the AK production function, our approach is de facto identical.)

Second, we need to assume that $\delta = 0$ (no depreciation of productive capital) and $n = 0$ (constant population). The latter makes the effective discount rate in our utility function (7) identical to Turnovsky's ρ.

Third, Turnovsky assumes that the government maintains the balanced budget in each period, and thus has no debt. Therefore, we need to assume that $\xi = 0$ (so that public deficit is always equal to zero) and (to ensure zero public debt for all $t \geq 0$) we have to remove any initial public debt by setting $d_0 = d_{D0} = d_{F0} = 0$.

Fourth, we need to make the structure of taxes identical to Turnovsky's. We distinguish 5 different tax rates in Eq. (8), whereas Turnovsky applies 3. However, since there is no public debt now, τ_D effectively disappears (regardless of its value, the tax revenue is zero, because the tax base is zero). If we assume that $\tau_L = \tau_K$, and replace both with a new symbol τ_Y, then tax revenue defined by Eq. (8) becomes:

$$T = \tau_Y(wL + w_K K) + \tau_C C + \tau_Z r_Z Z = \tau_Y Y + \tau_C C + \tau_Z r_Z Z, \tag{8a}$$

which is identical with Turnovsky's assumption (4.7b).

Fifth, Turnovsky assumes that the tax revenue is completely rebated to the private sector (so that taxes have purely distortionary role, and no other functions). We can accomplish that by eliminating public consumption, i.e. assuming that $\sigma_C = 0$. Our Eqs. (9) and (12) are now reduced to: $G = T = G_T$, which is exactly what Turnovsky assumes. Obviously, to be able to solve the model, we must also remove g_C from the utility function (7). The simplest way to that is set $\kappa = 0$.

These five simple steps reduce our model to the model presented in chapter 4 of Turnovsky (2009). It follows that the budget constraint of the private sector (14) is now reduced to the following:

$$\dot{z} = c(1+\tau_C) + i\left(1 + \frac{\chi}{2}\frac{i}{k}\right) + (1+\tau_Z)r_Z z - (1-\tau_Y)y - g_T, \tag{14a}$$

which is identical to Turnovsky's equation (4.3a). The budget constraint of the social planner (16) is now reduced to:

$$\dot{z} = c + i\left(1 + \frac{\chi}{2}\frac{i}{k}\right) + r_Z z - y, \tag{16a}$$

which is identical to Turnovsky's "national resource constraint" (4.7c).

Obviously, solving such a reduced model replicates all mathematical results and conclusions of Turnovsky. All the theoretical results of this paper can also be directly "reduced" to the results of Turnovsky by plugging all the assumptions described above into our formulae, i.e. the following: $\delta = 0$, $n = 0$, $\sigma_C = 0$, $\kappa = 0$, $\xi = 0$, $d_0 = d_{D0} = d_{F0} = 0$, $\tau_L = \tau_K = \tau_Y$ together with (1a). One can easily verify that plugging them into Eqs. (A.12), (A.14), (A.9) reduces them to Turnovsky's Eqs. (4.9a), (4.9b) and (4.9c). Similarly, our Eqs. (F.10), (F.12) and (F.9) become Turnovsky's Eqs. (4.9a'), (4.9b') and (4.9c'), and etc. (Turnovsky's equation (4.3a) contains a little typo: rather than $+T_i$ it should be $-T_i$.)

Our model can therefore be regarded as an extension (generalization) of Turnovsky's model. As mentioned in the Introduction, apart from some relatively minor issues (e.g. introducing depreciation of productive capital, public consumption and population growth) there are 2 important generalizations which qualitatively change the theoretical results and conclusions:

1. The introduction of government deficit financed by public debt composed of domestic and foreign debt, coupled with a more detailed structure of taxes and public consumption.

2. The introduction of two separate interest rates (for private and public sector); both linear functions of the debt-to-GDP ratios.

It's worth to compare our results with Turnovsky's and draw some conclusions. Let us start with the decentralized economy.

By comparing (19) with its counterpart in Turnovsky's model, (4.13), one can easily notice that 5 parameters show up that are absent in (4.13): n, δ, κ, ξ, ω. Recall also that the steady-state rate of growth is $\bar{\varphi} = (\bar{q} - 1)/\chi - (n + \delta)$. It implies that most (but, importantly, not all – more on that in the next 2 paragraphs) of the extensions incorporated in our model do influence (in a non-trivial way) the long-run rate of growth (per capita) of the decentralized economy. To make sure, the list includes:

1. the rate of growth of population n,
2. the rate of depreciation of productive capital δ,
3. the elasticity of substitution between public and private consumption κ,
4. the size of public deficit ξ,
5. the structure of public debt: the share of foreign creditors ω.

Note also that Eq. (19) contains τ_K rather than τ_Y in Turnovsky's equation (4.13). Recall that Turnovsky is taxing both labor and capital at the same rate equal to τ_Y, whereas we apply two distinct rates. It implies that the long-run rate of growth is independent of the tax rate on labor. The reason is simple: labor is supplied exogenously, and in competitive equilibrium always fully employed. In fact, only 1 out of 5 different tax rates included in our model affects $\bar{\varphi}$, namely the tax rate on capital income τ_K (see also Table 1).

Importantly, though, our extensions do not change one important conclusion from Turnovsky's model: the long-run domestic growth rate $\bar{\varphi}$ is independent of parameters determining (what Turnovsky calls) "external" borrowing costs, i.e. ε_Z, ε_D, p_Z, and p_D. The long-run domestic growth rate $\bar{\varphi}$ is determined entirely by "internal" conditions, i.e. the technological and demographic parameters A, α, χ, δ, n, parameters of the utility function ρ, γ, and κ, and 3 fiscal parameters: the tax rate τ_K, the size of public deficit ξ, and the share of foreign creditors in public debt ω.

Let us analyze in more detail the relationships between the costs of borrowing and the equilibrium in the decentralized economy. Table 3 summarizes the long-run effects of changes in the costs of borrowing.

Similarly as in Turnovsky's model, given that $\bar{\varphi}$ is independent of ε_Z and p_Z, it follows from equation (21) that the net cost of borrowing from abroad by the private sector, $\bar{r}_Z(1 + \tau_Z)$, is independent of ε_Z and p_Z. Therefore, a sudden increase in the base interest rate ε_Z and/or the risk premium p_Z, which obviously raises the cost of foreign credit and discourages borrowing from abroad, over time leads to a reduction in the equilibrium debt-to-GDP ratio $\underline{\bar{z}}$ (one can see it directly from Eq. (22)). Moreover, this reduction in $\underline{\bar{z}}$ exactly offsets the increase in ε_Z and p_Z, so that the overall net unit cost of foreign lending to the private sector, $\bar{r}_Z(1 + \tau_Z)$ remains unchanged.

One could expect that an analogous mechanism of the long-run adjustment applies to the public sector, and indeed it does apply, but only in the model of the benevolent social planner. In the decentralized economy, the government behaves in accordance with rigid budgetary rules listed in section 2.4, so it does not adjust the level of borrowing to any changes in the cost of external and domestic credit. To see why,

Table 3: How the steady state in the decentralized economy is influenced by the costs of borrowing

(.)	$\partial(.)/\partial\varepsilon_Z$	$\partial(.)/\partial p_Z$	$\partial(.)/\partial\varepsilon_D$	$\partial(.)/\partial p_D$
$\bar{q}$	0	0	0	0
$\underline{\bar{i}}$	0	0	0	0
$\bar{\varphi}$	0	0	0	0
$\bar{r}_Z$	0	0	0	0
$\underline{\bar{z}}$	$-1/p_Z < 0$	$-\underline{\bar{z}}/p_Z < 0$	0	0
$\bar{r}_D$	0	0	1	$\xi/(n+\bar{\varphi}) > 0$
$\underline{\bar{d}}_F$	0	0	0	0
$\underline{\bar{d}}_D$	0	0	0	0
$\underline{\bar{c}}$	$\dfrac{(\bar{r}_Z - n - \bar{\varphi})}{p_Z(1+\sigma_C)} > 0$	$\dfrac{(\bar{r}_Z - n - \bar{\varphi})\underline{\bar{z}}}{p_Z(1+\sigma_C)} > 0$	$\dfrac{-\underline{\bar{d}}_F}{1+\sigma_C} < 0$	$\dfrac{-\xi\underline{\bar{d}}_F}{(1+\sigma_C)(n+\bar{\varphi})} < 0$
Ω	*analytical formula does not exist*	*analytical formula does not exist*	*analytical formula does not exist*	*analytical formula does not exist*

recall that $\bar{\varphi}$ is independent of ε_D and p_D, and so Eqs. (23) and (24) imply that the equilibrium debt-to-GDP ratios $\underline{\bar{d}}_F$ and $\underline{\bar{d}}_D$ are independent of ε_D and p_D. It follows from Eq. (2) that a sudden increase in the base interest rate ε_D and/or the risk premium p_D (which obviously raises the cost of borrowing by the government) does not discourage the government's borrowing. (Recall that we are still analyzing the decentralized economy.) Unlike the private sector, in the model of decentralized economy the government is rigid (more precisely, the representative agent takes government's actions as given) – it does not react to any changes in the cost of borrowing. (Of course, it is completely different in the model of the benevolent social planner, that we will soon turn to.)

Consumption-to-GDP ratio $\underline{\bar{c}}$ is also influenced by the cost of borrowing. Interestingly, an increase in private financial parameters (the base interest rate ε_Z and/or the risk premium p_Z) leads to higher $\underline{\bar{c}}$, whereas an increase in public parameters (ε_D and p_D) reduces it. The first mechanism was explained by Turnovsky (2009), p. 73–74. In short, an increase in ε_Z and/or p_Z reduces the equilibrium debt-to-GDP ratio $\underline{\bar{z}}$, but the overall net unit cost of foreign borrowing, $\bar{r}_Z(1+\tau_Z)$, remains unchanged. Therefore, the total cost of servicing private foreign debt is lower, leaving more resources available for consumption, and thus raising the equilibrium consumption-to-GDP ratio $\underline{\bar{c}}$.

The second mechanism – related to public sector – is not included in Turnovsky's model. An increase in the base interest rate ε_D results in a proportional (linear) increase in the interest rate faced by the government, $\bar{r}_D$, but leaves the level of public debt (in relation to GDP; both domestic and foreign) unchanged. Therefore, the total cost of servicing public debt raises. It has negative consequences for the budget constraint of the nation as a whole – in particular, given all rules of fiscal spending and taxation, the government is forced to cut financial transfers to households g_T, which reduces resources for private consumption. Note that this mechanism works only through foreign public debt: if $\underline{\bar{d}}_F$ is zero (which requires setting $\omega = 0$), then private consumption is independent of ε_D. An intuition is straightforward: the interest on domestic public debt becomes the revenue of the private sector, so whatever the interest rate on domestic public debt, it does not matter for the national budget constraint.

An increase in the risk premium p_D has qualitatively identical influence on the long-run equilibrium – the same mechanisms and intuition applies.

Finally, note that the welfare Ω in the decentralized economy depends on all 4 financial parameters, but investigating these relationships is only possible by numerical methods (though intuition allows to expect that the cost of borrowing should be negatively correlated with the welfare of the nation).

Let us now look at the benevolent social planner. The most important difference compared to the decentralized economy is that the equilibrium rate of growth $\bar{\varphi}^*$ is now a function of virtually all parameters of the model, which is clearly visible from Eq. (30), except for all the tax rates (recall that they are fully neutral for the benevolent social planner). It implies that most of the extensions incorporated in our model do influence (in a non-trivial way) the long-run rate of growth (per capita) of the 'centrally planned' economy. Again, to make sure, the list of factors that do influence the equilibrium rate of growth $\bar{\varphi}^*$ includes:

1. the rate of growth of population n,
2. the rate of depreciation of productive capital δ,
3. the elasticity of substitution between public and private consumption κ,
4. the size of public deficit ξ,
5. the structure of public debt: the share of foreign creditors ω.

Moreover, the rate of growth $\bar{\varphi}^*$ is dependent on all parameters determining borrowing costs, i.e. ε_Z, ε_D, p_Z, and p_D. (Technically, it is an obvious consequence of the fact that the benevolent social planner takes into account all externalities, whereas the representative agent does not.) It's also the case in Turnovsky's model, although he applies a single interest rate for both public and private sector. In Turnovsky's words: "an increase in the cost of borrowing, whether in the form of a higher foreign interest rate or a higher risk premium, will have an adverse effect on the long-run growth rate". Unlike in Turnovsky's model, in our extended model, due to the complexity of Eq. (30), we cannot analytically determine the direction of the relationship between the rate of growth $\bar{\varphi}^*$ and financial parameters: ε_Z, ε_D, p_Z and p_D. Therefore, in the next section we will turn to calibration and simulation of the model.

Finally, note that our propositions regarding the replication of the first-best equilibrium in section 6 constitute a generalization of Turnovsky's (2009) results presented in his section 4.1.4. In principle, they have similar economic interpretation (they allow the representative agent to internalize all externalities), and so there is no need to replicate Turnovsky's comments on that issue.

8 Model calibration for Poland

Tables 4 and 5 present the base set of parameters and the initial values (endowments) together with a concise explanation of the sources and methods of calibration. The calibration was based on macroeconomic statistics for the period 2000 – 2013, published by the Eurostat, the National Bank of Poland, the Central Statistical Office of Poland, the Kiel Institute for the World Economy, and existing research regarding OECD countries.

9 The baseline scenario

9.1 The steady state (the balanced growth path)

Table 6 contains the calculation results obtained with the base set of parameters and endowments. The table presents only the steady state. The transitory dynamics, i.e., the trajectories of selected variables are presented in the appendix (section K).

It is worth to compare the decentralized economy with the first-best equilibrium. The key difference is the size of investment and consumption. The social planner invests much more than the representative agent – almost 27% of GDP, compared to a mere 16%. Thanks to intense capital formation the social planner is able to achieve and maintain a very high GDP growth rate of 4.9% per year, compared to a mere 1.3% in the decentralized economy.

Interestingly, in the first-best equilibrium the private sector borrows from abroad much more than in the decentralized economy. This is due to the fact that the social planner takes into account positive externalities of capital accumulation, which significantly increases the sense (the profitability) of borrowing abroad for investment purposes. Needless to say, this effect is partly offset by another externality – this time negative – the social planner knows that borrowing more from abroad raises the cost of borrowing (the real interest rate). Nevertheless, the net effect is positive, resulting in a significantly higher level of private foreign debt compared to the decentralized economy.

Conversely, the public sector in the first-best equilibrium borrows almost 4 times less (97% of GDP) than in the decentralized economy (377%). This is primarily due to a significant difference in the rate of GDP growth, but also in part due to the fact that the social planner takes into account large negative effects of raising external public debt (huge increase in the real interest rate on public debt reaching as much as 13% in the decentralized economy), which translates into a lower willingness for

Table 4: The summary of calibration for Poland

Parameters & endowments	Sources and methods of the calibration
	Technology
$\alpha = 2/3$, $\beta = 1/3$	The review of empirical literature. Sources: Mankiw, Romer, Weil (1992), Bernanke, Gurkaynak (2002), Konishi, Nishiyama (2002), Willman (2002), Balisteri et.al. (2003), Turnovsky (2009), Growiec (2012). Note: in our model the capital K is interpreted as an aggregate of physical capital and human capital. See Turnovsky (2009) or Rebelo (1991).
$A = 1/3$	OECD statistics and the database published by the Kiel Institute for the World Economy
$\chi = 13$	The review of empirical and theoretical literature. Sources: Holt et.al. (1960), Peck (1974), Hayashi (1982), Auerbach, Kotlikoff (1987), Gilchrist and Himmelberg (1995), Ortigueira, Santos (1997), Caballero, Engel (1999), Barro and Sala-i-Martin (2004), Cooper, Haltiwanger (2006).
$\delta = 4\%$	Eurostat statistics and a review of empirical literature. Sources: Easterly, Rebelo (1993), Arrazola, de Hevia (2004), Manuelli, Seshadri (2005), Cichy (2008), Turnovsky (2009). Note: Since the capital K is interpreted as an aggregate of physical capital and human capital, this depreciation rate is calculated as an average of the two depreciation rates related to human capital (approx. 1.5%) and physical capital (approx. 6.5%).
	The utility function and demographics
$\kappa = 0.27$	The review of empirical and theoretical literature. Sources: Turnovsky (1999) and (2004), Park, Philippopoulos (2004), Dhont, Heylen (2009).
$\rho = 0.04$	The metanalysis by Nijkamp, Percoco (2006) of 42 previous analyses, and European Commission (2002).
$\gamma = -1$	The comprehensive meta-analysis by Havranek *et.al.* (2013) of 169 previous analyses.
$n = 0\%$	Demographic forecasts for Poland published by the Central Statistical Office of Poland.
	Fiscal policy
$\sigma_C = 28.7\%$	According to Eurostat, during the period 2000 – 2013 public consumption in Poland, as a share of GDP, amounted to an average of 18.1%, while private consumption was on average 62.9% of GDP. Thus, $\sigma_C = 18.1\%/62.9\% = 28.7\%$.
$\xi = 4.8\%$	The average deficit of the public sector in Poland in the period 2000 – 2012 (according to Eurostat methodology).
$\omega = 0.4$	The average share of foreign debt in public debt in Poland during the period 2000 – 2012.
$\tau_K = 21.2\%$, $\tau_L = 32.8\%$, $\tau_D = 19\%$, $\tau_C = 19.4\%$	Eurostat statistics: the average taxation rates (*implicit tax rates*) in Poland in the period 2000 – 2010 (the latest available data)
$\tau_Z = 20\%$	We have tried to calculate this rate based on the balance of payments statistics for Poland , however the results turned out to fluctuate wildly year to year. Thus, we assumed the arithmetic average of τ_K and τ_D.

the government to run into debt.

Admittedly, the first-best share of private consumption in GDP along the balanced growth path is lower than in the decentralized economy. Nevertheless, thanks to much higher GDP growth, the social planner delivers higher welfare: $\Omega^* > \Omega$. Figure 1 presents the trajectories of consumption in both economies. After a brief period of sacrifices (approximately 13 years), during which the per capita consumption in the

Table 5: The summary of calibration for Poland

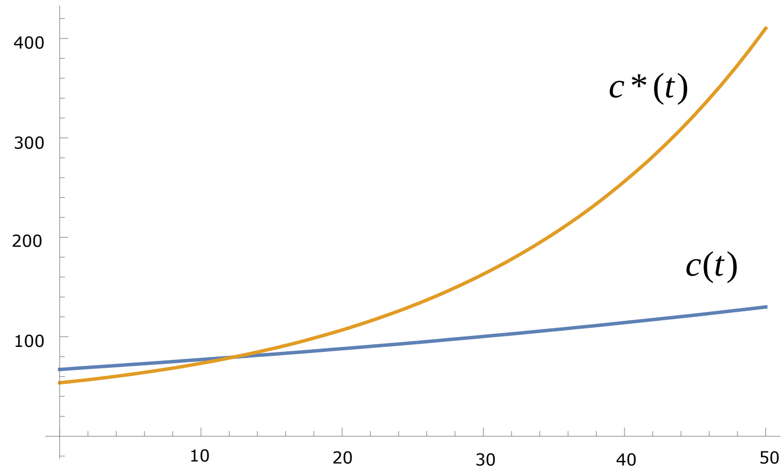
Parameters & endowments	Sources and methods of the calibration
The interest rates: the base interest rates and risk premiums	
$p_D = 0.03$	The review of empirical literature regarding OECD countries. Sources: Chinn, Frankel (2003), Faini (2006), Kinoshita (2006), Ardagna et.al. (2007), Laubach (2009), Poghosyan (2013).
$\varepsilon_D = 1.85\%$	Calibrated so as to be consistent with the (real) average 10-year Treasury bond yields in Poland in the period 2001 – 2012 (equal to 3.29%), the average public-debt-to-GDP ratio (48.1%), and $p_D = 0.03$, i.e., calculated from the equation: $3.29\% = \varepsilon_D + 0.03 \cdot 48.1\%$.
$\varepsilon_Z = 3.37\%$ $p_Z = 0.05$	First, the average real cost of foreign borrowing to the private sector in Poland was calculated based on the balance of payments statistics. In the period 2000 – 2012 it was on average 5.99%. Second, it was disaggregated into the base interest rate and the risk premium, so as to turn out identical proportions as in the public sector, i.e., by assuming that $\varepsilon_Z/\varepsilon_D = p_Z/p_D = 5.99/3.29$.
The initial values (endowments)	
$k_0 = 300$	The initial stock of capital per capita is set arbitrarily (as a numeraire); 300 is convenient, as it yields $y_0 = 100$, and hence the initial values of all the other variables are identical to their percentage shares of GDP.
$\underline{z}_0 = 59.5\%$ $\underline{d}_{F0} = 29.2\%$	Statistical data for Poland published by the National Bank of Poland (NBP): net international investment position (NIIP) of the private sector and the public sector in 2012.
$\underline{d}_{D0} = 26.4\%$	The difference between the public debt (source: the NBP) and the NIIP of the public sector in 2012.

Table 6: The balanced growth path in the baseline scenario

Variable	Decentralized economy		Benevolent social planner	
The balanced growth rate (BGR)	$\bar{\varphi}$	1.274%	$\bar{\varphi}^*$	4.943%
Private consumption (% of GDP)	$\underline{\bar{c}}$	45.63%	$\underline{\bar{c}}^*$	41.33%
Initial private consumption (% of GDP)	$\underline{c}_0$	50.03%	$\underline{c}_0^*$	53.72%
Investment (% of GDP)	$\underline{\bar{i}}$	15.82%	$\underline{\bar{i}}^*$	26.83%
The real interest rate on private foreign debt	$\bar{r}_Z$	5.74%	$\bar{r}_Z^*$	9.30%
Private foreign debt (% of GDP)	$\underline{\bar{z}}$	47.46%	$\underline{\bar{z}}^*$	118.51%
The real interest rate on public debt	$\bar{r}_D$	13.15%	$\bar{r}_D^*$	4.76%
Foreign debt of the government (% of GDP)	$\underline{\bar{d}}_F$	150.7%	$\underline{\bar{d}}_F^*$	38.8%
Domestic debt of the government (% of GDP)	$\underline{\bar{d}}_D$	226.1%	$\underline{\bar{d}}_D^*$	58.3%
Total public debt (% of GDP)	$\underline{\bar{d}}$	376.8%	$\underline{\bar{d}}^*$	97.1%
Welfare (utility)	Ω	–0.1195	Ω^*	–0.1081
Lost consumption indicator (LCI)		8.16%		

first-best equilibrium is lower than in the decentralized economy, the country enters into an infinitely long period of prosperity with consumption per capita higher than in the decentralized economy.

Figure 1: The trajectories of consumption per capita in the baseline scenario



For the well-known mathematical reasons, the values of utility functions Ω^* and Ω cannot be easily compared (for example, by calculating the difference or quotient). Therefore, in order to obtain an intuitive view of the welfare difference between the two types of economies, we shall apply the lost consumption indicator (LCI) that measures by what percentage higher both types of consumption (private and public) in the second-best equilibrium would have to be in every moment of the infinite time horizon to reach the first-best utility level. It is straightforward to demonstrate that in our model the LCI can be calculated as follows: $\mathrm{LCI} = (\Omega^*/\Omega)^{1/(\gamma(1+\kappa))} - 1$. In the baseline scenario, the LCI is equal to approximately 8.2%, which can be roughly interpreted as the welfare cost of all the externalities that economic agents fail to internalize.

9.2 A few words of caution

The results presented in table 6 diverge from the actual statistics reported for the period 2000 − 2013, which were used for the calibration. It does not imply, though, that the calibration was incorrect: if one compares these results with statistical data regarding, for example, the GDP growth rate, it becomes clear that the baseline scenario for the decentralized economy differs from actual data in minus, whereas the first-best scenario differs in plus. For example, in the period 2000 − 2013 Polish GDP grew on average at the rate of 3.7%, whereas in the baseline scenario we have 1.3% for the decentralized economy and 4.9% for the social planner. This may mean that in fact economic agents in Poland have been internalizing a substantial part of externalities "by themselves". The comparison of GDP growth rates suggests that they might have internalized as much as 2/3 of the externalities, as we have:

$(3.7-1.3)/(4.9-1.3) = 2.4/3.6 = 2/3$. Therefore, when interpreting the results of all our simulations, one has to keep in mind that they represent the extreme cases: the decentralized economy, where agents take into account 0% of the external effects, and the social planner which incorporates 100% of these effects. Clearly, the truth (the real world) is located somewhere between these extremes, and calculations suggest that it lies twice closer to the social planner than to the decentralized economy.
As an illustration of these words of caution, let us consider the estimated welfare cost of externalities equal to 8.2% (the LCI in the baseline scenario). If we take these words of caution seriously, the welfare cost is in reality only approximately 2.7%.

10 The replication of the balanced growth path (the partial replication)

Now, we shall determine the values of the three parameters of fiscal policy that allow replication of the first-best steady state in the baseline scenario. The replicating values of two parameters were calculated directly from Eqs. (38) and (39), whereas the replicating value of the rate of taxation of capital income τ_K^{rep} was calculated numerically, in accordance with the procedure outlined in appendix G. We obtained the following values:

$$\sigma_C^{rep} = 0.27, \tau_Z^{rep} = 63.75\%, \tau_K^{rep} = -67.98\%. \tag{40}$$

Therefore, partial replication requires strongly negative tax rate on capital income (in practice, subsidizing investment in productive capital) coupled with high positive tax rate on interest paid by private borrowers to foreign lenders (to discourage foreign financing). Qualitatively, these conclusions are similar to those presented by Turnovsky (2009).
Recall that partial replication (of the steady state only) is not synonymous with full replication of the entire first-best trajectories. Therefore, the parameter values (40) bring the decentralized economy closer, but not necessarily exactly to the first-best solution. Nevertheless, according to our calculations, partial replication reduces the LCI from 8.1% (the baseline scenario) to a mere 0.7%. Full replication would, of course, reduce it to zero, but it is probably not worth the effort. Recall that full replication requires an application of tax rates $\tau_Z^{opt}(t)$ and $\tau_K^{opt}(t)$ that (continuously) change over time. (Moreover, calculating the trajectory $\tau_K^{opt}(t)$ is a complex numerical problem − see section 6). Furthermore, even if we would determine both these trajectories, it is hard to imagine their implementation. Updating the tax rates even once a year would be cumbersome and expensive (e.g., the menu costs). It is not impossible that the total economic costs would exceed 0.7% of the lost welfare.

11 The quest for the global optimum

At this point we embark on a more general question: we search for such values of fiscal policy parameters that maximize welfare measured by Ω^*. It follows from the properties of the model that the procedure of searching for the optimal parameters can be divided into two stages: first, find the optimal parameter values for the social planner; second, solve the problem of (partial) replication finding the appropriate values of two tax rates: τ_K and τ_Z.
To have a fixed point of reference, the results will be compared with the first-best solution in the baseline scenario. We apply the LCI measure, though in the opposite direction: we calculate by what percentage should the first-best consumption flow in the baseline scenario be increased in order to deliver the same level of welfare as in the analyzed (new) scenario. To emphasize the difference, let us call this measure the gained consumption indicator (GCI). It is straightforward to demonstrate that the GCI in our model can be calculated as follows: $\mathrm{GCI} = \left(\Omega/\Omega^B\right)^{1/(\gamma(1+\kappa))} - 1$, where Ω is the value of welfare in the analyzed (new) scenario, whereas $\Omega^B = -0.1081$ is the value of welfare in the first-best baseline scenario.

11.1 Stage 1: The optimal fiscal policy

At this stage, looking at the economy from the position of the benevolent social planner, we are searching for the optimal values of all parameters controlled by the government and the central bank that influence the welfare. It follows from section 5 that three parameters matter: σ_C, ξ and ω. Note that utility maximization requires the equality: $\sigma_C = \kappa$, and this condition is independent of the values of the remaining two parameters. In order to find their optimal values, we apply an algorithm solving one optimal control problem of the social planner for each point of a 2-dimensional grid (with selected precision), as we have 2 parameters. The optimal combination of ξ and ω is 2.41% and 100%, i.e. the optimal deficit amounts to 2.41% of GDP, and the optimum foreign lenders' share in public debt is 100% (an edge solution). This somewhat surprising result hinges on the assumed value of risk premium for the public sector: $p_D = 0.03$, which is based on the review of empirical literature regarding OECD countries. The higher the value of p_D, the lower the optimal value of ω. For sufficiently high value of p_D it may even be zero. The corresponding GCI is 2.57%, so this choice allows the social planner to gain over 2.5% of consumption compared to the baseline first-best scenario. Let us call this variant scenario A.
Finally, it is worth noting that the GCI value is relatively small, which suggests that the size of budget deficit and the structure of its financing far less important than the choice of tax rates for the replication of the first-best solution. We turn to this issue in the next stage.

11.2 Stage 2: Partial replication of the first-best steady state in scenario A

The two tax rates, τ_K and τ_Z, must be adjusted in such a way that the decentralized economy replicates the first-best steady state in scenario A. Using proposition 7, we obtain:

$$\tau_Z^{rep} = 63.82\%, \tau_K^{rep} = -64.15\%. \tag{41}$$

Partial replication reduces the LCI from about 16.7% (the LCI in the decentralized economy in scenario A) to just 0.7%. Full replication would reduce it to zero, but – as we argued in section 10 – it is probably not worth the effort.

11.3 Stage 2: 'Almost optimal' tax rates

Replication of the first-best steady state in scenario A requires substantial subsidies to capital (investment) coupled with very high positive tax rate on interest paid by private sector to foreign lenders (to discourage foreign financing). This kind of policy might be impossible to implement in practice, for numerous reasons: political, ethical, moral, etc. (This could well be an example of the trade-off between efficiency and equity.) Therefore, let us shortly discuss slightly more realistic scenarios. Assume that the government may consider reducing the capital income tax rate to 10% or zero, or – at best – grant small subsidies amounting to, say, 10% or 20%. Assuming that public consumption as well as ξ and ω are at their optimal levels equal to 27%, 2.41% and 100%, respectively, for each of several hypothetical levels of τ_K we have calculated the optimal (welfare maximizing) tax rate τ_Z. The results are presented in table 7.

Table 7: 'Almost optimal' tax rates compared to scenario A

Assumed tax rate τ_K	The optimal tax rate τ_Z	LCI in comparison to scenario A	Balanced growth rate ($\bar{\varphi}$)
−20%	15.1%	2.7%	3.29%
−10%	2.5%	4.8%	2.88%
0	−10.9%	7.5%	2.45%
10%	−33.6%	10.6%	2.00%
20%	−56.0%	12.6%	1.54%

If we keep the capital tax rate in Poland at its current level ($\tau_K = 20\%$), then, from the point of view of welfare, it would be optimal to set the tax rate on interest paid by private sector to foreign lenders at $\tau_Z = -56\%$. Thus, if $\tau_K = 20\%$, then the government should encourage the private sector to borrow capital abroad by subsidizing an inflow of foreign capital to Poland. The value of $\tau_Z = -56\%$ means that the compensation of foreign investors (e.g., profits transferred abroad, interest on

bonds, loans, etc.) not only should not be taxed, but the government should increase these payments by an extra bonus of 56%. This situation is, however, unfavorable from the point of view of welfare, because in comparison to the solution obtained by the benevolent social planner in scenario A, the country is losing as much as 12.6% of consumption (LCI), and the long-run GDP growth rate is only 1.54%.
Without doubt, it would be far better to reverse the situation: set $\tau_K = -20\%$ (to encourage the private sector to invest by subsidizing capital income), and simultaneously discourage borrowing abroad by setting $\tau_Z = 15.1\%$. In that case the LCI in comparison to scenario A is only 2.7%, and the rate of GDP growth in the steady state is as much as 3.29%.

12 Conclusions and discussion

We have reached several theoretical conclusions regarding fiscal policy:

1. All income and consumption taxes are neutral for the benevolent social planner (they do not influence the first-best solution of the model), but not for the decentralized economy.

2. Other parameters of fiscal policy influence both economies: the trajectories of many variables (as well as welfare) depend on the share of public consumption in GDP, the size of public deficit, and the structure of public debt (the share of foreign lenders). However, as an analytical solution to the model of the social planner does not exist, an analysis of the relationships between these parameters and the BGR and welfare requires numerical methods.

3. The social planner can induce individual economic agents to internalize all the externalities by proper adjustment of fiscal policy, which allows the decentralized economy to replicate the first-best solution. In our model, replication requires 3 instruments of fiscal policy: the share of public consumption in GDP, the tax rate on interest paid by private sector to foreign lenders, and the tax rate on capital income. The optimal value of the first of these parameters is constant over time, the optimal value of the second changes over time according to the formula derived in the paper. The optimal value of the third changes over time and additional difficulty is the fact that there is no analytical formula for the optimal value of this parameter – not only for the whole trajectory (as a function of time), but even for the single value of this parameter in a selected moment of time. Therefore, from a practical point of view, the replication is a pretty difficult task.

4. Though full replication of the trajectories generated by social planner is a complex numerical problem, it is possible to solve analytically a simplified problem of partial replication (of the steady state only). The necessary and

sufficient condition for this is that the three above-mentioned parameters of fiscal policy must be at certain replicating levels, constant over time. We derive the analytical formulas for the first two of them, and we demonstrate that the replicating value of the third is a unique, feasible solution of a fifth-order polynomial equation.

The main empirical conclusions for Poland can be summarized as follows:

1. The optimal (welfare maximizing) values of fiscal policy parameters are as follows. The optimal level of public deficit amounts to 2.41% of GDP; the optimum foreign lenders' share in public debt is 100%. (We treat this share as an instrument of fiscal policy, assuming that the government can somehow control it. We admit that in reality this may, however, be difficult.) The optimal tax rate on capital income is minus 64.15%, whereas the optimal taxation of the interest on private external debt is 63.82%.

These optimal values should be treated with caution, for at least two reasons. First, these values hinge on the calibration of the model based on the period 2000-2013 – in particular on the average level of base interest rates and risk premiums in that period. It's unlikely that these financial parameters remain at the same level in the future. Moreover, although most parameters have been calibrated on the basis of statistical data regarding Poland, several important parameters do not have their counterparts in the available data. Therefore, they have been calibrated on the basis of the average values observed in other OECD countries, or on the basis of the so-called consensus – values that are widely accepted in the literature.
Second, we found that the obtained baseline scenario diverges from the actual statistics recorded in Poland in the period 2000 – 2013, which were used for the calibration. In particular, the baseline scenario for the decentralized economy differs from actual data in minus, whereas the first-best baseline scenario differs in plus. This may mean that in reality economic agents in Poland do internalize a large part of externalities. By comparing the actual average GDP growth rate with values obtained in the baseline scenario, we found that they probably internalize as much as 2/3 of the external effects. Therefore, if we would like to determine a scenario reflecting the actual economic situation in Poland (that could be even viewed as a forecast), we should consider a "weighted average" of the baseline scenarios for the two types of the economy. In other words, a realistic, reliable scenario (forecast) is probably located somewhere between the second-best and the first-best baseline scenario. Moreover, our calculations suggest that it is located twice closer to the social planner than to the decentralized economy.
Due to these two objections, all empirical results presented above should be considered with caution. This also applies to section 11, in which we tried to determine the optimal values of fiscal policy parameters. With the emphasis we stress that the values obtained in stage 1 deserve much more confidence than those obtained in stage 2. In stage 1, the optimal size of public consumption, the deficit-to-GDP ratio and

the share of foreign lenders in public debt have been established on the basis of an analysis of the social planner only. Thus, the optimal values obtained in stage 1 do not depend on what percentage of externalities economic agent internalize. As long as we neglect the first objection mentioned above, these optimal values can be regarded as the true optimal recipe for the fiscal policy in Poland.
The results of stage 2 are far more problematic and less trustworthy, as the obtained replicating values of the two tax rates crucially depend on what part of externalities agents internalize "by themselves". Our results of stage 2 hinge on the assumption that they internalize nothing. However, if economic agents in Poland indeed internalize a significant part of these effects – replicating tax rates are quite different, far less radical. On the one hand, the subsidies to capital income need not be so high, and, on the other hand, the taxation of interest on private foreign debt can be much lower.

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A The solution of the optimization problem for the representative agent

Obviously, the shadow price of debt is negative: $\lambda_1' < 0$. Thus, following Turnovsky (2009), we replace it with $\lambda_1 = -\lambda_1'$, which allows us to use the ratio of shadow prices $q = \lambda_2/\lambda_1 = -\lambda_2/\lambda_1' > 0$ that can be interpreted as the market price of capital in relation to the market price of private foreign debt (bonds).
The optimal solution must satisfy the following (necessary and sufficient) conditions, including two transversality conditions:

$$\forall t \quad \partial H_c/\partial c = 0, \tag{A.a}$$

$$\forall t \quad \partial H_c/\partial i = 0, \tag{A.b}$$

$$\dot{\lambda}_1' = -\partial H_c/\partial z + \lambda_1'(\rho - n), \tag{A.c}$$

$$\dot{\lambda}_2' = -\partial H_c/\partial k + \lambda_2'(\rho - n), \tag{A.d}$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda_1'(t)z(t) = 0, \tag{A.e}$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda_2'(t)z(t) = 0, \tag{A.f}$$

Condition (A.a) can be written as

$$\lambda_1(1+\tau_C) = c^{\gamma-1}g_C^{\kappa\gamma}, \tag{A.1}$$

which means that the shadow price of wealth (in the form of bonds), adjusted for the size of consumption tax must be (for all t) equal to the marginal utility of private consumption. Log-differentiating this equation with respect to t yields:

$$\hat{\lambda}_1 = (\gamma - 1)\hat{c} + \kappa\gamma\hat{g}_C. \tag{A.2}$$

Throughout the paper hats over variables denote rates of growth, e.g. $\hat{c} = \dot{c}/c$, etc. Note that from Eq. (12), it follows that private and public consumption grow at identical rates, say ψ. Thus $\hat{g}_C = \hat{c} = \psi$.
Condition (A.c) can be written as:

$$\hat{\lambda}_1 = \rho - (1+\tau_Z)r_Z. \tag{A.3}$$

Substituting Eq. (A.3) into Eq. (A.2), and using $\hat{g}_C = \hat{c} = \psi$, we can calculate the growth rate of consumption (both private and public) per capita:

$$\psi = \frac{\dot{c}}{c} = \frac{(1+\tau_Z)r_Z - \rho}{A_1} = \frac{r_Z - \rho + \tau_Z r_Z}{A_1}, \tag{A.4}$$

where $A_1 = 1 - \gamma(1+\kappa)$. Importantly, the trajectory of the interest rate $r_Z(t)$ is not necessarily constant over time. Thus, the trajectory of private consumption can only be expressed in a general form:

$$c(t) = c_0 \cdot \exp\left(\int_0^t \psi(s)ds\right). \tag{A.5}$$

Condition (A.b) can be written as:

$$q = \lambda_2/\lambda_1 = 1 + \chi i/k. \tag{A.6}$$

The ratio of the shadow prices $q = \lambda_2/\lambda_1$ can be broadly interpreted as the market price of capital in relation to the market price of foreign bonds. According to Eq. (A.6), it must be equal to the marginal cost of an additional unit of investment (adjusted for the adjustment cost). Dividing both sides of Eq. (5) by k, and using Eq. (A.6), we obtain the growth rate of capital and output per capita:

$$\varphi = \hat{k} = \hat{y} = (q-1)/\chi - (n+\delta). \tag{A.7}$$

This growth rate is not necessarily constant, as it is related to the trajectory $q(t)$. Therefore, at this stage, the trajectory $k(t)$ must be written in a general form:

$$k(t) = k_0 \exp\left(\int_0^t \varphi(s)ds\right). \tag{A.8}$$

To determine the path of $q(t)$, we need to use Eq. (A.d). Having regard to Eqs. (A.3) and (A.6), and using Eq. (4), it can be written as: $\dot{\lambda}_2 = -\lambda_1\left[(q-1)^2/2\chi + (1-\tau_K)\alpha A - (1-\omega)\xi A\right] + \lambda_2(\rho+\delta)$. Dividing both sides by λ_2, and taking into account Eqs. (A.3) and (A.6) together with Eq. (1), after minor manipulation we obtain:

$$\dot{q} = \left[(1+\tau_Z)\left(\varepsilon_Z + p_Z \underline{z}\right) + \delta\right] \cdot q - (1-\tau_K)\alpha A + (1-\omega)\xi A - (q-1)^2/2\chi, \tag{A.9}$$

where $\underline{z} = z/y$.

To derive the steady state it's convenient to replace the original (per capita) variables with their shares in GDP. (Turnovsky (2009) uses similar approach in chapter 4, only instead of shares in GDP he applies ratios to capital. As we use the AK production function, our approach is in fact identical.) Let us denote these shares with an underline, e.g., $\underline{c} = c/y$, $\underline{d}_D = d_D/y$, etc. Obviously,

$$\hat{\underline{c}} = \hat{c} - \hat{y} = \psi - \varphi, \tag{A.10}$$

$$\hat{\underline{z}} = \hat{z} - \hat{y} = \hat{z} - \varphi. \tag{A.11}$$

Substituting Eqs. (1), (A.4) and (A.7) into Eq. (A.10) yields:

$$\dot{\underline{c}} = \left[\frac{(1+\tau_Z)r_Z - \rho}{A_1} - \frac{q-1}{\chi} + n + \delta\right] \cdot \underline{c}, \tag{A.12}$$

where $r_Z = \varepsilon_Z + p_Z\underline{z}$. The derivation of the equation of motion of $\underline{z}$ requires several substitutions and manipulations. First, substituting $g_C = \sigma_C c$ into Eq. (16), dividing both sides by z, and finally using Eq. (A.6), we obtain:

$$\hat{z} = (1+\sigma_C)\frac{\underline{c}}{\underline{z}} + \frac{q^2-1}{2A\chi}\cdot\frac{1}{\underline{z}} + [r_Z - n] + r_D\frac{\underline{d}_F}{\underline{z}} - (1+\omega\xi)\frac{1}{\underline{z}}. \tag{A.13}$$

From Eq. (A.11), it follows that $\dot{\underline{z}} = \hat{\underline{z}}\cdot\underline{z} = (\hat{z}-\varphi)\underline{z}$. Using Eqs. (A.13) and (A.7), it can be rearranged to:

$$\dot{\underline{z}} = (1+\sigma_C)\underline{c} + \frac{q^2-1}{2A\chi} - (1+\omega\xi) + \left(r_Z - \frac{q-1}{\chi} + \delta\right)\underline{z} + r_D\underline{d}_F, \tag{A.14}$$

where $r_Z = \varepsilon_Z + p_Z\underline{z}$ and $r_D = \varepsilon_D + p_D\underline{d}$. Eqs. (A.12), (A.14) and (A.9) jointly determine the evolution of 3 variables: $\underline{c}$, $\underline{z}$ and q. However, apart from these three variables, the right-hand sides of these equations contain 2 additional variables: $\underline{d}_D$ and $\underline{d}_F$. Therefore, in order to close the system of differential equations, we need to append 2 additional equations describing the dynamics of $\underline{d}_D$ and $\underline{d}_F$. From Eqs. (10) and (11), it follows that: $\hat{\underline{d}}_F = \hat{D}_F - \hat{Y} = \omega\xi/\underline{d}_F - n - \varphi$, $\hat{\underline{d}}_D = (1-\omega)\xi/\underline{d}_D - n - \varphi$. As $\dot{\underline{d}}_F = \hat{\underline{d}}_F\cdot\underline{d}_F$ and $\dot{\underline{d}}_D = \hat{\underline{d}}_D\cdot\underline{d}_D$, we have:

$$\dot{\underline{d}}_F = (-n-\varphi)\underline{d}_F + \omega\xi \tag{A.15}$$

$$\dot{\underline{d}}_D = (-n-\varphi)\underline{d}_D + (1-\omega)\xi \tag{A.16}$$

Obviously, $\dot{\underline{d}} = \dot{\underline{d}}_F + \dot{\underline{d}}_D$. Eqs. (A.12), (A.14), (A.9), (A.15) and (A.16) constitute a nonlinear autonomous system of differential equations of the following form:

$$\begin{bmatrix} \dot{\underline{c}} \\ \dot{\underline{z}} \\ \dot{q} \\ \dot{\underline{d}}_F \\ \dot{\underline{d}}_D \end{bmatrix} = \begin{bmatrix} f^1(\underline{c}, \underline{z}, q, \underline{d}_F, \underline{d}_D) \\ f^2(\underline{c}, \underline{z}, q, \underline{d}_F, \underline{d}_D) \\ f^3(\underline{c}, \underline{z}, q, \underline{d}_F, \underline{d}_D) \\ f^4(\underline{c}, \underline{z}, q, \underline{d}_F, \underline{d}_D) \\ f^5(\underline{c}, \underline{z}, q, \underline{d}_F, \underline{d}_D) \end{bmatrix}. \tag{A.17}$$

B The rationale for the assumption that $a_3 < 0$

Notice that $a_3 < 0$, if:

$$(1-\tau_K)\alpha > (1-\omega)\xi. \tag{B.1}$$

(This condition is sufficient, though not necessary.) Let us estimate the value of the left-hand side on the basis of realistic, empirical data. The rate of capital tax everywhere in the world is lower than 50% (in most countries, much lower), while the share of capital in output is estimated at approximately 1/3. Thus $(1-\tau_K)\alpha > 1/6$. Therefore, inequality (B.1) could be violated only, if $(1-\omega)\xi > 1/6$, which requires that the budget deficit (in % of GDP) multiplied by the share of foreign creditors in public debt $(1-\omega)$ exceeds 1/6. This is not possible in light of real-world data. Even in an extreme case of $(1-\omega)=1$, it would require budget deficit above 16.7% of the GDP. In the long run, such a high level of deficit is never observed. Therefore, from an empirical perspective, it is obvious that condition (B.1) is satisfied, and hence $a_3<0$.

Note: The capital K is defined broadly, and includes both physical capital and human capital. (See Table 4). Therefore, in our calibration we apply $\alpha=2/3$. Notice that with this value of α our justification for the assumption $a_3<0$ not only still holds, but gets twice stronger.

C Details of Proposition 1

The system of equations (17) is nonlinear, and hence we will only investigate the local stability of the equilibrium applying a standard method of first-order linearization about the equilibrium. Accordingly, non-linear functions f^i can be approximated as follows:

$$f^i(\underline{c},\underline{z},q,\underline{d}_F,\underline{d}_D) \approx \left.\frac{\partial f^i}{\partial \underline{c}}\right|_E \cdot \tilde{\underline{c}} + \left.\frac{\partial f^i}{\partial \underline{z}}\right|_E \cdot \tilde{\underline{z}} + \left.\frac{\partial f^i}{\partial q}\right|_E \cdot \tilde{q} + \left.\frac{\partial f^i}{\partial \underline{d}_F}\right|_E \cdot \tilde{\underline{d}}_F + \left.\frac{\partial f^i}{\partial \underline{d}_D}\right|_E \cdot \tilde{\underline{d}}_D,$$

$(i=1,\ldots,5)$, where symbols with tilde denote deviations from the steady state, i.e., $\tilde{\underline{c}}=\underline{c}-\bar{\underline{c}}$, $\tilde{\underline{z}}=\underline{z}-\bar{\underline{z}}$, $\tilde{q}=q-\bar{q}$, $\tilde{\underline{d}}_F=\underline{d}_F-\bar{\underline{d}}_F$, $\tilde{\underline{d}}_D=\underline{d}_D-\bar{\underline{d}}_D$. The linear approximation of the system of equations (18) about the equilibrium has the following form:

$$\begin{bmatrix} \dot{\underline{c}} & \dot{\underline{z}} & \dot{q} & \dot{\underline{d}}_F & \dot{\underline{d}}_D \end{bmatrix}^T = M \begin{bmatrix} \tilde{c} & \tilde{z} & \tilde{q} & \tilde{\underline{d}}_F & \tilde{\underline{d}}_D \end{bmatrix}^T, \tag{C.1}$$

with the matrix of values of partial derivatives (Jacobian) calculated in the equilibrium:

$$M = \begin{bmatrix} 0 & \frac{(1+\tau_Z)p_Z\bar{\underline{c}}}{A_1} & -\frac{\bar{\underline{c}}}{\chi} & 0 & 0 \\ 1+\sigma_C & \bar{r}_Z+p_Z\bar{\underline{z}}-n-\bar{\varphi} & \frac{\bar{q}-A\bar{\underline{z}}}{A\chi} & \varepsilon_D+\frac{p_D\xi(1+\omega)}{n+\bar{\varphi}} & \frac{p_D\omega\xi}{n+\bar{\varphi}} \\ 0 & (1+\tau_Z)p_Z\bar{q} & (1+\tau_Z)\bar{r}_Z-n-\bar{\varphi} & 0 & 0 \\ 0 & 0 & \frac{-\omega\xi}{\chi(n+\bar{\varphi})} & -n-\bar{\varphi} & 0 \\ 0 & 0 & \frac{-(1-\omega)\xi}{\chi(n+\bar{\varphi})} & 0 & -n-\bar{\varphi} \end{bmatrix}.$$

The general solution of the linear system of equations (C.1) can be written as:

$$\begin{bmatrix} \underline{c} & \underline{z} & q & \underline{d}_F & \underline{d}_D \end{bmatrix}^T = \begin{bmatrix} \bar{\underline{c}} & \bar{\underline{z}} & \bar{q} & \bar{\underline{d}}_F & \bar{\underline{d}}_D \end{bmatrix}^T + \sum_{i=1}^{5} s_i e^{r_i t} v^i, \tag{C.2}$$

where r_i are the eigenvalues of the matrix M, v^i are its eigenvectors, and s_i are unknown constants dependent on the starting point (endowments). The local stability of the equilibrium depends on the signs of the eigenvalues of M. The product of these eigenvalues is equal to $\det M$, whereas their sum is equal to $\operatorname{tr} M$. It follows that:

$$\det M = -(n+\bar{\varphi})^2(1+\tau_Z)(1+\sigma_C)p_Z\bar{\underline{c}} \cdot \left[\bar{q}/\chi + \left((1+\tau_Z)\bar{r}_Z - n - \bar{\varphi}\right)/A_1\right],$$

$$\operatorname{tr} M = \bar{r}_Z + p_Z\bar{\underline{z}} + (1+\tau_Z)\bar{r}_Z - 4(n+\bar{\varphi}).$$

Despite all the assumptions made so far, it is not possible to determine the sign of $\operatorname{tr} M$. However, the transversality condition (26) implies that $\det M < 0$, which entails that M has an odd number of negative eigenvalues, i.e., 1, 3 or 5 such values. (Strictly speaking, negative real parts, as some eigenvalues may be complex conjugate numbers.) The first of these possibilities is rejected, as Eqs. (A.15) and (A.16) imply that two variables, $\underline{d}_F$ and $\underline{d}_D$, are globally stable. Furthermore, 5 negative eigenvalues would imply local stability of all 5 variables in the system of equations (C.1), which is virtually impossible in the light properties of Turnovsky (2009, chapter 4) model, as well as in the light of our own analyses performed for simplified versions of the model. Therefore, the only viable possibility seems to be 3 negative eigenvalues, which we henceforth assume (We have confirmed the viability of this assumption in numerous simulations. Again, strictly, we assume 3 eigenvalues with negative real parts, out of which one is real, and two may be either real or complex conjugate numbers.) Under this assumption, the equilibrium has the form of the stable saddle path. As there are 2 positive eigenvalues, 2 out of 5 variables must "jump" to accommodate any shock instantly, whereas the remaining 3 variables evolve continuously over time. For obvious reasons, the "jump" variables are consumption $\underline{c}$ and q, whereas all three debt indicators must be continuous.

If we denote positive eigenvalues as r_4 and r_5, then $s_4 = s_5 = 0$, and the solution (C.2) boils down to Eq. (28), where r_i $(i = 1, 2, 3)$ are the negative eigenvalues, and v^i are the corresponding eigenvectors of matrix M.

D Proof that the transversality conditions (A.e)-(A.f) are satisfied if, and only if, (26)

1. Let us start with condition (A.f). Substituting $\lambda_2(t) = q(t)\lambda_1(t)$, this condition can be written in an equivalent form:

$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda_1(t)q(t)k(t) = 0. \tag{D.1}$$

From Eqs. (A.7) and (A.7), it follows that the trajectory of capital has the following form:

$$k(t) = k_0 e^{\int_0^t \left(\frac{q(s)-1}{\chi}\right) ds} e^{-(n+\delta)t}. \tag{D.2}$$

Meanwhile, Eq. (A.3) implies that $\dot{\lambda}_1/\lambda_1 = \rho-(1+\tau_Z)r_Z(\underline{z})$, where $r_Z(\underline{z}) = \varepsilon_Z+p_Z\underline{z}$. Thus, the trajectory $\lambda_1(t)$ is of the form:

$$\lambda_1(t) = \lambda_1(0)e^{\rho t}e^{-(1+\tau_Z)}\int_0^t (r_Z(\underline{z}(s))\,ds. \tag{D.3}$$

Using Eqs. (D.2) and (D.3), condition (D.1) can be written as:

$$\lambda_1(0)k_0 \cdot \lim_{t\to\infty}\left\{q(t)e^{-\delta t}e^{-(1+\tau_Z)\int_0^t(r_Z(\underline{z}(s))ds}e^{\int_0^t\left(\frac{q(s)-1}{\chi}\right)ds}\right\} = 0,$$

which is equivalent to:

$$\lambda_1(0)k_0\lim_{t\to\infty}\left\{q(t)e^{-\left(\varepsilon_Z(1+\tau_Z)+\delta+\frac{1}{\chi}\right)t}e^{-(1+\tau_Z)p_Z\int_0^t\underline{z}(s)ds}e^{\frac{1}{\chi}\int_0^t q(s)ds}\right\} = 0. \tag{D.4}$$

In order to examine this condition, we need to know the trajectories of variables $\underline{z}(s)$ and $q(s)$. Because the model is non-linear, we will use the approximate trajectories obtained by solving the linearized model. From the system of equations (C.2) we know that:

$$\underline{z}(t) = \bar{\underline{z}} + s_1e^{r_1t}v_2^1 + s_2e^{r_2t}v_2^2 + s_3e^{r_3t}v_2^3 = \bar{\underline{z}} + \sum_{i=1}^{3}s_iv_2^i\,e^{r_it}. \tag{D.5}$$

$$q(t) = \bar{q} + s_1e^{r_1t}v_3^1 + s_2e^{r_2t}v_3^2 + s_3e^{r_3t}v_3^3 = \bar{q} + \sum_{i=1}^{3}s_iv_3^i\,e^{r_it}. \tag{D.6}$$

We are interested in stable equilibria only, thus we assume that:

$$\forall i \quad (r_i \geq 0 \quad \Rightarrow \quad s_i = 0)\,. \tag{D.7}$$

(All nonnegative eigenvalues of the matrix M correspond to zero constants s_i.) Using Eq. (D.5), we obtain

$$\int_0^t \underline{z}(s)ds = \bar{\underline{z}}t - \sum_{i=1}^{3}\frac{s_iv_2^i}{r_i} + \sum_{i=1}^{3}\frac{s_iv_2^i}{r_i}\,e^{r_it},$$

which implies that:

$$\begin{aligned} e^{-(1+\tau_Z)p_Z\int_0^t\underline{z}(s)ds} &= e^{-(1+\tau_Z)p_Z\bar{\underline{z}}t}e^{(1+\tau_Z)p_Z\sum_{i=1}^{3}\frac{s_iv_2^i}{r_i}}\cdot \\ &\cdot e^{-(1+\tau_Z)p_Z\sum_{i=1}^{3}\frac{s_iv_2^i}{r_i}\,e^{r_it}}. \end{aligned} \tag{D.8}$$

Similarly, on the basis of Eq. (D.6), we compute: $\int_0^t q(s)ds = \bar{q}t - \sum_{i=1}^{3} \frac{s_i v_3^i}{r_i} + \sum_{i=1}^{3} \frac{s_i v_3^i}{r_i} e^{r_i t}$, which implies that

$$e^{\frac{1}{\chi}\int_0^t q(s)ds} = e^{\frac{\bar{q}}{\chi}t} e^{-\frac{1}{\chi}\sum_{i=1}^{3} \frac{s_i v_3^i}{r_i}} e^{\sum_{i=1}^{3} \frac{s_i v_3^i}{\chi r_i} e^{r_i t}}. \tag{D.9}$$

Using Eqs. (D.8) and (D.9), we can rewrite condition (D.4) as:

$$\begin{aligned} &\lambda_1(0) k_0 e^{(1+\tau_Z) p_Z \sum_{i=1}^{3} \frac{s_i v_2^i}{r_i} - \frac{1}{\chi}\sum_{i=1}^{3} \frac{s_i v_3^i}{r_i}} \cdot \\ &\cdot \lim_{t\to\infty} \left\{ q(t) e^{-\left(\varepsilon_Z(1+\tau_Z) + \delta + \frac{1}{\chi} + p_Z \underline{(1+\tau_Z)\bar{z}} - \frac{\bar{q}}{\chi}\right)t} \cdot \right. \\ &\left. \cdot\, e^{-(1+\tau_Z) p_Z \sum_{i=1}^{3} \frac{s_i v_2^i}{r_i} e^{r_i t}} e^{\sum_{i=1}^{3} \frac{s_i v_3^i}{\chi r_i} e^{r_i t}} \right\} = 0. \end{aligned} \tag{D.10}$$

Assumption (D.7) implies that

$$\lim_{t\to\infty} e^{-(1+\tau_Z) p_Z \sum_{i=1}^{3} \frac{s_i v_2^i}{r_i} e^{r_i t}} = 1$$

and

$$\lim_{t\to\infty} e^{\sum_{i=1}^{3} \frac{s_i v_3^i}{\chi r_i} e^{r_i t}} = 1.$$

Therefore, condition (D.10) is satisfied if, and only if:

$$\lim_{t\to\infty} \left\{ q(t) e^{-\left(\varepsilon_Z(1+\tau_Z) + \delta + \frac{1}{\chi} + p_Z(1+\tau_Z)\underline{\bar{z}} - \frac{\bar{q}}{\chi}\right)t} \right\} = 0.$$

Using Eq. (D.6) and the equality $\bar{r}_Z = \varepsilon_Z + p_Z \underline{\bar{z}}$, we can rewrite this condition as:

$$\begin{aligned} &\bar{q} \lim_{t\to\infty} e^{\left(\frac{\bar{q}-1}{\chi} - \bar{r}_Z(1+\tau_Z) - \delta\right)t} + \\ &+ \lim_{t\to\infty} \left\{ \left(\sum_{i=1}^{3} s_i v_3^i e^{r_i t}\right) e^{\left(\frac{\bar{q}-1}{\chi} - \bar{r}_Z(1+\tau_Z) - \delta\right)t} \right\} = 0. \end{aligned} \tag{D.11}$$

It follows from Eq. (D.7) that $\lim_{t\to\infty} \sum_{i=1}^{3} s_i v_3^i e^{r_i t} = 0$ and

$$e^{\left(\frac{\bar{q}-1}{\chi} - \bar{r}_Z(1+\tau_Z) - \delta\right)t} > 0$$

(positive and finite number), and hence the second part of the sum in Eq. (D.11) is zero. Therefore, for Eq. (D.11) to hold, we must have

$$\lim_{t\to\infty} e^{\left(\frac{\bar{q}-1}{\chi} - \bar{r}_Z(1+\tau_Z) - \delta\right)t} = 0.$$

This equality holds if, and only if, $\dfrac{\bar{q}-1}{\chi} - \bar{r}_Z(1+\tau_Z) - \delta < 0$, which can be written in a more convenient form:

$$(1+\tau_Z)\bar{r}_Z > \bar{\varphi} + n. \tag{D.12}$$

2. The second transversality condition is (A.f). As $\lambda_1'(t) = -\lambda_1(t)$ and $z(t) = \underline{z}(t) \cdot y(t) = \underline{z}(t) \cdot Ak(t)$, this condition is equivalent to:

$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda_1(t)\underline{z}(t)Ak(t) = 0. \tag{D.13}$$

Eq. (22) implies that $\lim\limits_{t\to\infty} \underline{z}(t) = \bar{\underline{z}} \neq +\infty$, thus Eq. (D.13) can be rewritten as:

$$A\bar{\underline{z}} \lim_{t\to\infty} e^{-(\rho-n)t}\lambda_1(t)k(t) = 0.$$

Notice that this condition is equivalent to Eq. (A.f), as long as we neglect the special case of $\bar{\underline{z}} = 0$. It follows that the necessary and sufficient condition for both transversality conditions to hold is inequality (D.12).

E Proof of proposition 2

The level of welfare in the economy is described by integral (7). As $g_C(t) = \sigma_C c(t)$, we can express its value for the obtained optimal solution as: $\Omega = \frac{1}{\gamma}{\sigma_C}^{\kappa\gamma} \int_0^\infty (c_t)^{\gamma(1+\kappa)} e^{-(\rho-n)t}dt$. Substituting the trajectory of private consumption (A.5) yields:

$$\Omega = \frac{1}{\gamma}{\sigma_C}^{\kappa\gamma} c_0^{\gamma(1+\kappa)} \int_0^\infty \exp\left(\gamma(1+\kappa)\int_0^t \psi(s)ds\right) e^{-(\rho-n)t}dt. \tag{E.1}$$

As we have seen, there is no way to derive an explicit formula for the trajectory $\underline{z}(t)$ which in turn determines the trajectory $r_Z(t)$, which finally determines the trajectory $\psi(t)$. Therefore, in order to estimate the value of the integral Ω, we use a linear approximation of the model. From Eq. (28), we know that (around the equilibrium) $\underline{z}(t) \approx \bar{\underline{z}} + \sum_{i=1}^3 s_i e^{r_i t} v_2^i$. Using this in Eq. (1), we obtain: $r_Z(t) \approx \bar{r}_Z + p_Z \sum_{i=1}^3 s_i e^{r_i t} v_2^i$. Next, using this in Eq. (A.4), and substituting the resulting formula into Eq. (E.1) yields Eq. (29).

F The solution of the social planner's optimization problem

The optimal solution must satisfy the following (necessary and sufficient) conditions, including four transversality conditions:

$$\forall t \quad \partial H_c/\partial c = 0, \tag{F.a}$$

$$\forall t \quad \partial H_c/\partial i = 0, \tag{F.b}$$

$$\forall t \quad \partial H_c/\partial g_C = 0, \tag{F.c}$$

$$\dot{\lambda}_1' = -\partial H_c/\partial z + \lambda_1'(\rho - n), \tag{F.d}$$

$$\dot{\lambda}_2 = -\partial H_c/\partial k + \lambda_2(\rho - n), \tag{F.e}$$

$$\dot{\lambda}_3 = -\partial H_c/\partial d_F + \lambda_3(\rho - n), \tag{F.f}$$

$$\dot{\lambda}_4 = -\partial H_c/\partial d_D + \lambda_4(\rho - n), \tag{F.g}$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda_1'(t)z(t) = 0, \tag{F.h}$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda_2(t)k(t) = 0, \tag{F.i}$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda_3(t)d_F(t) = 0, \tag{F.j}$$

$$\lim_{t\to\infty} e^{-(\rho-n)t}\lambda_4(t)d_D(t) = 0. \tag{F.k}$$

Condition (F.a) can be written as:

$$\lambda_1 = c^{\gamma-1} g_C^{\kappa\gamma}, \tag{F.1}$$

which is a counterpart of Eq. (A.1), and has a similar interpretation. An important difference, however, is the absence of consumption tax. Condition (F.c) is:

$$\lambda_1 = \kappa\, c^{\gamma} g_C^{\kappa\gamma-1}, \tag{F.2}$$

which means that the shadow price of wealth (in the form of bonds) must be equal to the marginal utility of public consumption. Equating the right-hand sides of Eqs. (F.1) and (F.2) immediately yields $g_C = \kappa\, c$, which implies that the two types of consumption must grow at identical rates: $\hat{g}_C = \hat{c} = \psi$. Recall that an identical rule applied for the decentralized economy, but it resulted from the assumption (12), whereas now it follows from the necessary conditions of optimality.
Condition (F.d) can be written as:

$$\hat{\lambda}_1 = \rho - r_Z - p_Z \underline{z}, \tag{F.3}$$

which differs from its counterpart, Eq. (A.3). Differentiating Eq. (F.1) with respect to time t yields: $\hat{\lambda}_1 = (\gamma - 1)\hat{c} + \kappa\gamma\hat{g}_C$. Substituting Eq. (F.3), and using $\hat{g}_C = \hat{c} = \psi$, after minor manipulation yields the growth rate of (both types of) per capita consumption:

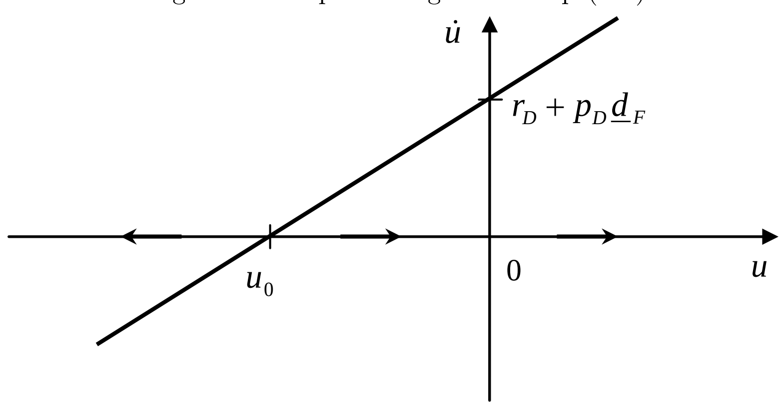
$$\psi = \frac{\dot{c}}{c} = \frac{r_Z + p_Z \underline{z} - \rho}{A_1}. \tag{F.4}$$

Notice that this formula is significantly different from its counterpart in the decentralized economy, Eq. (A.4). Unlike in the decentralized economy, the growth rate of consumption does not depend on the tax rate on interest on the private sector's external debt of, but it depends on the external-debt-to-GDP ratio of the private sector $\underline{z}$, as well as the risk premium p_Z. So far we don't know whether in the optimal solution the interest rate $r_Z(t)$ and the debt ratio $\underline{z}(t)$ are constant over time. Thus, the optimal trajectory of private consumption (per capita) can only be written in the same general form as in the decentralized economy, i.e., as in Eq. (A.5). Similarly, condition (F.b) yields an equation identical to Eq. (A.6). Thus, the growth rate of capital and the trajectory of capital are described by Eqs. (A.7) and (A.8), as before. Now, consider two additional necessary conditions: (F.f) and (F.g). The former can be written as: $\dot{\lambda}_3 = \lambda_1 \left(p_D \underline{d}_F + r_D\right) + \lambda_3 \rho$. Dividing both sides by λ_3, and using Eq. (F.3) yields: $\dot{\lambda}_3/\lambda_3 - \dot{\lambda}_1/\lambda_1 = \left(p_D \underline{d}_F + r_D\right) \lambda_1/\lambda_3 + r_Z + p_Z \underline{z}$. At this point, it is convenient to introduce another ratio of shadow prices (analogous to q): $u = \lambda_3/\lambda_1$. Using this, we can transform this equation to the following form:

$$\dot{u} = r_D + p_D \underline{d}_F + \left(r_Z + p_Z \underline{z}\right) u. \tag{F.5}$$

The phase diagram for this differential equation is presented in fig. 2. A positive slope of the function $\dot{u}(u)$ follows from the fact that $r_Z + p_Z \underline{z} > 0$, whereas a positive value at $u = 0$ follows from the fact that $r_D + p_D \underline{d}_F > 0$.

Figure 2: The phase diagram for Eq. (F.5)



Obviously, it is an unstable saddle point. Thus, the optimal solution must satisfy the following condition: $\forall t \;\; \dot{u} = 0$, i.e., for any time t,

$$\lambda_3 = -\lambda_1 \cdot \frac{r_D + p_D \underline{d}_F}{r_Z + p_Z \underline{z}}. \tag{F.6}$$

An identical analysis of condition (F.g) leads to a similar conclusion:

$$\lambda_4 = -\lambda_1 \cdot \frac{p_D \underline{d}_F}{r_Z + p_Z \underline{z}}. \tag{F.7}$$

To determine the trajectory $q(t)$, we use Eq. (F.e), which can be written as:

$$\begin{aligned} \dot{\lambda}_2 &= -\left(\lambda_1' \left[-\frac{\chi}{2}\frac{i^2}{k^2} + \frac{\partial r_Z}{\partial k} z - (1+\omega\xi)A + \frac{\partial r_D}{\partial k} d_F\right]\right) + \\ &\quad + \lambda_2(n+\delta) - \lambda_3 \omega\xi A - \lambda_4(1-\omega)\xi A + \lambda_2(\rho - n). \end{aligned} \tag{F.8}$$

As $y = Ak$, from Eqs. (1) and (2), it follows that: $\dfrac{\partial r_Z}{\partial k} = -\dfrac{p_Z z}{Ak^2}$, $\dfrac{\partial r_D}{\partial k} = -\dfrac{p_D d}{Ak^2}$. Substituting these formulae into Eq. (F.8), and using Eq. (A.6) together with $\lambda_1 = -\lambda_1'$ yields:

$$\begin{aligned} \dot{\lambda}_2 &= -\lambda_1 \left[\tfrac{(q-1)^2}{2\chi} + (1+\omega\xi)A + Ap_Z \tfrac{z^2}{y^2} + Ap_D \tfrac{d \cdot d_F}{y^2}\right] + \\ &\quad \lambda_2(\rho+\delta) - \lambda_3 \omega\xi A - \lambda_4(1-\omega)\xi A. \end{aligned}$$

Finally, dividing both sides by λ_2, and using Eqs. (F.3), (F.6), (F.7), after transformation we obtain:

$$\begin{aligned} \dot{q} &= -\left[A + \omega\xi A + \frac{(q-1)^2}{2\chi} + Ap_Z \underline{z}^2 + Ap_D \underline{d}_F \underline{d}\right] + \\ &\quad + \left(r_Z + p_Z \underline{z} + \delta\right) q + \xi A \frac{\omega r_D + p_D \underline{d}_F}{r_Z + p_Z \underline{z}}, \end{aligned} \tag{F.9}$$

where $\underline{d} = \underline{d}_F + \underline{d}_D$. It is the social planner's counterpart of Eq. (A.9), though far more complex – it is possible to demonstrate by substitutions that instead of a quadratic function in q, this time we have a fifth-order polynomial in q.

Obviously, along the balanced growth path all debt-to-GDP ratios ($\underline{z}$, $\underline{d}_F$, $\underline{d}_D$) must be constant – otherwise, we would have $\dot{q} \neq 0$, and hence $\varphi \neq const$. Now, we will prove more – namely, that the stationary state is achieved when all variables in relation to production reach the constant values. As before, we will use the ratios to GDP: $\underline{c}$ and $\underline{z}$. Recall that the growth rates of these ratios are expressed by Eqs. (A.10) and (A.11). Using Eqs. (1), (A.6) and (F.4), Eq. (A.10) can be written as:

$$\dot{\underline{c}} = \left[\frac{r_Z + p_Z \underline{z} - \rho}{A_1} - \frac{q-1}{\chi} + n + \delta\right] \cdot \underline{c}, \tag{F.10}$$

where $r_Z = \varepsilon_Z + p_Z \underline{z}$. The derivation of dynamics equation of the debt ratio $\underline{z}$ requires several algebraic transformations. First, dividing both sides of Eq. (16) by z, and using Eq. (A.16) together with $g_C = \kappa\, c$, we obtain:

$$\hat{z} = (1+\kappa)\frac{\underline{c}}{\underline{z}} + \frac{q^2-1}{2A\chi} \cdot \frac{1}{\underline{z}} + [r_Z - n] + r_D \frac{\underline{d}_F}{\underline{z}} - (1+\omega\xi)\frac{1}{\underline{z}}. \tag{F.11}$$

From Eq. (A.11), it follows that: $\dot{\underline{z}} = \hat{\underline{z}} \cdot \underline{z} = (\hat{z} - \varphi)\underline{z}$. Together with Eqs. (F.11) and (A.6), it can be rearranged to:

$$\dot{\underline{z}} = (1+\kappa)\underline{c} + \frac{q^2-1}{2A\chi} - (1+\omega\xi) + \left(r_Z - \frac{q-1}{\chi} + \delta\right)\underline{z} + r_D \underline{d}_F, \tag{F.12}$$

where $r_Z = \varepsilon_Z + p_Z \underline{z}$, $r_D = \varepsilon_D + p_D \underline{d}$. Just like in the decentralized economy, Eqs. (F.10), (F.12) and (F.9) jointly determine the evolution of $\underline{c}$, $\underline{z}$ and q. In order to close this system of differential equations we must append Eqs. (A.15) and (A.16) describing the dynamics of $\underline{d}_D$ and $\underline{d}_F$. In that way we have obtained a nonlinear autonomous system of 5 differential equations in 5 variables, as in Eq. (17).

G Proof of proposition 3

The steady state is derived by nullifying the right-hand sides of Eqs. (F.10), (F.12), (F.9), (A.15) and (A.16), which results in the system of 5 equations of the form: $f^i(\underline{c}, \underline{z}, q, \underline{d}_F, \underline{d}_D) = 0$ $(i = 1, \ldots, 5)$, which can be written as:

$$\frac{\bar{r}_Z + p_Z \bar{\underline{z}} - \rho}{A_1} - \bar{\varphi} = 0, \tag{G.1}$$

$$(1+\kappa)\bar{\underline{c}} + \frac{\bar{q}^2 - 1}{2A\chi} - (1+\omega\xi) + (\bar{r}_Z - n - \bar{\varphi})\,\bar{\underline{z}} + \bar{r}_D \bar{\underline{d}}_F = 0, \tag{G.2}$$

$$\begin{aligned}(\bar{r}_Z + p_Z \bar{\underline{z}} + \delta)\cdot \bar{q} \quad &- \quad A(1+\omega\xi) - \tfrac{(\bar{q}-1)^2}{2\chi} - A p_Z \bar{\underline{z}}^2 - A p_D \bar{\underline{d}}_F \bar{\underline{d}} + \\ &+ \quad \xi A \frac{\omega \bar{r}_D + p_D \bar{\underline{d}}_F}{\bar{r}_Z + p_Z \bar{\underline{z}}} = 0,\end{aligned} \tag{G.3}$$

$$(-n - \bar{\varphi})\bar{\underline{d}}_F + \omega\xi = 0, \tag{G.4}$$

$$(-n - \bar{\varphi})\bar{\underline{d}}_D + (1-\omega)\xi = 0, \tag{G.5}$$

where $\bar{r}_Z = \varepsilon_Z + p_Z \bar{\underline{z}}$, $\bar{r}_D = \varepsilon_D + p_Z \bar{\underline{d}}$, $\bar{\underline{d}} = \bar{\underline{d}}_F + \bar{\underline{d}}_D$, $\bar{\varphi} = (\bar{q}-1)/\chi - (n+\delta)$. First, notice that Eq. (G.1) immediately yields the steady-state rate of growth of all per capita variables:

$$\hat{\bar{y}} = \hat{\bar{k}} = \hat{\bar{c}} = \hat{\bar{z}} = \bar{\varphi}^* = (\bar{r}_Z + p_Z \bar{\underline{z}} - \rho)/A_1.$$

From Eq. (G.1), it follows that:

$$\bar{r}_Z + p_Z \bar{\underline{z}} = \rho + \bar{\varphi} A_1. \tag{G.6}$$

Using Eq. (1), we obtain:

$$\bar{r}_Z = (\rho + \varepsilon_Z + \bar{\varphi} A_1)/2, \tag{G.7}$$

$$\bar{\underline{z}} = (\rho - \varepsilon_Z + \bar{\varphi} A_1)/2p_Z. \tag{G.8}$$

In addition, it follows from Eq. (2) that: $\bar{r}_D = \varepsilon_D + p_D \bar{\underline{d}}$. Using Eqs. (G.6)-(G.8), equality (G.3) can be written as:

$$\begin{aligned}&(\rho + \bar{\varphi}A_1 + \delta)\cdot \bar{q} - A(1+\omega\xi) - \tfrac{(\bar{q}-1)^2}{2\chi} - \tfrac{A}{4p_Z}(\rho + \bar{\varphi}A_1 - \varepsilon_Z)^2 - A p_D \bar{\underline{d}}_F \bar{\underline{d}} + \\ &+ \xi A \frac{\omega(\varepsilon_D + p_D \underline{d}) + p_D \bar{\underline{d}}_F}{\rho + \bar{\varphi} A_1} = 0.\end{aligned} \tag{G.9}$$

Eqs. (G.4) and (G.5) determine the steady-state values of the government debt indicators, which are expressed by identical formulae as in the decentralized economy: Eqs. (23) and (24). Substituting these formulae into Eq. (G.9) yields:

$$\begin{aligned}&(\rho+\bar{\varphi}A_1+\delta)\cdot\bar{q}-A(1+\omega\xi)-\tfrac{(\bar{q}-1)^2}{2\chi}-\tfrac{A}{4p_Z}\left(\rho+\bar{\varphi}A_1-\varepsilon_Z\right)^2-\\&-Ap_D\frac{\omega\xi^2}{(n+\bar{\varphi})^2}+\xi A\frac{\omega\left(\varepsilon_D+p_D\frac{\xi}{n+\bar{\varphi}}\right)+p_D\frac{\omega\xi}{n+\bar{\varphi}}}{\rho+\bar{\varphi}A_1}=0.\end{aligned}\tag{G.10}$$

It follows from Eq. (A.6) that $\bar{q}=1+\chi(\bar{\varphi}+n+\delta)$. Replacing $\bar{q}$ in Eq. (G.10) with this formula yields an equation with one unknown, $\bar{\varphi}$:

$$\begin{aligned}&(\rho+\bar{\varphi}A_1+\delta)\cdot[1+\chi(\bar{\varphi}+n+\delta)]-A(1+\omega\xi)-\tfrac{\chi(\bar{\varphi}+n+\delta)^2}{2}+\\&-\tfrac{A}{4p_Z}\left(\rho+\bar{\varphi}A_1-\varepsilon_Z\right)^2-Ap_D\frac{\omega\xi^2}{(n+\bar{\varphi})^2}+\\&+\xi A\frac{\omega(\varepsilon_D+p_D\frac{\xi}{n+\bar{\varphi}})+p_D\frac{\omega\xi}{n+\bar{\varphi}}}{\rho+\bar{\varphi}A_1}=0.\end{aligned}\tag{G.11}$$

Multiplying both sides by the expression $4p_Z(\rho+\bar{\varphi}A_1)(n+\bar{\varphi})^2$, after rearrangement we obtain a fifth-order polynomial equation (30).

The coefficients of Eq. (30)**:**

$$\begin{aligned}w_0 &= 2p_Z\left\{2A\xi^2(2n-\rho)\omega p_D-n^2\left[-\rho\left(2\rho-n^2\chi+\delta^2\chi+2n\rho\chi+2\delta(1+\rho\chi)\right)+\right.\right.\\&\quad\left.\left.+2A(\rho+\xi\rho\omega)-2A\xi\omega\varepsilon_D\right]\right\}-An^2\rho(\rho-\varepsilon_Z)^2,\\ w_1 &= 2\left\{2p_Z\left[2A\xi^2\omega p_D-n\left(-\rho\left(\delta^2\chi-2n^2\chi+\rho(2+3n\chi)+2\delta(1+\rho\chi)\right)+\right.\right.\right.\\&\quad\left.\left.\left.2A(\rho+\xi\rho\omega)-2A\xi\omega\varepsilon_D\right)\right]-An\rho(\rho-\varepsilon_Z)^2\right\}+A_1\left\{2\left(-n^2\left(-4\rho+n^2\chi-\right.\right.\right.\\&\quad\left.\left.-\delta^2\chi-4n\rho\chi-2\delta(1+2\rho\chi)+2A(1+\xi\omega)-2A\,\xi^2\omega p_D\right)p_Z-\\&\quad\left.-An^2(\rho-\varepsilon_Z)(3\rho-\varepsilon_Z)\right\},\\ w_2 &= 2p_Z\left[\rho\left(\delta^2\chi-6\chi n^2+\rho(2+6n\chi)+2\delta(1+\rho\chi)\right)-2A(\rho+\rho\xi\omega)+\right.\\&\quad\left.+2A\xi\omega\,\varepsilon_D\right]+n^2A_1^2\left(4p_Z(1+n\chi+\delta\chi)-A(3\rho-2\varepsilon_Z)\right)-2A_1n\left[-2\,p_Z\left(4\rho-\right.\right.\\&\quad\left.\left.-2n^2\chi+\delta^2\chi++6n\rho\chi+2\delta(1+2\rho\chi)-2A(1+\xi\omega)\right)+A(\rho-\varepsilon_Z)(3\rho-\varepsilon_Z)\right]\\&\quad-A\rho(\rho-\varepsilon_Z)^2,\end{aligned}$$

$$w_3 = -An^2A_1^3 - 4(2n-\rho)\rho\chi p_Z + 2nA_1^2\left[2p_Z\left(2+3n\chi+2\delta\chi\right) - A\left(3\rho - 2\varepsilon_Z\right)\right] +$$
$$+A_1\left[2p_Z\left(4\rho - 6n^2\chi + \delta^2\chi + 12n\rho\chi + 2\delta(1+2\rho\chi) - 2A(1+\xi\omega)\right) - \right.$$
$$\left. A(\rho-\varepsilon_Z)(3\rho-\varepsilon_Z)\right],$$
$$w_4 = -2AnA_1^3 - 2\rho\chi p_Z + 8(\rho-n)A_1\chi p_Z + A_1^2\left[4p_Z\left(1+3n\chi+\delta\chi\right) - \right.$$
$$\left. -A\left(3\rho - 2\varepsilon_Z\right)\right],$$
$$w_5 = -A_1\left(AA_1^2 + 2\chi p_Z - 4A_1\chi p_Z\right).$$

H Details of Proposition 4

The linear approximation of the system of equations (17) about the equilibrium has the same form as in the decentralized economy, i.e., Eq. (C.1). This time, however, symbols with tilde denote deviations from the first-best steady-state, i.e., $\tilde{\underline{c}} = \underline{c} - \bar{\underline{c}}^*$, $\tilde{\underline{z}} = \underline{z} - \bar{\underline{z}}^*$, $\tilde{q} = q - \bar{q}^*$, $\tilde{\underline{d}}_F = \underline{d}_F - \bar{\underline{d}}_F^*$, $\tilde{\underline{d}}_D = \underline{d}_D - \bar{\underline{d}}_D^*$. The Jacobian matrix M^* is (Formally, the steady-state values in Eq. (H.1) should be denoted by stars, but for clarity of exposition, this time we skipped stars):

$$M^{*T} = \begin{bmatrix} 0 & 1+\kappa & 0 & 0 & 0 \\ \frac{2p_Z\bar{\underline{c}}}{A_1} & \rho+\bar{\varphi}A_1 - n - \bar{\varphi} & \begin{array}{c} 2p_Z(\bar{q} - A\bar{\underline{z}}) + \\ -\xi A\frac{\omega\bar{r}_D + p_D\bar{\underline{d}}_F}{2p_Z(\rho+\bar{\varphi}A_1)^2} \end{array} & 0 & 0 \\ -\frac{\bar{\underline{c}}}{\chi} & \frac{\bar{q}-A\bar{\underline{z}}}{A\chi} & \rho+\bar{\varphi}A_1 - n - \bar{\varphi} & -\frac{\bar{\underline{d}}_F}{\chi} & -\frac{\bar{\underline{d}}_D}{\chi} \\ 0 & \begin{array}{c} p_D(2\bar{\underline{d}}_F + \bar{\underline{d}}_D) + \\ +\varepsilon_D \end{array} & \begin{array}{c} -Ap_D \cdot \\ \cdot\left[2\bar{\underline{d}}_F + \bar{\underline{d}}_D - \frac{\xi A(1+\omega)}{\rho+\bar{\varphi}A_1}\right] \end{array} & -n-\bar{\varphi} & 0 \\ 0 & p_D\bar{\underline{d}}_F & -Ap_D\left[\bar{\underline{d}}_F - \frac{\xi\omega}{\rho+\bar{\varphi}A_1}\right] & 0 & -n-\bar{\varphi} \end{bmatrix} \tag{H.1}$$

The general solution of the linear system of equations (C.1) can be written as:

$$\begin{bmatrix} \underline{c}^* & \underline{z}^* & q^* & \underline{d}_F^* & \underline{d}_D^* \end{bmatrix}^T = \begin{bmatrix} \bar{\underline{c}}^* & \bar{\underline{z}}^* & \bar{q}^* & \bar{\underline{d}}_F^* & \bar{\underline{d}}_D^* \end{bmatrix}^T + \sum_{i=1}^{5} s_i e^{r_i t} v^i, \tag{H.2}$$

where r_i are the eigenvalues of the matrix M^*, v^i are its eigenvectors, and s_i are unknown constants dependent on the starting point (endowments). It follows that:

$$
\begin{aligned}
\det M^* &= \frac{-(1+\kappa)(n+\bar{\varphi}^*)\underline{\bar{c}}^*}{2p_Z\chi A_1(\rho+\bar{\varphi}^* A_1)^3}\cdot[expr]\,,\\
\operatorname{tr} M^* &= 2\left[\rho+\bar{\varphi}^* A_1 - 2(n+\bar{\varphi}^*)\right],
\end{aligned}
$$

where *expr* is such a complex expression, that it is impossible to analytically determine its sign (see below). Moreover, even the sign of $\operatorname{tr} M^*$ cannot be predetermined without making additional assumptions. It is clear, therefore, that the stability of equilibrium can only be investigated by numerical methods, after calibrating the model.

To be able to continue our discussion, let us assume that the matrix M^* has the same properties as in the decentralized economy, i.e., it has exactly 3 eigenvalues with negative real parts. In this case, the equilibrium has the form of the stable saddle path, and solution (H.2) can be written as (32).

The formula for *expr*:

$$
\begin{aligned}
expr = {}& 4\bar{\varphi}^3(n+\bar{\varphi})(\bar{q}-A\underline{\bar{z}}+\chi\bar{\varphi})A_1^4 p_Z^2 +\\
& -4\rho^2 p_Z^2\left(n(n-\rho)\rho\chi+(2n-\rho)\rho\chi\bar{\varphi}+\right.\\
& \quad +\rho\chi\bar{\varphi}^2 + Ap_D\left(\omega\underline{\bar{d}}_D\left(\xi(1+A+A\omega)-\rho\underline{\bar{d}}_D\right)+\right.\\
& \qquad \left.\left.+\left(A\,\xi\omega(1+\omega)-\rho(1+3\omega)\,\underline{\bar{d}}_D\right)\underline{\bar{d}}_F - 2\rho\omega\ \underline{\bar{d}}_F^2\right)\right)+\\
& -4\bar{\varphi}^2 A_1^3 p_Z^2\left\{-3\rho\bar{q}(n+\bar{\varphi})+3A\rho\underline{\bar{z}}(n+\xi+\bar{\varphi})+\right.\\
& \quad +\bar{\varphi}\left(n(n-4\rho)\chi+\chi\bar{\varphi}(2n-4\rho+\bar{\varphi})+\right.\\
& \qquad \left.-Ap_D\left(\omega\underline{\bar{d}}_D^2+(1+3\omega)\underline{\bar{d}}_D\underline{\bar{d}}_F+2\omega\ \underline{\bar{d}}_F^2\right)\right)+\\
& -A_1\left[8\rho(3n\rho-2\rho^2)\chi\bar{\varphi}^2 p_Z^2+12\rho^2\chi\bar{\varphi}^3 p_Z^2+\right.\\
& \quad +\bar{\varphi}\left(-4\rho p_Z^2\left(-n(3n\rho-4\rho^2)\chi+\rho^2\bar{q}+\right.\right.\\
& \qquad -A\rho^2\underline{\bar{z}}+Ap_D\left(\omega\underline{\bar{d}}_D\left(-2\xi(1+A+A\omega)+3\rho\underline{\bar{d}}_D\right)+\right.\\
& \qquad\quad +\left(-2A\xi\omega(1+\omega)+3\rho(1+3\omega)\underline{\bar{d}}_D\right)\underline{\bar{d}}_F+\\
& \qquad\quad \left.\left.+6\rho\omega\ \underline{\bar{d}}_F^2\right)\right)+A\xi\rho\left(p_D\underline{\bar{d}}_F+\omega\bar{r}_D\right)\Big)+\\
& \qquad +n\rho\left(-4\rho^2\bar{q}p_Z^2+A\left(4\rho^2\underline{\bar{z}}p_Z^2+\xi\left(p_D\underline{\bar{d}}_F+\omega\bar{r}_D\right)\right)\right)\Big]+\\
& -\bar{\varphi}A_1^2\left[4\left(\xi^2-6\rho^2+6n\rho\right)\chi\bar{\varphi}^2\ p_Z^2+12\rho\chi\bar{\varphi}^3 p_Z^2+\right.\\
& \quad +\bar{\varphi}\left(4p_Z^2\left(n(-6\rho^2+3n\rho)\ \chi-3\rho^2\bar{q}+\right.\right.\\
& \qquad +3A\rho^2\underline{\bar{z}}+Ap_D\left(\omega\underline{\bar{d}}_D\left(\xi(1+A+A\omega)-3\rho\underline{\bar{d}}_D\right)+\right.
\end{aligned}
$$

$$+\left(A\xi\omega(1+\omega)-3\rho(1+3\omega)\bar{\underline{d}}_D\right)\bar{\underline{d}}_F-6\rho\omega\;\bar{\underline{d}}_F^2\Big)\Big)+$$
$$+A\xi\;\left(p_D\bar{\underline{d}}_F+\omega\bar{r}_D\right)\Big)+n\left(-12\rho^2\;\bar{q}p_Z^2+\right.$$
$$\left.+A\;\left(12\rho^2\bar{\underline{z}}p_Z^2+\xi\left(p_D\bar{\underline{d}}_F+\omega\bar{r}_D\right)\right)\right]\Big\}\,.$$

I Proof that the transversality conditions (F.h)-(F.k) are satisfied if, and only if, (31)

1. Let us start with condition (F.i). Substituting $\lambda_2(t)=q^*(t)\cdot\lambda_1(t)$ this condition can be written in an equivalent form:

$$\lim_{t\to\infty}e^{-(\rho-n)t}\lambda_1(t)q^*(t)k^*(t)=0. \tag{I.1}$$

Recall that both Eqs. (A.7) and (A.7) hold for the social planner. It follows that the trajectory of capital is of the form:

$$k^*(t)=k_0e^{\int_0^t\left(\frac{q^*(s)-1}{\chi}\right)ds}e^{-(n+\delta)t}. \tag{I.2}$$

Meanwhile, Eq. (F.3) implies that $\dot{\lambda}_1/\lambda_1=\rho-r_Z(\underline{z}^*)-p_Z\underline{z}^*(t)$, where $r_Z(\underline{z})=\varepsilon_Z+p_Z\underline{z}^*(t)$. Thus, $\dot{\lambda}_1/\lambda_1=\rho-\varepsilon_Z-2p_Z\underline{z}^*(t)$. Therefore, the trajectory $\lambda_1(t)$ has the following form:

$$\lambda_1(t)=\lambda_1(0)e^{(\rho-\varepsilon_Z)t}e^{-2p_Z\int_o^t\left(\underline{z}^*(s)\right)ds}. \tag{I.3}$$

Having regard to Eqs. (I.2) and (I.3) condition (I.1) can be written as:

$$\lambda_1(0)k_0\lim_{t\to\infty}\left\{q^*(t)e^{-(\varepsilon_Z+\delta)t}e^{-2p_Z\int_0^t\left(\underline{z}^*(s)\right)ds}e^{\int_0^t\left(\frac{q^*(s)-1}{\chi}\right)ds}\right\}=0,$$

which is equivalent to:

$$\lambda_1(0)k_0\lim_{t\to\infty}\left\{q^*(t)e^{-\left(\varepsilon_Z+\delta+\frac{1}{\chi}\right)t}e^{-2p_Z\int_0^t\underline{z}^*(s)ds}e^{\frac{1}{\chi}\int_0^t q^*(s)ds}\right\}=0. \tag{I.4}$$

In order to examine this condition we need to know the trajectories of variables $\underline{z}^*(t)$ and $q^*(t)$. Because the model is non-linear, we will use the approximate trajectories obtained by solving the linearized model. From Eqs. (H.2) we know that:

$$\underline{z}^*(t)=\bar{\underline{z}}^*+s_1e^{r_1t}v_2^1+s_2e^{r_2t}v_2^2+s_3e^{r_3t}v_2^3=\bar{\underline{z}}^*+\sum_{i=1}^3s_iv_2^i\,e^{r_it}, \tag{I.5}$$

$$q^*(t) = \bar{q}^* + s_1 e^{r_1 t} v_3^1 + s_2 e^{r_2 t} v_3^2 + s_3 e^{r_3 t} v_3^3 = \bar{q}^* + \sum_{i=1}^{3} s_i v_3^i \, e^{r_i t}. \tag{I.6}$$

We are interested in stable equilibria only, and hence we assume that:

$$\forall i \quad (r_i \geq 0 \quad \Rightarrow \quad s_i = 0). \tag{I.7}$$

(All nonnegative eigenvalues of the matrix M must be associated with zero constants s_i.) Using Eq. (I.5), we obtain

$$\int_0^t \underline{z}^*(s)ds = \bar{\underline{z}}^* t - \sum_{i=1}^{3} \frac{s_i v_2^i}{r_i} + \sum_{i=1}^{3} \frac{s_i v_2^i}{r_i} \, e^{r_i t},$$

which implies that

$$e^{-2p_Z \int_0^t \underline{z}^*(s)ds} = e^{-2p_Z \bar{\underline{z}}^* t} e^{2p_Z \sum_{i=1}^{3} \frac{s_i v_2^i}{r_i}} e^{-2p_Z \sum_{i=1}^{3} \frac{s_i v_2^i}{r_i} \, e^{r_i t}}. \tag{I.8}$$

Analogously, on the basis of Eq. (I.6), we compute:

$$\int_0^t q^*(s)ds = \bar{q}^* t - \sum_{i=1}^{3} \frac{s_i v_3^i}{r_i} + \sum_{i=1}^{3} \frac{s_i v_3^i}{r_i} \, e^{r_i t},$$

which implies that:

$$e^{\frac{1}{\chi} \int_0^t q^*(s)ds} = e^{\frac{\bar{q}^*}{\chi} t} e^{-\frac{1}{\chi} \sum_{i=1}^{3} \frac{s_i v_3^i}{r_i}} e^{\sum_{i=1}^{3} \frac{s_i v_3^i}{\chi r_i} \, e^{r_i t}}. \tag{I.9}$$

Using Eqs. (I.8) and (I.9), we can rewrite condition (I.4) as:

$$\begin{aligned} &\lambda_1(0) k_0 e^{2p_Z \sum_{i=1}^{3} \frac{s_i v_2^i}{r_i} - \frac{1}{\chi} \sum_{i=1}^{3} \frac{s_i v_3^i}{r_i}} \cdot \\ &\cdot \lim_{t \to \infty} \left\{ q^*(t) e^{-\left(\varepsilon_Z + \delta + \frac{1}{\chi} + 2p_Z \bar{z}^* - \frac{\bar{q}^*}{\chi} \right) t} \cdot \right. \\ &\left. \cdot e^{-2p_Z \sum_{i=1}^{3} \frac{s_i v_2^i}{r_i} \, e^{r_i t}} e^{\sum_{i=1}^{3} \frac{s_i v_3^i}{\chi r_i} \, e^{r_i t}} \right\} = 0. \end{aligned} \tag{I.10}$$

Assumption (I.7) implies that

$$\lim_{t \to \infty} e^{-2p_Z \sum_{i=1}^{3} \frac{s_i v_2^i}{r_i} \, e^{r_i t}} = 1$$

and

$$\lim_{t \to \infty} e^{\sum_{i=1}^{3} \frac{s_i v_3^i}{\chi r_i} \, e^{r_i t}} = 1.$$

Therefore, condition (I.10) is satisfied if, and only if:

$$\lim_{t\to\infty}\left\{q^*(t)e^{-\left(\varepsilon_Z+\delta+\frac{1}{\chi}+2p_Z\bar{\underline{z}}^*-\frac{\bar{q}^*}{\chi}\right)t}\right\}=0.$$

Using Eq. (I.6) together with the equality $\bar{r}_Z^*=\varepsilon_Z+p_Z\bar{\underline{z}}^*$ we can rewrite this condition as:

$$\begin{aligned}&\bar{q}\lim_{t\to\infty}e^{\left(\frac{\bar{q}-1}{\chi}-r_Z^*-p_Z\bar{\underline{z}}^*-\delta\right)t}+\\&+\lim_{t\to\infty}\left\{\left(\sum_{i=1}^3 s_i v_3^i\, e^{r_i t}\right)e^{\left(\frac{\bar{q}-1}{\chi}-r_Z^*-p_Z\bar{\underline{z}}^*-\delta\right)t}\right\}=0.\end{aligned} \tag{I.11}$$

It follows from Eq. (I.7) that $\lim\limits_{t\to\infty}\sum_{i=1}^3 s_i v_3^i\, e^{r_i t}=0$ and

$$e^{\left(\frac{\bar{q}-1}{\chi}-\bar{r}_Z^*-p_Z\bar{\underline{z}}^*-\delta\right)t}>0$$

(positive and finite number), and hence the second part of the sum in Eq. (I.11) is zero. Therefore, for Eq. (I.11) to hold, we must have:

$$\lim_{t\to\infty}e^{\left(\frac{\bar{q}-1}{\chi}-r_Z^*-p_Z\bar{\underline{z}}^*-\delta\right)t}=0.$$

This equality holds if, and only if,

$$\bar{r}_Z^*+p_Z\bar{\underline{z}}^*>\frac{\bar{q}^*-1}{\chi}-\delta=\bar{\varphi}^*+n.$$

It's straightforward to show that it is equivalent to:

$$\rho>n+\gamma(1+\kappa)\bar{\varphi}^*. \tag{I.12}$$

2. The second transversality condition is (F.h). As $\lambda_1'(t)=-\lambda_1(t)$ and $z^*(t)=\underline{z}^*(t)y^*(t)=\underline{z}^*(t)Ak^*(t)$, this condition is equivalent to:

$$\lim_{t\to\infty}e^{-(\rho-n)t}\lambda_1(t)\underline{z}^*(t)Ak^*(t)=0. \tag{I.13}$$

Eq. (G.8) implies that $\lim\limits_{t\to\infty}\underline{z}^*(t)=\bar{\underline{z}}^*\neq\pm\infty$, thus Eq. (I.13) can be rewritten as:

$$A\bar{\underline{z}}^*\lim_{t\to\infty}e^{-(\rho-n)t}\lambda_1(t)k^*(t)=0. \tag{I.14}$$

Notice that this condition is equivalent to Eq. (F.i), as long as we neglect the special case of $\bar{\underline{z}}^*=0$.

3. The third transversality condition is (F.j). As $d_F^*(t) = \underline{d}_F^*(t) \cdot y^*(t)$ and $y^*(t) = Ak^*(t)$, this condition is equivalent to:

$$A \lim_{t\to\infty} e^{-(\rho-n)t}\lambda_3(t)\underline{d}_F^*(t)k^*(t) = 0. \tag{I.15}$$

From Eq. (F.6), it follows that $\forall t$

$$\lambda_3(t) = -\lambda_1(t) \cdot \frac{r_D^*(t) + p_D\underline{d}_F^*(t)}{r_Z^*(t) + p_Z\underline{z}^*(t)}. \tag{I.16}$$

Using equalities $r_D^*(t) = \varepsilon_D + p_D\underline{d}^*(t)$ and $r_Z^*(t) = \varepsilon_Z + p_Z\underline{z}^*(t)$, we can rewrite Eq. (I.16) as:

$$\lambda_3(t) = -\lambda_1(t) \cdot \frac{\varepsilon_D + p_D\underline{d}_D^*(t) + 2p_D\underline{d}_F^*(t)}{\varepsilon_Z + 2p_Z\underline{z}^*(t)}. \tag{I.17}$$

Having regard to Eq. (I.17), condition (I.15) can be converted into the following form:

$$A \cdot \lim_{t\to\infty} e^{-(\rho-n)t}\lambda_1(t)\frac{\varepsilon_D + p_D\underline{d}_D^*(t) + 2p_D\underline{d}_F^*(t)}{\varepsilon_Z + 2p_Z\underline{z}^*(t)}\underline{d}_F^*(t)k^*(t) = 0. \tag{I.18}$$

Eq. (G.8) implies that $\lim_{t\to\infty}\underline{z}^*(t) = \bar{\underline{z}}^* \neq +\infty$. Similarly, Eqs. (23) and (24) that are valid not only for the decentralized economy, but also for the social planner, imply that $\lim_{t\to\infty}\underline{d}_F^*(t) = \bar{\underline{d}}_F^* \neq +\infty$, and $\lim_{t\to\infty}\underline{d}_D^*(t) = \bar{\underline{d}}_D^* \neq +\infty$. As by assumption these trajectories of public and private debt are positive, their limits cannot be equal to $-\infty$. These four facts imply that

$$\lim_{t\to\infty}\frac{\varepsilon_D + p_D\underline{d}_D^*(t) + 2p_D\underline{d}_F^*(t)}{\varepsilon_Z + 2p_Z\underline{z}^*(t)}\underline{d}_F^*(t) \neq \pm\infty$$

(this limit is finite). For this reason, condition (I.18) can be written in an equivalent form:

$$A\frac{\varepsilon_D + p_D\bar{\underline{d}}_D^*(t) + 2p_D\bar{\underline{d}}_F^*(t)}{\varepsilon_Z + 2p_Z\bar{\underline{z}}^*(t)}\bar{\underline{d}}_F^*(t) \cdot \lim_{t\to\infty} e^{-(\rho-n)t}\lambda_1(t) \cdot k^*(t) = 0. \tag{I.19}$$

Notice that this condition is equivalent to Eq. (F.i).
4. Analogously, we can demonstrate that the fourth transversality condition (F.k) is also equivalent to Eq. (F.i).
5. It follows that that the necessary and sufficient condition for all four transversality conditions to hold is inequality (31).

J Proof of proposition 5

Using condition $g_C = \kappa c$, we can express the level of welfare in the economy as: $\Omega^* = \frac{1}{\gamma}\kappa^{\kappa\gamma}\int_0^\infty (c_t)^{\gamma(1+\kappa)} e^{-(\rho-n)t}dt$. Substituting Eq. (A.5) yields:

$$\Omega^* = \frac{1}{\gamma}\kappa^{\kappa\gamma}c_0^{\gamma(1+\kappa)}\int_0^\infty e^{\left(\int_0^t \psi^*(s)ds\right)\gamma(1+\kappa)} e^{-(\rho-n)t}dt. \tag{J.1}$$

In order to evaluate the value of this integral, we use the linear approximation of the model. It follows from Eq. (32) that around the steady state:

$$\underline{z}^*(t) \approx \bar{\underline{z}}^* + \sum_{i=1}^{3} s_i e^{r_i t} v_2^i. \tag{J.2}$$

Substituting Eq. (J.2) into Eq. (1) yields a linear approximation of the trajectory $r_Z^*(t)$:

$$r_Z^*(t) = \varepsilon_Z + p_Z\underline{z}^*(t) \approx \varepsilon_Z + p_Z\bar{\underline{z}}^* + p_Z\sum_{i=1}^{3} s_i e^{r_i t} v_2^i = \bar{r}_Z^* + p_Z\sum_{i=1}^{3} s_i e^{r_i t} v_2^i. \tag{J.3}$$

Substituting Eqs. (J.2) and (J.3) into Eq. (F.4), we obtain an approximation of $\psi^*(t)$:

$$\psi^*(t) = \frac{\varepsilon_Z + 2p_Z\underline{z}^*(t) - \rho}{A_1} \approx \frac{1}{A_1}\left[\bar{r}_Z^* + p_Z\bar{\underline{z}}^* + 2p_Z\sum_{i=1}^{3} s_i e^{r_i t} v_2^i +\right] - \frac{\rho}{A_1}.$$

Thus

$$\int_0^t \psi^*(s)ds = \frac{\left(\bar{r}_Z^* + p_Z\bar{\underline{z}}^* - \rho\right)t}{A_1} + \frac{2p_Z}{A_1}\sum_{i=1}^{3}\frac{s_i v_2^i}{r_i}\left(e^{r_i t} - 1\right). \tag{J.4}$$

Finally, substituting Eq. (J.4) into Eq. (J.1) yields Eq. (33).

K Proof of proposition 6

Proposition 6 implies that the rates of taxes and public consumption are replicating the first-best trajectories if, and only if, for each $t \geq 0$ the following equalities hold:

$$\underline{c}(t) = \underline{c}^*(t), \tag{K.1}$$

$$\underline{z}(t) = \underline{z}^*(t), \tag{K.2}$$

$$q(t) = q^*(t), \tag{K.3}$$

$$\underline{d}_F(t) = \underline{d}_F^*(t), \tag{K.4}$$

$$\underline{d}_D(t) = \underline{d}_D^*(t), \tag{K.5}$$

$$g_C(t) = g_C^*(t), \tag{K.6}$$

as these 6 conditions directly implicate the equality (identity) of all other trajectories. Indeed, Eq. (K.3) together with Eq. (A.7) which holds for both types of economies imply that

$$\varphi(t) = \varphi^*(t). \tag{K.7}$$

Eq. (K.7) together with Eq. (A.8) (which is true for both types of economies) imply that $k(t) = k^*(t)$ and

$$y(t) = y^*(t). \tag{K.8}$$

From (K.1) and (K.8), it follows that

$$c(t) = c^*(t). \tag{K.9}$$

Eqs. (K.2) and (1) implicate the following: $r_Z(t) = r_Z^*(t)$. Meanwhile, Eqs. (K.4), (K.5), (K.8) together with identities $d_F(t) = \underline{d}_F(t) \cdot y(t)$ and $d_D(t) = \underline{d}_D(t) \cdot y(t)$ for both types of the economy imply that

$$d_F(t) = d_F^*(t), \tag{K.10}$$

$$d_D(t) = d_D^*(t). \tag{K.11}$$

which in turn implies that $d(t) = d^*(t)$. From Eqs. (K.2), (K.8) and an identity $z(t) \equiv \underline{z}(t) \cdot y(t)$ for both economies it follows that

$$z(t) = z^*(t). \tag{K.12}$$

Finally, Eqs. (K.4), (K.5) and (2) imply that $r_D(t) = r_D^*(t)$.
All these equations ensure that the decentralized economy replicates the first-best solution. What we have demonstrated above is that it will suffice to ensure that the six conditions (K.1)–(K.6) are fulfilled, as they constitute the necessary and sufficient conditions of full replication. Now, let us examine what particular values of fiscal policy parameters will ensure that.
First, note that because Eq. (K.6) implies Eq. (K.9), the condition (36) must hold.
Second, note that using the formula for the trajectories of consumption in both types of economies, i.e., Eq. (A.5), we can rewrite Eq. (K.8) in an equivalent form:

$$c(t) = c_0 \cdot e^{\int_0^t \psi(s)ds} = c_0^* \cdot e^{\int_0^t \psi^*(s)ds} = c^*(t). \tag{K.13}$$

This equality must hold for any moment t. Obviously, it holds if, and only if, $c_0 = c_0^*$ and $\psi(t) = \psi^*(t)$. Substituting Eqs. (A.4) and (F.4) we can write the latter as:

$$\frac{(1+\tau_Z)r_Z(t) - \rho}{1 - \gamma(1+\kappa)} = \frac{r_Z^*(t) + p_Z \underline{z}^*(t) - \rho}{1 - \gamma(1+\kappa)}.$$

Using Eq. (1) we can rewrite this as:

$$\frac{(1+\tau_Z)r_Z(t)-\rho}{1-\gamma(1+\kappa)} = \frac{\varepsilon_Z + 2p_Z\underline{z}^*(t)-\rho}{1-\gamma(1+\kappa)}.$$

This condition holds if, and only if, $(1+\tau_Z)r_Z(t) = \varepsilon_Z + 2p_Z\underline{z}^*(t)$. Using Eq. (1) together with Eq. (K.12) it can be transformed into: $(1+\tau_Z)\left[\varepsilon_Z + p_Z\underline{z}^*(t)\right] = \varepsilon_Z + 2p_Z\underline{z}^*(t)$, which can be reduced to the form (37).

Next, we will demonstrate that the rate of tax on capital income must vary in time, but it is impossible to provide an explicit analytical formula for its trajectory $\tau_K^{opt}(t)$. Using the (approximate) formulae for the trajectories $q(t)$ and $q^*(t)$, we can write condition (K.3) as:

$$\bar{q} + \sum_{i=1}^{3} s_i e^{r_i t} v_3^i = \bar{q}^* + \sum_{i=1}^{3} s_i^* e^{r_i^* t} w_3^i. \tag{K.14}$$

It must hold for any time t. Setting $t \to \infty$ yields:

$$\bar{q} = \bar{q}^*. \tag{K.15}$$

(Obviously, the equality of trajectories $q(t)$ and $q^*(t)$ implies the equality of their limit values). From Eqs. (K.14) and (K.15) it follows that the replication requires the fulfillment of the equality:

$$\sum_{i=1}^{3} s_i e^{r_i t} v_3^i = \sum_{i=1}^{3} s_i^* e^{r_i^* t} w_3^i \tag{K.16}$$

Notice that the right-hand side of this equation is independent of all tax rates, because the matrix M given by Eq. (H.1) is independent of them. Therefore the right-hand side of Eq. (K.16) can be considered as a given number, while the left side is dependent on five fiscal policy parameters, i.e., τ_K, τ_Z, σ_C, ω, ξ . The parameters τ_Z and σ_C are already fixed by conditions (36) and (37). The parameters ω and ξ are also fixed by conditions (K.10) and (K.11) – these equalities hold if, and only if, ω and ξ are exactly the same as assumed by the social planner. Therefore, the only parameter that remains free to manipulate is τ_K. It follows from this that in order to ensure condition (K.16) the parameter τ_K must be properly adjusted for each moment of time $t \geq 0$. Notice that on the left-hand side of Eq. (K.16) we have the (negative) eigenvalues of the matrix M, which cannot be described by analytical formulae, as they are the roots of the polynomial of the 5^{th} order. Therefore, it is not possible to solve equation (K.16), even for a given moment of time (some selected t). For this reason it is impossible to derive an analytical formula for the optimal trajectory $\tau_K^{opt}(t)$– not only for the whole trajectory, but even for a single value of τ_K in a selected moment of time t. Individual (momentary) values of the trajectory $\tau_K^{opt}(t)$ can only be calculated numerically, for a given set of values of all model parameters.

L Proof of proposition 7

The formulae describing the steady states of both economies directly implicate that the replication of the steady state occurs if, and only if, the following (necessary and sufficient) conditions are satisfied:

$$\underline{\bar{z}} = \underline{\bar{z}}^{*}. \tag{L.1}$$

$$\underline{\bar{c}} = \underline{\bar{c}}^{*}. \tag{L.2}$$

$$\bar{\varphi} = \bar{\varphi}^{*}. \tag{L.3}$$

Let us consider these conditions one by one. From Eq. (1), it follows that if condition (L.1) holds, then $\bar{r}_Z = \bar{r}_Z^*$, and vice versa. Put simply, condition (L.1) is equivalent to: $\bar{r}_Z = \bar{r}_Z^*$. This fact together with the formulae for the rates of growth of consumption in both types of economies imply that if condition (L.1) holds, then $\bar{\psi} = \bar{\psi}^*$. Previously, we have proved that $\psi(t) = \psi^*(t)$ if, and only if, $\tau_Z(t) = \tau_Z^{opt}(t)$. It implies that $\bar{\psi} = \bar{\psi}^*$ if, and only if, $\tau_Z = \lim_{t\to\infty} \tau_Z^{opt}(t)$. It follows that if condition (L.1) holds, then the tax rate τ_Z is equal to (39), which is the second necessary condition of partial replication.

The first necessary condition can be derived from condition (L.2). Recall that $\bar{q}^* = 1 + \chi(\bar{\varphi}^* + n + \delta)$ and $\bar{q} = 1 + \chi(\bar{\varphi} + n + \delta)$. It follows that condition (L.3) is equivalent to the equality: $\bar{q} = \bar{q}^*$. Meanwhile, if $\bar{\varphi} = \bar{\varphi}^*$, then $\underline{\bar{r}}_D = \underline{\bar{r}}_D^*$ and $\underline{\bar{d}}_F = \underline{\bar{d}}_F^*$. Taking all of this into account, it's easy to see that if (L.3) holds, then condition (L.2) is satisfied if, and only if, (38), which is the first necessary condition of partial replication. Note that these two conditions coincide with the conditions of full replication, (36) and (37). Thus, technically, they could be derived from these two conditions by assuming that the economy governed by the social planner is on the balanced growth path from the very beginning (from $t = 0$).

Finally, we need to analyze the last necessary and sufficient condition of partial replication, (L.3). Recall that the social planner's rate of balanced growth $\bar{\varphi}^*$ is a real, positive solution of a fifth-order polynomial equation, and hence there is no analytical formula for $\bar{\varphi}^*$. Meanwhile, the balanced growth rate in the decentralized economy, $\bar{\varphi}$, is given by a simple formula, which through appropriate substitutions can be written as:

$$\bar{\varphi} = \frac{-\rho - \delta - \frac{\gamma(1+\kappa)}{\chi} + \gamma(1+\kappa)(n+\delta) + \frac{\sqrt{\Delta}}{2\chi} - 1 + 2\gamma(1+\kappa)}{1 - 2\gamma(1+\kappa)} = const., \tag{L.4}$$

where Δ is a linear function of τ_K given by Eqs. (19) and (20). Therefore, the rate of taxation τ_K, which assures the equality $\bar{\varphi} = \bar{\varphi}^*$ can only be identified numerically (there is no analytical formula of this rate). In order to do this, having chosen all values of parameters, one needs to find such a value of τ_K, for which $\bar{\varphi}$ calculated in accordance with Eq. (L.4) is at the same time an appropriate solution of Eq. (30), i.e., it satisfies Eq. (30) and is equal to $\bar{\varphi}^*$.

Technical note. The easiest procedure to calculate the replicating rate τ_K^{rep} is this: derive the function $\bar{\varphi}(\tau_K)$ by plugging Eqs. (19) and (20) into Eq. (L.4). Then, substitute the function $\bar{\varphi}(\tau_K)$ into Eq. (30), and solve it numerically for τ_K.

M The transitory dynamics in the baseline scenario

Yellow lines represent the first-best solution, whereas blue lines – the decentralized economy in the baseline scenario. The trajectories were obtained on the basis of the linear approximation of the model. For each type of the economy, first the steady state was calculated, next the matrices M and M^*, their eigenvalues and eigenvectors, and finally all the trajectories in accordance with formulae presented in sections 3 and 4. The time index t is in years. All numerical calculations have been carried out with Mathematica 9.0.

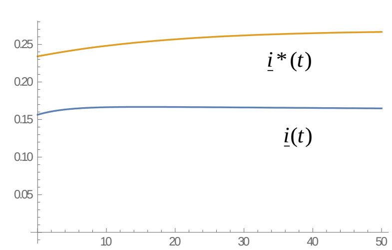


Fig. M1. Investment (% of GDP)

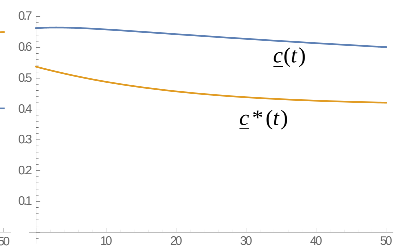


Fig. M2. Private consumption (% of GDP)

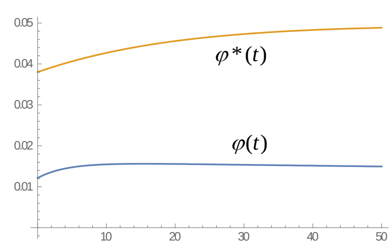


Fig. M3. The growth rate of GDP per capita

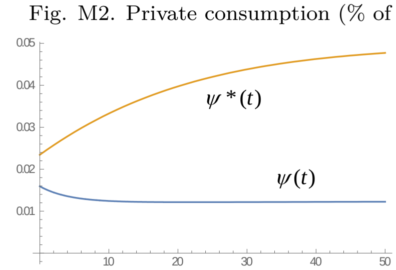


Fig. M4. The rate of growth of private consumption per capita

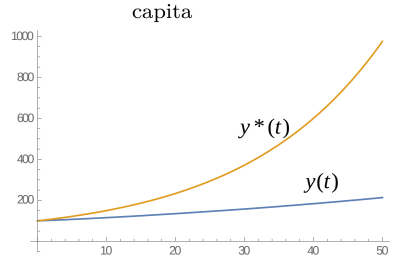


Fig. M5. GDP per capita

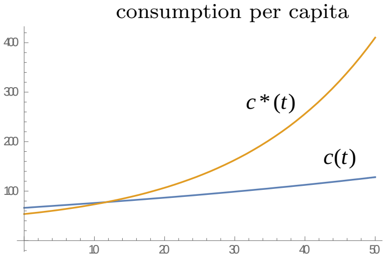


Fig. M6. Private consumption per capita

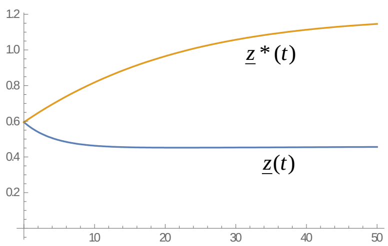


Fig. M7. Private foreign debt (% of GDP)

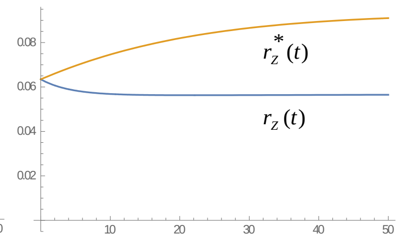


Fig. M8. The interest rate on private foreign debt

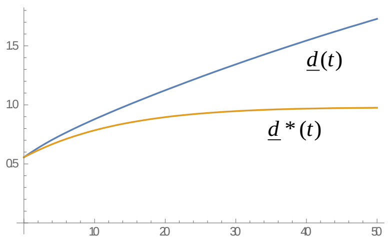


Fig. M9. Public debt (% of GDP)

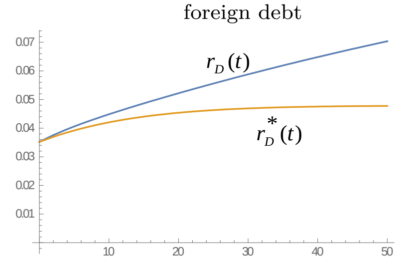


Fig. M10. The interest rate on public debt.

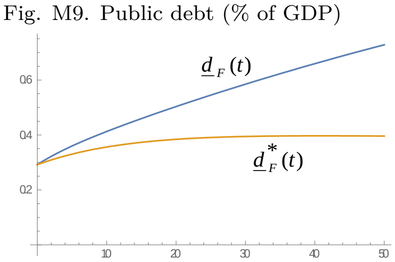


Fig. M11. Foreign public debt (%of GDP)

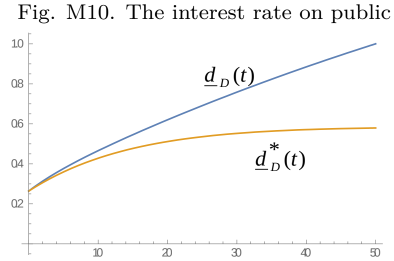


Fig. M12. Domestic public debt (% of GDP)