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On applications of computer algebra systems in queueing theory calculations

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Abstract. In the present paper, the most important aspects of computer algebra systems applications in complicated calculations for classical queueing theory models and their novel modifications are discussed. We mainly present huge computational possibilities of *Mathematica* environment and effective methods of obtaining symbolic results connected with most important performance characteristics of queueing systems. First of all, we investigate effective solutions of computational problems appearing in queueing theory such as: finding final probabilities for Markov chains with a huge number of states, calculating derivatives of complicated rational functions of one or many variables with the use of classical and generalized L'Hospital's rules, obtaining exact formulae of Stieltjes convolutions, calculating chosen integral transforms used often in the above-mentioned theory and possible applications of generalized density function of random variables and vectors in these computations. Some exemplary calculations for practical models belonging both to classical models and their generalizations are attached as well.

Key words: classical queueing models, queueing systems with random volume customers and sectorized memory buffer, generalized L'Hospital's rule, Stieltjes convolution, Laplace and Laplace–Stieltjes transforms.

1. INTRODUCTION

Queueing theory is the field of applied mathematics that has been experiencing great development in recent years. This scientific direction, started in the 20's of the twentieth century by A. K. Erlang, had initially important meaning mainly for telecommunication engineers (they used obtained theoretical results to calculate performance characteristics of telephone exchanges that were needed in telecommunication systems designing process) [1] but its importance was also noticed by scientists from technical computer science area because it introduced some models that could be used (sometimes, after some modifications) in the process of real-life computer systems analyzing or designing (e.g. computer networks). The number of published papers investigating such models has been still increasing since the moment of a big headway and popularization of computer systems in the 90's of previous century. As an example, we can mention some chosen publications from last years [2–5]. In these works authors analyze systems with random volume customers (customers coming to the queueing systems transport information that is written down in memory buffer of the system until customer ends his service, so it is assumed that customers are additionally characterized by some random volume - see also monograph [6]). The very interesting, novel approach appears also in papers [7, 8] that investigate models in which the above-mentioned customer's volume is multidimensional. The main problems analyzed for queueing models are connected with calculating characteristics of the number of customers present in the system (especially in the steady state), characteristics of the total volume of customers and loss probability (if the memory buffer is limited, we have additional losses of customers that cannot be accepted for servicing due to their, too big, volume). In addition, the need of constructing such models is confirmed in projects of some technical devices [9, 10].

In the process of mathematical analysis of both classical models and, especially, their generalizations, we face the problem of complicated symbolic computations. The general results often contain functions that are very complex and inconvenient from the numerical point of view as they contain such mathematical concepts like: generating functions, integral transforms or convolutions. Moreover, in obtained formulae we usually find very complicated rational functions of one or many variables what does not let calculate needed numerical characteristics in easy way. For example, we need to calculate derivatives of these functions, often using L'Hospital's rule many times, which makes computations are hardly possible without computer algebra systems help. The next computational problem, appearing even in simple classical models described by Markov chains, is solving systems of linear equations in which we have a huge number of variables.

In fact, computer algebra systems give fantastic tools to lead complicated symbolic computations successfully. E. g. *Mathematica* environment delivers many implemented useful functions letting calculate integral transforms and their inversions or derivatives of complicated rational functions as well as solve systems of linear equations with huge number of variables [11]. The big advantage of computer algebra systems is also storing previous results in memory and the possibility to use them again in the next steps of computations despite of their complexity. These facts confirm that computer algebra systems are effective tools that may be helpful in the process of complicated queueing models analyzing.

The main purpose of the presented paper is to discuss some smart computational techniques using computer algebra systems that can help researchers, whose scope of scientific activities is related to the study of queuing systems, in obtaining of significant results connected with performance characteristics of investigated models of such systems. We want to show proven methods that make complicated calculations become

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easier compared to manual ones and let shorten time intended for such complex computations. These techniques were developed and used by author during his many years of practice in analyzing of the above-mentioned models and can certainly help to facilitate analogous analyzes of queuing systems conducted by other scientists.

This paper presents main computational problems appearing during mathematical analysis of queueing systems models and their possible solutions with the help of *Mathematica* environment. The rest of the paper is organized as follows. In the next Sec. 2, we present short description of most important classical queueing models and their modifications, and possible applications of these models in the area of telecommunication or computer systems analyzing and designing process. In Sec. 3, we discuss main mathematical concepts used in classical queueing theory that can make large computational problems. Sec. 4 presents these problems and their possible solutions together with some numerical examples done with the help of computer algebra systems. In Sec. 5 , we focus on problems of calculating performance characteristics for the queueing systems with random volume customers (and also with sectorized memory buffer) connected with much more complicated computations containing functions of many variables as well as possible solutions of these problems. The last Sec. 6 contains conclusions and final remarks.

2. CLASSICAL QUEUEING MODELS, THEIR MODIFICATIONS AND POSSIBLE APPLICATIONS

First mathematical models of real-life queueing systems were constructed by Danish mathematician and engineer A. K. Erlang. In his works (epecially [12]) he offered a model describing telephone exchange working process based on assumptions that arriving customers form Poisson entrance flow (this assumption was checked in experiments), customer's service time is exponentially distributed and system contains n devices that work independently and has no queue i.e. all customers (telephone calls) arriving to the the system in whiles when all devices are busy, are lost (blocked). The main obtained practical result was probability that a coming call will be blocked. Methods used by Erlang allowed to obtain numbers of customers distribution for simple models for which we can use Markov chains or their small modifications. Results were used in the process of analyzing or designing of some simple telecommunication systems e.g. helped to choose the proper number of needed devices in such way to make sure that probability of block will be small enough. These models denoted as $M/M/n/m$ in Kendall's modified notation were first ones describing real telecommunication systems.

Later on, more complicated and more practical models were analyzed. One of the most important results were those connected with $M/G/1/\infty$ queueing model (single-server queueing system with infinite queue) obtained independently by F. Pollaczek and A. Khintchine [13, 14]. The analysis of this model demanded introducing some modifications and new mathematical approach because theory of Markov chains was not sufficient in this case and it was necessary to introduce more general semimarkovian stochastic processes. During the entire period of the 20th century there appeared many new papers analyzing more and more complicated but realistic models describing telecommunication or (later) computer systems. In such way results for the queueing models of the $M/G/1/n$, $M/G/n/0$, $M/G/\infty$ or $GI/M/n/\infty$-types were obtained [1,6], as well as those for single-server queueing system with egalitarian processor sharing investigated by S. Yashkov - see [15] (in this system all customers are served at the same time) or systems with vacations (assuming random rests is servers' working) investigated e.g. by T. Lee and B. Doshi - see [16, 17]. Obtained results had practical meaning because, based on calculated characteristics, engineers could e.g. (on the level of system designing) shorten mean queue length in the case of systems with finite number of waiting positions in the queue or decrease mean waiting time for systems with infinite queue.

Together with the headway in computer science, some scientists started using queueing models also in this area, adapting classical results onto computer systems. In the 70's of previous century, first papers appeared that introduced new concept - customer's volume (assuming that customers transport some information) and extended classical research. The main purpose in this case was to calculate characteristics of customers' total volume (sum of the volumes of all customers present in the system) but investigations initially assumed that character of dependency between customer's volume (size) and his service time does not have influence on total customers' volume characteristics, and, what is interesting, considered them as independent (what was not true in most of real-life computer systems) - see e.g. books written by M. Schwarz [18, 19]. First works that introduced new mathematical approach and finally took into consideration the above-mentioned dependence between customer's volume and his service time were those by A. Alexandrov and B. Kaz and by B. Sengupta [20,21]. Obtained results let calculate number of customers distribution, values of loss probability or approximate needed size of memory buffers (in practical case when they are limited, assuming some admissible degree of loss probability) or (in the case when we assume that customer's total volume is unlimited) characteristics of total volume. The computations became more complicated because results were obtained mainly in the terms of Laplace and Laplace–Stieltjes transforms. Very important results were presented by O. Tikhonenko - see e.g. monograph [6]. Recent investigations have been concerning on systems with nonidentical servers and sectorized memory - see again [5,7,8] and also [22] in which very complicated mathematical approach is used (multidimensional generalizations of concepts such as integral transforms or Stieltjes convolutions).

Nowadays computer or telecommunication systems have become more and more complex and their analysis is often very complicated and exhaustive because we have to take into account many technical aspects of their working process. On the other hand, obtained performance characteristics of such systems let understand their working process better or even design them in more effective way. Models describing real-life systems are also very complicated from the mathematical point of view and some parts of their analysis would not be possible without help of automatic tools offered by computer algebra systems. It is worth highlighting that *Mathematica* environment has seemed to be the best tool in such computations in recent years. Author of this paper has been used some interesting computational techniques with the help of *Mathematica* environment in his research for many years and would like to present them to other scientists whose area of research is connected with queueing systems analysis what can be very useful.

3. MATHEMATICAL CONCEPTS USED IN CLASSICAL QUEUEING THEORY

A. Main assumptions and purposes of analysis

In classical queueing theory, we usually analyze systems for which we assume that:

- we know the distribution function of time intervals between neighboring moments of customers' arrival (denoted as $A(t)$);
- we know the distribution function of the customer's service time (denoted as $B(t)$);
- system usually contains n servers, where $n \geq 1$ (n can also be infinite);
- system usually has waiting room (queue) with m positions in which customers are patiently waiting when all servers are busy, $m \geq 0$ and the value of m can also be infinite.

In this paper, simple queueing models are denoted with the use of modified Kendall's notation $G/G/n/m$, where first two letters are notations for the type of functions $A(t)$ and $B(t)$ (for example, M means that function is exponentially distributed), whereas n and m are the numbers of servers and waiting positions (the length of a queue), respectively - in this article we assume that, in the case of queue length, we do not take into account these customers that are on service. For example, classical Erlang system (without a waiting room) is denoted here as $M/G/n/0$.

The main purposes of classical models analysis are:

1. Calculating characteristics of the number of customers distribution i.e. obtaining functions $P_k(t) = \mathbf{P}\{\eta(t) = k\}$, $k = \overline{0, n+m}$, where $\eta(t)$ is a random process describing number of customers present in the system in time instant t. In many practical applications, we calculate only steady-state characteristics of the system $p_k = \mathbf{P}\{\eta = k\}$, where η is the steady-state number of customers present in the system ($\eta(t) \Rightarrow \eta$ in the sense of a weak convergence). It is obvious that (in the case when stationary mode exists) $p_k = \lim_{t\to\infty} P_k(t)$. In addition, we also calculate moments of random variable η (e.g. mean number of customers present in the system in the steady state);
2. Calculating characteristics of customer's waiting time at least in stationary mode i.e. obtaining formula describing function $W(t)$, that is waiting time distribution in the steady state, and its moments (mean waiting time and variance) for systems with infinite number of waiting positions;
3. Calculating other important characteristics e.g. busy period distribution function and its moments (important especially for single-server systems).

Unfortunately, we rather seldom are able to obtain these results in exact forms. Distributions of the number of customers are sometimes presented in the terms of generating functions and the other characteristics often in the terms of Laplace–Stieltjes transforms. Moreover, even in cases when we only have to solve some algebraic system of equations describing the behavior of a simple markovian system i.e. obtain steady-state number of customers distribution (using theory of Markov chains with continuous time) we often face the problem of enormous number of states and manual way of getting results is hardly possible. Now we introduce these concepts and discuss their basic properties that let calculate some main performance characteristics of the classical queueing systems and their more complicated modifications.

B. Markov chains with continuous time

During simplest classical queueing models analysis (e.g. of the $M/M/n/m$-type) we often use the concept of a Markov chain with continuous time. No we will remind this idea.

Definition 1 *Assume that $T = [0, \infty)$ and X is a finite or countable set. Stochastic process $\{\xi(t),\ t \in T\}$ taking values from the set X is called Markov chain if, for every number $n = 1, 2, \ldots$, all $t_1, t_2, \ldots, t_n \in T$, where $t_1 < t_2 < \ldots < t_n$, and all $x_0, x_1, \ldots, x_n \in X$ it satisfies the following condition:*

$$\mathbf{P}\{\xi(t_n) = x_n | \xi(t_0) = x_0, \xi(t_1) = x_1, \ldots, \xi(t_{n-1}) = x_{n-1}\} = \\ = \mathbf{P}\{\xi(t_n) = x_n | \xi(t_{n-1}) = x_{n-1}\}.$$

Elements of the set X are usually called the states of the process $\xi(t)$.

Markov chains are very convenient tools used in analysis of some chosen queueing models thanks mainly to their property of "forgetting about the past" what lets us simply write out differential equations describing the behavior of the system and obtain even non-stationary number of customers distribution (obtained equations are linear so we can e.g. use Laplace transforms apparatus in this case). Finding of steady-state number of customers distribution is also very simple because then (based on of previously obtained differential equations) we only have to solve some linear system of algebraic equations that are obtained in the process of calculating the limits of proper functions (if $t \to \infty$). In most cases we can use some non-complicated recursive methods (even in the case of infinite number of states, then we usually obtain birth-death processes, a good example is the model of $M/M/n/\infty$ queueing system) [1]. As a simple example, consider system of the $M/M/n/m$-type, where n and m are finite and assume that a is a parameter of an entrance flow and μ is a parameter of exponentially distributed customer's service time (in all systems of this type process $\eta(t)$ denoting number of customers present in system in time instant t is a Markov chain with continuous time). In this case the only possible transitions are those from state **n** to **n+1** ($n = \overline{0, n+m-1}$) (describing an arrival of a new customer) and from state **n** to **n-1** ($n = \overline{1, n+m}$) (customer's service termination). The stochastic graph describing working process of this system is presented in Fig. 1. System

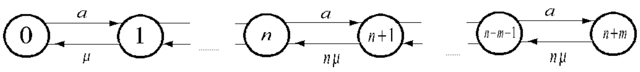


Fig. 1. Stochastic graph for $M/M/n/m$ queueing system

of steady-state equations describing its behavior is presented below:

$$\begin{cases} ap_{k-1} = k\mu p_k, 1 \le k \le n; \\ ap_{k-1} = n\mu p_k, n < k \le n+m. \end{cases}$$

The solution has the form [1]:

$$p_k = \begin{cases} \frac{(n\rho)^k p_0}{k!}, 1 \le k \le n; \\ \frac{n^n \rho^k p_0}{n!}, n < k \le n+m, \end{cases}$$

where $\rho = \frac{a}{n\mu}$. The value of p_0 we get from the normalization condition $\sum_{i=0}^{n+m} p_i = 1$ and it e.g. for $\rho \neq 1$ has a form:

$$p_0 = \left[\sum_{k=0}^{n} \frac{(n\rho)^k}{k!} + \frac{n^n \rho^{n+1}(1-\rho^m)}{n!(1-\rho)} \right]^{-1}.$$

Unfortunately, there is some class of more interesting queueing models e.g. queueing models with non-identical servers for which the above-mentioned systems of linear equations are much more complicated and we cannot use simple techniques in their solving but have to use some more advanced matrix methods and the help of computer algebra systems. This conception will be investigated in details in the next section.

C. Generating function

Definition 2 *Assume that we consider discrete random variable η taking countable number of values k ($k = 0, 1, \ldots$) with probabilities $p_k = \mathbf{P}\{\eta = k\}$ (of course, $\sum_{k=0}^{\infty} p_k = 1$ and $p_k > 0$). Generating function of the random variable η is an analytic function $P(z)$ defined as follows:*

$$P(z) = p_0 + \sum_{k=1}^{\infty} p_k z^k, \tag{1}$$

where z is a complex variable that satisfies condition $|z| \le 1$. Notice that in the case when η takes only finite number of values, the above definition is also correct as we have then only some non-zero values of p_k and the others are zeros.

It is obvious that $P(1) = \sum_{k=0}^{\infty} p_k = 1$. In addition, function P has the following important practical properties:

1. Mean value $\mathbf{E}\eta$ of the random variable η can be calculated as follows:

$$\mathbf{E}\eta = P'(1). \tag{2}$$

Indeed, we have obviously $P'(z) = \sum_{k=1}^{\infty} k p_k z^{k-1}$, so

$$P'(1) = \sum_{k=1}^{\infty} k p_k = \sum_{k=0}^{\infty} k p_k = \mathbf{E}\eta.$$

Analogously, we can calculate higher moments (using derivatives of function P), e.g.

$$\mathbf{E}\eta^2 = P''(1) + P'(1).$$

2. We can calculate p_i probabilities using formula

$$p_i = \frac{P^{(i)}(0)}{i!}. \tag{3}$$

In fact, $P^{(i)}(z) = \sum_{k=1, k\neq i}^{\infty} k(k-1)\ldots(k-i+1) p_k z^{k-i} + i! p_i$, whence $P^{(i)}(0) = i! p_i$.

D. Laplace–Stieltjes transform

Definition 3 *Assume that we consider random variable ξ taking non-negative values described by distribution function $A(t)$. Laplace–Stieltjes transform of the random variable ξ (or distribution function $A(t)$) is an analytic function $\alpha(q)$ defined as follows:*

$$\alpha(q) = \int_0^{\infty} e^{-qt}\, dA(t), \tag{4}$$

where q is a complex variable and $Re\ q \ge 0$.

Note that Laplace–Stieltjes transform exists for different types of non-negative random variables: discrete, absolute continuous and the others. In the case of absolute continuous random variables (if we assume existence of density function $a(t)$) formula (4) takes the following form (here we have simply Riemann's integral instead of Stieltjes' one):

$$\alpha(q) = \int_0^{\infty} e^{-qt} a(t)\, dt. \tag{4a}$$

In the case of discrete random variables taking values x_k with probabilities $p_k = \mathbf{P}\{\xi = x_k\}$ we have

$$\alpha(q) = \sum_k p_k e^{-qx_k}. \tag{4b}$$

It is clear that $\alpha(0) = \int_0^{\infty} dA(t) = 1$. Moreover, function α has the following properties:

1. Moments of random variable ξ can be calculated using the following formula:

$$\beta_i = \mathbf{E}\xi^i = (-1)^i \alpha^{(i)}(0). \tag{5}$$

It means that, especially, $\mathbf{E}\xi = -\alpha'(0)$ and $\mathbf{E}\xi^2 = \alpha''(0)$. Indeed, $\alpha^{(i)}(q) = (-1)^i \int_0^{\infty} t^i e^{-qt}\, dA(t)$, whence

$$\alpha^{(i)}(0) = (-1)^i \int_0^{\infty} t^i\, dA(t) = (-1)^i \mathbf{E}\xi^i.$$

2. If $\xi_1, \ldots, \xi_n$ are independent non-negative random variables and $\alpha_1(q), \ldots, \alpha_n(q)$ are their Laplace–Stieltjes transforms then transform $\alpha(q)$ of their sum $\xi = \xi_1 + \ldots + \xi_n$ has the form of product:

$$\alpha(q) = \prod_{i=1}^{n} \alpha_i(q). \tag{6}$$

To prove this property, notice that

$$\alpha(q) = \mathbf{E}e^{-q\xi} = \mathbf{E}e^{-q(\xi_1+\ldots+\xi_n)} = \mathbf{E}\left(\prod_{i=1}^{n} e^{-q\xi_i}\right).$$

Because random variables $\xi_1, \ldots, \xi_n$ are independent, we have obviously that $\mathbf{E}\left(\prod_{i=1}^{n} e^{-q\xi_i}\right) = \prod_{i=1}^{n} \mathbf{E}e^{-q\xi_i}$ which ends the proof.

3. We have the following well known relation between Laplace–Stieltjes transform $\alpha(q)$ and Laplace transform $\mathscr{L}(q)$ (we ommit simple proof of this relation):

$$\alpha(q) = \int_0^\infty e^{-qt} dA(t) = q \int_0^\infty e^{-qt} A(t)\, dt = q\mathscr{L}(q). \quad (7)$$

In queueing theory investigations we additionally use next two practical concepts. First of them is connected with the need of analyzing the sums of independent random variables distributions, and the second one with an idea of density function generalization.

E. Stieltjes convolution

In analytic investigations of queueing theory models, we often use distribution functions of the sums of independent non-negative random variables or vectors. It is clear (see e.g. [23]) that in the case when ξ_1 and ξ_2 are two independent non–negative random variables having distribution functions $F_1(x)$ and $F_2(x)$, respectively, then their sum's ($\xi = \xi_1 + \xi_2$) distribution function has the convolution form:

$$F(x) = \mathbf{P}\{\xi < x\} = \int_0^x F_1(x-u)\, dF_2(u). \quad (8)$$

If we consider the sum of n independent non-negative random variables having the same distribution function $F(x)$ then its distribution function $F_n(x)$ can be defined by the following recursive definition:

$$F_0(x) \equiv 1,\ F_n(x) = \int_0^x F_{n-1}(x-u)\, dF(u),\ n \geq 1. \quad (9)$$

Relation from the right side of the above formula is called n-fold Stiletjes convolution of distribution function $F(x)$. If analyzed random variable is absolute continuous (we assume existence of its density function $f(x)$) then we simply obtain:

$$F_n(x) = \int_0^x F_{n-1}(x-u) f(u)\, du. \quad (9a).$$

F. Generalized density of random variable - the use of Dirac-delta distribution

As we could see in previous subsections, the existence of density function (see formulae (4a) and (9a)) lets calculate some integral transforms and convolutions using Riemann's integral instead of more difficult Stieltjes' one. So if we investigate absolute continuous random variables (what sometimes happens in practice) calculations are usually less complicated. It can be proved (see e.g. [24]) that the concept of density function can be generalized onto random variables of other types if we use Dirac-delta distribution.

Indeed, let us consider e.g. discrete random variable ξ taking finite number of values $x_1, \ldots, x_k$ with probabilities $p_1, \ldots, p_k$, respectively (of course, $\sum_{i=1}^k p_i = 1$). Then its distribution function can be defined as follows:

$$F(x) = \sum_{i=1}^k p_i\, H(x - x_i), \quad (10)$$

where $H(x)$ is left–side continuous Heaviside's unitstep function. Considering the generalized definition of derivative (in the sense of distributions), where Dirac-delta distribution $\delta(x)$ is a derivative of Heaviside's unitstep function: $\delta(x) = \frac{\partial H(x)}{\partial x}$, we can introduce generalized density function of random variable ξ as follows:

$$f(x) = \sum_{i=1}^k p_i\, \delta(x - x_i). \quad (11)$$

Such approach does not change the ways of basic characteristics of random variables calculations (their moments, integral transforms and so on because we treat here integrals also in generalized sense) and helps to lead calculations in queueing theory more effectively.

All above-mentioned mathematical concepts generate some computational problems. In the next section, we discuss them and show some ways of their solving with the use of computer algebra systems.

4. EXAMPLES OF COMPUTATIONAL PROBLEMS AND THEIR SOLUTIONS WITH THE HELP OF COMPUTER ALGEBRA SYSTEMS

A. Techniques of finding steady-state final probabilities of Markov chains with continuous time and enormous number of states

Sometimes we face the problem of analyzing queueing systems in which we have huge number of possible states. Even if investigated models can be described by Markov chains it is sometimes very difficult to find steady-state final probabilities due to very many equations appearing. In analogous situations manual way of finding solution of linear system of equations that describes behavior of analyzed queueing system seems to be rather impossible. Then we have to use some more complicated matrix methods and the help of computer algebra systems e.g *Mathematica* environment.

A very good example of such situation are systems with non-identical servers. These systems are modeled with the use of Markov chains whose states are sequences of busy servers $(i_1, \ldots, i_k)$, $k = \overline{1,n}$ (numbers of busy servers are ordered increasingly) with additional state 0 (empty system). In this case number of customers in the steady state cannot be understood as the state of Markov chain so the number of states is very huge. For example, consider queueing system of the $M/M/n/0$-type (n is finite) in which all servers have different parameters of exponentially distributed customer's service time μ_i, $i = \overline{1,n}$. It can be easily shown that numbers of states increase exponentially with the number of servers n (for n servers we have 2^n states). The very interesting fact about this model is that although it is very complicated, its solution (steady-state number of customers distribution) is relatively simple in the case when **customers choose free server randomly** - see e.g [25]. The explanation is very simple: in this model, all transitions between states that differ in number of busy servers (and that difference equals 1) are possible e.g. for system $M/M/3/0$ we have the following transitions: $0 \to (1)$, $0 \to (2)$, $0 \to (3)$, $(1) \to (1,2)$, $(1) \to (1,3)$, $(2) \to (1,2)$, $(2) \to (2,3)$, $(3) \to (1,3)$, $(3) \to (2,3)$, $(1,2) \to (1,2,3)$, $(1,3) \to (1,2,3)$, $(2,3) \to (1,2,3)$, $(1,2,3) \to (2,3)$, $(1,2,3) \to (1,3)$,

$(1,2,3) \to (1,2)$, $(1,2) \to (1)$, $(1,2) \to (2)$, $(1,3) \to (1)$, $(1,3) \to (3)$, $(2,3) \to (2)$, $(2,3) \to (3)$, $(1) \to 0$, $(2) \to 0$ and $(3) \to 0$. The intensities of proper transitions equal a, $\frac{a}{2}$, $\frac{a}{3}$, μ_1, μ_2, μ_3, $\mu_1+\mu_2$, $\mu_1+\mu_3$, $\mu_2+\mu_3$ and $\mu_1+\mu_2+\mu_3$. The stochastic graph is complicated in this case and we will not present it.

The much more interesting and complicated systems are those with non-identical servers in which **customers choose fastest free server**. In such models we do not have all above-mentioned transitions because customers never come to the slower server. E.g. if we assume that $\mu_1 > \mu_2 > \mu_3$, transitions $0 \to (2)$, $0 \to (3)$, $1 \to (1,3)$, $2 \to (2,3)$, $3 \to (2,3)$ are impossible. The system of steady-state balance equations describing the system behavior is much more complicated although many coefficients are zeros. Then we can use some matrix methods and command **LinearSolve** from *Mathematica* environment especially that matrix appearing in this equation is sparse and Mathematica uses smart techniques of evaluating such matrices. Let us return to our example of queueing model of the $M/M/3/0$-type with non-identical servers and fastest server choice. In the steady state we can write out the following system of equations describing the behavior of the system:

$$ap_0 = \mu_{!1} p_1 + \mu_2 p_2 + \mu_3 p_3;$$

$$(a+\mu_1)p_1 = ap_0 + \mu_2 p_{12} + \mu_3 p_{13};$$

$$(a+\mu_2)p_2 = \mu_1 p_{12} + \mu_3 p_{23};$$

$$(a+\mu_3)p_3 = \mu_1 p_{13} + \mu_2 p_{23};$$

$$(a+\mu_1+\mu_2)p_{12} = ap_1 + ap_2 + \mu_3 p_{123};$$

$$(a+\mu_1+\mu_3)p_{13} = ap_3 + \mu_2 p_{123};$$

$$(a+\mu_2+\mu_3)p_{23} = \mu_1 p_{123}.$$

with normalization condition:

$$p_0 + p_1 + p_2 + p_3 + p_{12} + p_{13} + p_{23} + p_{123} = 1.$$

This system can be rewritten in more convenient matrix form:

$$\begin{bmatrix} \mu_1 & \mu_2 & \mu_3 & 0 & 0 & 0 & 0 \\ a+\mu_1 & 0 & 0 & -\mu_2 & -\mu_3 & 0 & 0 \\ 0 & a+\mu_2 & 0 & -\mu_1 & 0 & -\mu_3 & 0 \\ 0 & 0 & a+\mu_3 & 0 & -\mu_1 & -\mu_2 & 0 \\ a & a & 0 & -(a+\mu_1+\mu_2) & 0 & 0 & \mu_3 \\ 0 & 0 & a & 0 & -(a+\mu_1+\mu_3) & 0 & \mu_2 \\ 0 & 0 & 0 & 0 & 0 & -(a+\mu_2+\mu_3) & \mu_1 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_{12} \\ p_{13} \\ p_{23} \\ p_{123} \end{bmatrix} = \begin{bmatrix} ap_0 \\ ap_0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From the practical point of view this form of equation $AX = B$ has in this case only one solution $X = A^{-1}B$ (satisfied conditions for existence of the steady-state numbers of customers distribution) and we can solve it with the use of *Mathematica* environment (**LinearSolve** command) what is illustrated in Fig. 2. Notice that the exact form of general solutions is rather complicated but we can obtain values for fixed parameters a, μ_1, μ_2, μ_3 e.g. if $a=1$, $\mu_1=3$, $\mu_2=2$, $\mu_3=1$ we have $p_0=\frac{58}{85}$, $p_1=\frac{59}{306}$, $p_2=\frac{11}{255}$, $p_3=\frac{3}{170}$, $p_{12}=\frac{25}{612}$, $p_{13}=\frac{11}{1530}$, $p_{23}=\frac{7}{1020}$, $p_{123}=\frac{7}{685}$. It can be easily investigated that complexity of general solutions increase with the number of servers. For $n=2$ the solution is much less complicated, namely:

$$p_0 = \frac{\mu_1\mu_2(2a+\mu_1+\mu_2)}{(a+\mu_1)(a^2+2a\mu_2+\mu_2(\mu_1+\mu_2))};$$

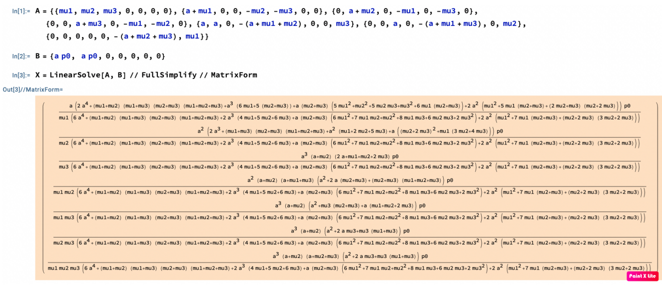


Fig. 2. Calculating number of customers distribution in $M/M/3/0$ queueing system with non-identical servers and fastest server choice in *Mathematica*

$$p_1 = \frac{a\mu_2(a+\mu_1+\mu_2)}{(a+\mu_1)(a^2+2a\mu_2+\mu_2(\mu_1+\mu_2))};$$

$$p_2 = \frac{a^2\mu_1}{(a+\mu_1)(a^2+2a\mu_2+\mu_2(\mu_1+\mu_2))};$$

$$p_{12} = \frac{a^2(a+\mu_2)}{(a+\mu_1)(a^2+2a\mu_2+\mu_2(\mu_1+\mu_2))}.$$

B. *Applications of derivatives and L'Hospital's rule for one-variable function. Exemplary results for $M/G/1/\infty$ queueing system*

In many situations number of customers distribution (even in steady-state) cannot be presented in exact form. We often obtain it in the terms of generating functions. But, based on properties of generating functions (see e.g. formula (2)), we could obtain some important characteristics of the steady-state number of customers distribution but we face some computational problems. For example, if we want to calculate mean value of the number of customers present in the system (in the steady state), we should calculate derivative $P'(1) = \lim_{z\to 1} P'(z)$ (function $P(z)$ is analytic if $|z| \le 1$) but note that usually in this case $P'(z) = \frac{N(z)}{D(z)}$, where both numerator and denominator of $P'(z)$ are zeros if $z=1$. So we have to use L'Hospital's rule (we have here undefined symbol $\frac{0}{0}$) until denominator is nonzero. Sometimes calculations become complicated (multiple use of l'Hospital's rule) and we can use *Mathematica* environment and the following algorithm:

1. Define function $P(z)$;
2. Calculate its derivative $P'(z)$;
3. Define separately numerator $N(z)$ and denominator $D(z)$ of $P'(z)$;
4. Calculate derivatives of $N(z)$ and $D(z)$ until $D(z)$ is nonzero at the point $z=1$ (because $P(z)$ is analytic then $N(z)$ will be then also nonzero at $z=1$);
5. Present final result making proper substitutions and simplify calculations.

Example 1 *Consider $M/G/1/\infty$ queueing system. The main result for this one is the well-known Pollaczek–Khinchine formula that defines generating function $P(z)$ of the steady-state number of customers present in the system [13, 14]:*

$$P(z) = p_0 + \sum_{k=1}^{\infty} p_k z^k = \frac{(1-\rho)(1-z)\beta(a-az)}{\beta(a-az)-z}, \qquad (12)$$

where $\beta(q)$ is Laplace–Stieltjes transform of the customer's service time (its distribution function is $B(t)$), a is an arrival rate (parameter of exponentially distributed function $A(t)$ defining time intervals between consecutive moments of customers' arrival to the system), β_1 is the mean value of the customer's service time and $\rho = a\beta_1 < 1$. In this case $P'(z) = \frac{N(z)}{D(z)}$, where

$$N(z) = (1-\rho)\left[(1-\beta(a-az))\beta(a-az)- \right.$$
$$\left. -(a-az)(\beta(a-az)-z)\beta'(a-az)\right]$$

and

$$D(z) = [\beta(a-az)-z]^2.$$

Both $N(z)$ and $D(z)$ are zeros at the point $z = 1$ because $\beta(0) = 1$. So we have to use l' Hospital' s rule (twice) and above-mentioned computational technique.

In Fig. 3. we present the realization of this method (we hide here partial results of calculations, presenting only final one).

```
In[1]:=  P = ((1 - ro) (1 - z) BETA[a - a z]) / (BETA[a - a z] - z)
In[2]:=  DP = D[P, z] // FullSimplify
In[3]:=  DENOMINATOR = Denominator[DP]
In[4]:=  NUMERATOR = Numerator[DP]
In[5]:=  NUMERATOR2 = D[NUMERATOR, z, z]
In[6]:=  DENOMINATOR2 = D[DENOMINATOR, z, z]
In[7]:=  N1 = NUMERATOR2 /. z → 1
In[8]:=  N2 = N1 /. {BETA[0] → 1, BETA′[0] → -BETA1, BETA″[0] → BETA2} // FullSimplify
In[9]:=  N3 = N2 /. BETA1 → ro / a // FullSimplify
In[10]:= D1 = DENOMINATOR2 /. z → 1
In[11]:= D2 = D1 /. {BETA[0] → 1, BETA′[0] → -BETA1, BETA″[0] → BETA2} // FullSimplify
In[12]:= D3 = D2 /. BETA1 → ro / a // FullSimplify
In[13]:= N3 / D3 // FullSimplify
Out[13]= a^2 BETA2/(2 - 2 ro) + ro
```

Fig. 3. Mean value of the number of customers in $M/G/1/\infty$ queueing system calculations in *Mathematica*

Obtained result

$$E\eta = P'(1) = \rho + \frac{a^2\beta_2}{2(1-\rho)},$$

where β_2 is the second moment of the customer's service time is the same as in classical queueing theory books - see e.g. [6]. Introduced method can be used for more complicated models in which formula for generating function is more complex (e.g. priority queues, queues with vacations). Moreover, the same technique can be applied in the case when we calculate the other queueing systems characteristics presented in the terms of Laplace–Stieltjes transforms. We will show that fact in next example.

Example 2 *It is rather known (see [1]) that Laplace–Stieltjes transform $w(q)$ of the steady-state waiting time of a customer (in the case of FIFO discipline) for $M/G/1/\infty$ queueing system equals*

$$w(q) = \frac{(1-\rho)q}{q-a+a\beta(q)}. \quad (13)$$

To calculate first moments of analyzed steady-state customer's waiting time, we have to calculate derivatives of the function $w(q)$ but this time at the point $q = 0$ (formula (5)), namely $w_1 = -w'(0)$, $w_2 = w''(0)$, facing analogous difficulties caused by undefined symbols (as $\beta(0) = 1$). Here we present (in Fig. 4.) the solution in Mathematica environment (notice that in the case of w_2 calculations, we have to use l'Hospital's rule three times). Obtained results after small modifications: $w_1 = \frac{a\beta_2}{2(1-\rho)}$, $w_2 = \frac{(a\beta_2)^2}{2(1-\rho)^2} + \frac{a\beta_3}{3(1-\rho)}$, where β_3 is the third moment of the customer's service time, are the same as in [1].

```
In[1]:=  w = ((1 - ro) q) / (q - a + a BET[q])
In[2]:=  Dw = -D[w, q] // FullSimplify
In[3]:=  Dw2 = D[w, q, q] // FullSimplify
In[4]:=  DENOMINATORDw = Denominator[Dw]
In[5]:=  NUMERATORDw = Numerator[Dw]
In[6]:=  DENOMINATORDw2 = D[DENOMINATORDw, q, q]
In[7]:=  NUMERATORDw2 = D[NUMERATORDw, q, q]
In[8]:=  N1 = NUMERATORDw2 /. q → 0
In[9]:=  N2 = N1 /. BET″[0] → BETA2
In[10]:= D1 = DENOMINATORDw2 /. q → 0
In[11]:= D2 = D1 /. {BET[0] → 1, BET′[0] → -BETA1}
In[12]:= D3 = D2 /. BETA1 → ro / a
In[13]:= N2 / D3 // FullSimplify
Out[13]= a BETA2/(2 - 2 ro)
In[14]:= DENOMINATORDw2 = Denominator[Dw2]
In[15]:= NUMERATORDw2 = Numerator[Dw2]
In[16]:= DENOMINATORDw23 = D[DENOMINATORDw2, q, q, q]
In[17]:= D1 = DENOMINATORDw23 /. q → 0
In[18]:= D2 = D1 /. {BET[0] → 1, BET′[0] → -BETA1}
In[19]:= D3 = D2 /. BETA1 → ro / a
In[20]:= NUMERATORDw23 = D[NUMERATORDw2, q, q, q]
In[21]:= N1 = NUMERATORDw23 /. q → 0
In[22]:= N2 = N1 /. {BET[0] → 1, BET′[0] → -BETA1, BET″[0] → BETA2, BET^(3)[0] → -BETA3}
In[23]:= N3 = N2 /. BETA1 → ro / a
In[24]:= N3 / D3 // FullSimplify // Apart
Out[24]= a^2 BETA2^2/(2 (-1 + ro)^2) - a BETA3/(3 (-1 + ro))
```

Fig. 4. First two moments of the customer's waiting time in $M/G/1/\infty$ queueing system calculations in *Mathematica*

Example 3 *Now we will present one more interesting example connected with calculating first two moments of the steady-state busy period for the same system. As it was proved e.g. in [6], Laplace–Stieltjes transform $\pi(q)$ of this random variable satisfies the following functional equation:*

$$\pi(q) = \beta(q+a-a\pi(q)). \quad (14)$$

It is impossible to obtain formula for $\pi(q)$ in exact form, but we can calculate the moments of analyzed busy period: $\pi_1 = -\pi'(0)$, $\pi_2 = \pi''(0)$ computing derivatives of the left and the right side of this equation. See the next Mathematica notebook (Fig. 5.). Here we obtain $\pi_1 = \frac{\beta_1}{1-\rho}$, $\pi_2 = \frac{\beta_2}{(1-\rho)^3}$ i.e. the same result as in [6].

```
In[1]:= RIGHT = BETA[q + a - a PI[q]]
In[2]:= DRIGHT = D[RIGHT, q]
In[3]:= R1 = DRIGHT /. q → 0
In[4]:= R2 = R1 /. PI[0] → 1
In[5]:= R3 = R2 /. {BETA′[0] → -BETA1, PI′[0] → -PI1}
In[6]:= L = -PI1
In[7]:= Result1 = Solve[R3 == L, PI1] // FullSimplify
In[8]:= Result1 /. a → ro / BETA1
Out[8]= {{PI1 → BETA1/(1 - ro)}}
In[9]:= DRIGHT2 = D[RIGHT, q, q]
In[10]:= R1 = DRIGHT2 /. q → 0
In[11]:= R2 = R1 /. {PI[0] → 1}
In[12]:= R3 = R2 /. {BETA′[0] → -BETA1, BETA″[0] → BETA2, PI′[0] → -PI1, PI″[0] → PI2}
In[13]:= L = PI2
In[14]:= Result2 = Solve[R3 == L, PI2] // FullSimplify
In[15]:= Result2 = Result2 /. PI1 → BETA1/(1 - ro) // FullSimplify
In[16]:= Result2 = Result2 /. a BETA1 → ro // FullSimplify
Out[16]= {{PI2 → -BETA2/(-1 + ro)^3}}
```

Fig. 5. First two moments of the busy period in $M/G/1/\infty$ queueing system calculations in *Mathematica*

C. Calculating convolutions

Computation of convolutions based on their definition (see formulae (8)-(9a)) is a very complicated and inconvenient process. It is often impossible to obtain exact form of convolution for any n, for many types of distribution functions $F(x)$ (even if $F(x)$ defines absolute continuous random variable). In queueing theory, especially when we consider models with random volume customers, we often need to obtain the general formula for distribution function of the sum of n independent random variables having the same distribution function what unfortunately demands calculating very complicated n-fold Stieltjes convolutions (they are often present in basic formulae for performance characteristics of above-mentioned queueing systems) based on recursive integral calculations.

In analogous cases *Mathematica* environment is also a wonderful tool that can help to calculate convolutions. On the base of mentioned in Sec. 3. properties of Laplace and Laplace–Stieltjes transforms (see formulae (6)-(7)), we may use the following method:

1. Calculate Laplace–Stieltjes transform (LST) of a single function $F(x)$:
$$\alpha(q) = \int_0^\infty e^{-qx}\, dF(x) = \int_0^\infty e^{-qx} f(x)\, dx,$$
where $f(x)$ is a density function of the random variable defined by distribution function $F(x)$ (in the case of random variables that are not absolute continuous, we consider the generalized density function discussed in section 3);
2. Calculate LST of a convolution $F_n(x)$ using formula (6):
$$\alpha_n(q) = [\alpha(q)]^n;$$
3. Obtain formula for Laplace transform of analyzed convolution using (7):
$$\mathscr{L}(q) = \frac{[\alpha(q)]^n}{q};$$
4. Finally, use **InverseLaplaceTransform** command from *Mathematica* environment to obtain exact formula of $F_n(x)$ convolution [11]:
$$F_n(x) = \mathscr{L}^{-1}(q).$$

It is clear that this way of calculating convolutions give a chance to obtain exact form of convolution only for fixed n (not for any n) but obtained results can help to predict general formulae that can be proved afterwards by induction method. Now we illustrate discussed method in the next example.

Example 4 *Consider random variable ξ that is uniformly distributed on interval $[a,b]$, $a \geq 0, b > a$. It is obvious that its density function has the form $f(x) = \frac{1}{b-a}$ if $x \in [a,b]$ and $f(x) = 0$, otherwise. In Fig. 6. we present the way of calculating convolutions $F_2(x)$ and $F_3(x)$ of analyzed random variable with the use of the above-mentioned computational algorithm. Obtained results let predict the general formula of convolution in the following form (what can be easily proved):*
$$F_n(x) = \left(\frac{-1}{b-a}\right)^n \sum_{l=0}^{n} \frac{(-1)^l\,[(b-a)l - bn + x]^n\, H[(b-a)l - bn + x]}{l!(n-l)!}, \tag{15}$$
where $H(x)$ is the left-side continuous Heaviside's unistep function.

```
In[1]:= f = 1 / (b - a)
Out[1]= 1/(-a + b)
In[2]:= LST = Integrate[E ^ (-q x) * f, {x, a, b}]
Out[2]= (e^(-a q) - e^(-b q))/((-a + b) q)
In[3]:= LT2 = LST ^ 2 / q
Out[3]= (e^(-a q) - e^(-b q))^2/((-a + b)^2 q^3)
In[4]:= LT3 = LST ^ 3 / q
Out[4]= (e^(-a q) - e^(-b q))^3/((-a + b)^3 q^4)
In[5]:= F2 = InverseLaplaceTransform[LT2, q, x] // FullSimplify
Out[5]= 1/(2 (a - b)^2) ((-2 a + x)^2 HeavisideTheta[-2 a + x] +
          (-2 b + x)^2 HeavisideTheta[-2 b + x] - 2 (a + b - x)^2 HeavisideTheta[-a - b + x])
In[6]:= F3 = InverseLaplaceTransform[LT3, q, x] // FullSimplify
Out[6]= 1/(6 (-a + b)^3) ((-3 a + x)^3 HeavisideTheta[-3 a + x] + (3 b - x)^3 HeavisideTheta[-3 b + x] -
          3 (a + 2 b - x)^3 HeavisideTheta[-a - 2 b + x] + 3 (2 a + b - x)^3 HeavisideTheta[-2 a - b + x])
```

Fig. 6. Convolution of uniform distributions calculations in *Mathematica*

We can easily find the model of a queueing system for which we have to calculate convolutions to obtain his main performance characteristics. As an example, we can mention general results for the $M/M/n/(m,V)$ queueing system [6] which is the generalization of the classical $M/M/n/m$ queue in which arriving customers are additionally characterized by some non-negative random volume ζ having distribution function $L(x)$, customer's service time is independent of his volume and exponentially distributed with parameter μ and customers' total volume is limited by value V. Fo such system, two main characteristics are those connected with steady-state number of customers and loss probability. We present them below.

$$p_k = \begin{cases} \frac{(n\rho)^k p_0}{k!} L_k(V), \ k = \overline{1,n}; \\ \frac{n^n \rho^k p_0}{n!} L_k(V), \ k = \overline{n+1, n+m}. \end{cases} \tag{16a}$$

$$p_{LOSS} = 1 - (n\rho)^{-1} \sum_{k=1}^{n-1} k p_k - \rho^{-1} \left(1 - \sum_{k=0}^{n-1} p_k\right), \tag{16b}$$

where p_0 can be obtained from normalization condition $\sum_{i=0}^{n+m} p_i = 1$. Formulae (16a) and (16b) seem to be very simple but notice that they contain Stieltjes convolutions $L_k(V)$.

5. THE OTHER PERFORMANCE CHARACTERISTICS OF THE QUEUEING SYSTEMS WITH RANDOM VOLUME CUSTOMERS AND NEXT COMPUTATIONAL PROBLEMS

As it was mentioned in Sec. 2., very interesting results in queueing theory were obtained during the analysis of models of queueing systems with random volume customers. For such models, we often deal with calculating of customers' total volume characteristics.

A. Calculating characteristics of total volume in $M/G/1/\infty$ queueing system

One of the most known results is the generalization of Pollaczek–Khinchine formula defining LST $\delta(s)$ of steady-state total volume of customers present in the system $M/G/1/\infty$:

$$\delta(s) = (1-\rho)\left[1 + \frac{\varphi(s) - \alpha(s, a - a\varphi(s))}{\beta(a - a\varphi(s)) - \varphi(s)}\right]. \tag{17}$$

From the practical point of view it is very important to calculate at least first two moments of the total volume $\delta_1 = -\delta'(0)$ and $\delta_2 = \delta''(0)$ (these values can be used to approximate loss characteristics in analogous systems but with limited customers' total volume - see [6]) but as you can see, formula (17) is very complicated because it contains complex rational functions of two variables as in analyzed models we assume existence of joint distribution function $F(x,t)$ of random vector containing two dependent components - the customer's volume ζ and his service time ξ together with double LST of this function: $\alpha(s,q) = \int_0^\infty \int_0^\infty e^{-sx-qt}\, dF(x,t)$ (whereas $\varphi(s)$ and $\beta(q)$ are single LSTs of the customer' s volume and his service time, respectively). Moreover, we also have here undefined symbols so we have to use L'Hospital's rule again, analogously as it was done in the previous section, but this time calculations are a bit difficult. During the calculations we also have to do much more substitutions because in this case (apart from moments φ_i and β_i of customer's volume and his service time) we additionally obtain mixed moments α_{ij} of the analyzed vector (ζ, ξ) as $\frac{\partial^{i+j}\alpha(s,q)}{\partial s^i \partial q^j}|_{s=0,q=0} = (-1)^{i+j}\alpha_{ij}$. In Fig. 7. we present the use of *Mathematica* environment in obtaining results connected with computations of characteristics δ_1 and δ_2 (hiding long partial results). Obtained results are identical as those presented in [6].

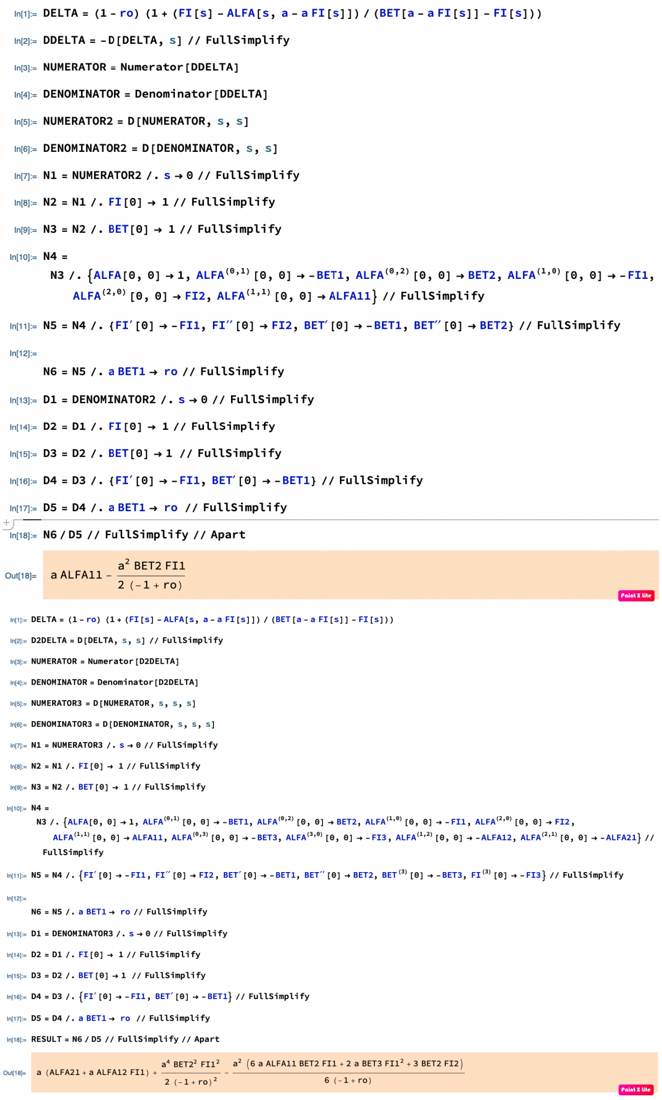


Fig. 7. Calculations of δ_1 and δ_2 for $M/G/1/\infty$ queueing system with random volume customers in *Mathematica*

B. Generalized L'Hospital's rule and its application in calculations for systems with sectorized memory

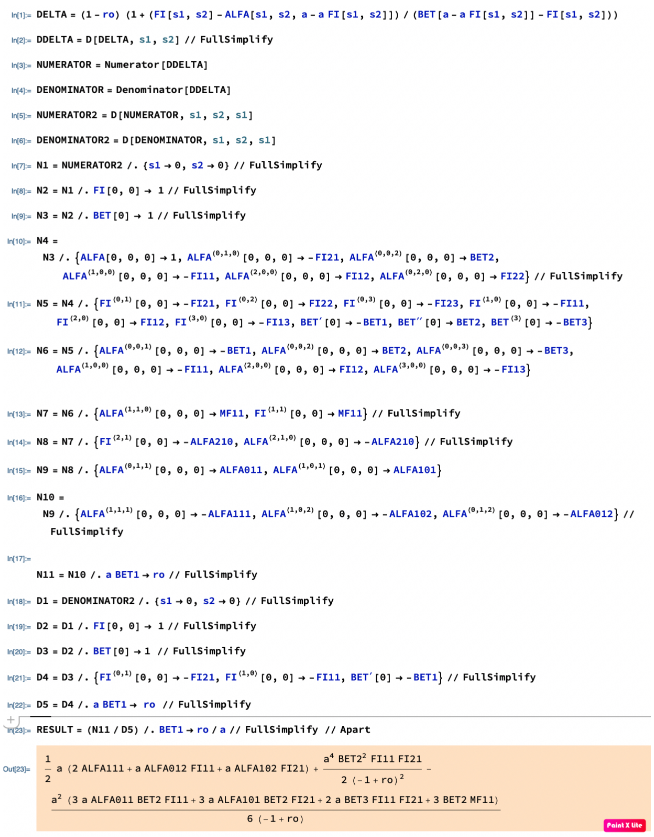
One of the newest approach in queueing systems analysis is concerned on those models that have sectorized memory. It means that we assume that arriving customers are characterized by some random volume vectors whose indications store data of different types. After customer is accepted on service (if all sectors of memory buffer have enough free memory to write new data to them) then multidimensional total customers' volume increases e.g. if at time moment t of new customer's arrival total volume contains two sectors and it equals $\boldsymbol{\sigma}(t) = (\sigma_1(t), \sigma_2(t))$ and customer is characterized by random volume vector (x,y) then we obviously have $\boldsymbol{\sigma}(t^+) = (\sigma_1(t) + x, \sigma_2(t) + y)$. Of course, after service termination (say, at time τ) total volume is released in the same way which means that $\boldsymbol{\sigma}(\tau^+) = (\sigma_1(\tau) - x, \sigma_2(\tau) - y)$ if served customer was characterized by random volume vector (x,y). As examples of recently published articles investigating such models we can mention again [7, 8, 22]. The most important performance characteristics of these systems are again those connected with total volume (usually we calculate mixed moments of multidimensional total volume) and loss probability (in the case when sectors of total volume are limited). Unfortunately, characteristics become multidimensional and from the computational point of view we face new problems. For example, Laplace–Stieltjes transform $\delta(s_1,\ldots,s_n)$ of total volume is at least two-dimensional and cancelling undefined symbols during total volume mixed moments calculations demands using of generalized L'Hospital's rule for functions containing many variables. The mathematical concept of such generalization is presented e.g. in [26, 27] but in our case situation is not so complex as it seems because LST of total volume $\delta(s_1,\ldots,s_n)$ is an analytic function if $Re\ s_1 \geq 0,\ldots, Re\ s_n \geq 0$ so $\lim_{s_1\to 0,\ldots,s_n\to 0} \frac{\partial^{i_1+\ldots+i_n}\delta(s_1,\ldots,s_n)}{\partial s_1^{i_1}\ldots\partial s_n^{i_n}}$ exists for all $i_1,\ldots,i_n$ and we can use this generalization in only one following way (with the help of *Mathematica* environment):

1. Define function $\delta(s_1,\ldots,s_n)$;
2. Calculate the following relation (containing partial derivatives) $(-1)^{i_1+\ldots+i_n}\frac{\partial^{i_1+\ldots+i_n}\delta(s_1,\ldots,s_n)}{\partial s_1^{i_1}\ldots\partial s_n^{i_n}}$;
3. Define separately numerator $N(s_1,\ldots,s_n)$ and denominator $D(s_1,\ldots,s_n)$ of the previous result;
4. Calculate partial derivatives of $N(s_1,\ldots,s_n)$ and $D(s_1,\ldots,s_n)$ until $D(s_1,\ldots,s_n)$ is nonzero at the point $(s_1,\ldots,s_n) = (0,\ldots,0)$. As it was proved in [26, 27], we should calculate derivatives in some ordered cycle e.g. by variables $s_1, s_2,\ldots,s_n,s_1,s_2,\ldots$. It means that calculating partial derivatives using the same variable e.g. by $s_1,s_1,\ldots$ is not effective;
5. Present final result making needed substitutions and simplify calculations.

We will present this technique analyzing system $M/G/1/\infty$ with sectorized memory containing two sectors. It is rather clear that formula defining two-dimensional LST of steady-state total customers' volume is the generalization of formula (17) (taking into consideration that arriving customers are characterized by 2-dimensional random volume vectors) and has the form

$$\delta(s_1,s_2) = (1-\rho)\left[1+\frac{\varphi(s_1,s_2)-\alpha(s_1,s_2,a-a\varphi(s_1,s_2))}{\beta(a-a\varphi(s_1,s_2))-\varphi(s_1,s_2)}\right]. \tag{17a}$$

Now we will calculate mixed $(1+1)$-th moment of the steady-state two-dimensional total volume $\delta_{11} = (-1)^{1+1}\frac{\partial^2\delta(s_1,s_2)}{\partial s_1\partial s_2}\big|_{s_1=0,s_2=0}$. The computations in *Mathematica* environment are presented in the next Fig. 8. Obtained result (after some small simplifications) was presented in [7]. Note that calculations in this case are very long, complex and exhausted (we do not show here partial results of our computations, believe that they are extremely long, obtaining the final result demands making of many substitutions as in the formula we have mixed moments α_{ijk} of $(i+j+k)$-th order of the joint distribution function of two-dimensional customer's volume vector and his service time $F(x_1,x_2,t)$. In this case $\alpha_{ijk} = (-1)^{i+j+k}\frac{\partial^{i+j+k}\alpha(s_1,s_2,q)}{\partial s_1^i\partial s_2^j\partial q^k}\big|_{s_1=0,s_2=0,q=0}$, where $\alpha(s_1,s_2,q)$ is three-dimensional LST of a vector (ζ_1,ζ_2,ξ) whose indications are two indications of customer's volume vector and his service time) but *Mathematica* environment help to obtain results effectively in about five minutes while making computations without computer algebra systems is almost impossible or takes a long time and needs many pages of difficult calculations in which it is easy to make mistakes, simply lose or give up. So previously planned computations can give a chance to obtain new interesting scientific results that would not be got without the help of computer algebra systems or simply would take much more time. Thus, presented technique seems to be unique and very practical.



```
In[1]:= DELTA = (1 - ro) (1 + (FI[s1, s2] - ALFA[s1, s2, a - a FI[s1, s2]]) / (BET[a - a FI[s1, s2]] - FI[s1, s2]))
In[2]:= DDELTA = D[DELTA, s1, s2] // FullSimplify
In[3]:= NUMERATOR = Numerator[DDELTA]
In[4]:= DENOMINATOR = Denominator[DDELTA]
In[5]:= NUMERATOR2 = D[NUMERATOR, s1, s2, s1]
In[6]:= DENOMINATOR2 = D[DENOMINATOR, s1, s2, s1]
In[7]:= N1 = NUMERATOR2 /. {s1 → 0, s2 → 0} // FullSimplify
In[8]:= N2 = N1 /. FI[0, 0] → 1 // FullSimplify
In[9]:= N3 = N2 /. BET[0] → 1 // FullSimplify
In[10]:= N4 =
  N3 /. {ALFA[0, 0, 0] → 1, ALFA^(0,1,0)[0, 0, 0] → -FI21, ALFA^(0,0,2)[0, 0, 0] → BET2,
    ALFA^(1,0,0)[0, 0, 0] → -FI11, ALFA^(2,0,0)[0, 0, 0] → FI12, ALFA^(0,2,0)[0, 0, 0] → FI22} // FullSimplify
In[11]:= N5 = N4 /. {FI^(0,1)[0, 0] → -FI21, FI^(0,2)[0, 0] → FI22, FI^(0,3)[0, 0] → -FI23, FI^(1,0)[0, 0] → -FI11,
    FI^(2,0)[0, 0] → FI12, FI^(3,0)[0, 0] → -FI13, BET′[0] → -BET1, BET″[0] → BET2, BET^(3)[0] → -BET3}
In[12]:= N6 = N5 /. {ALFA^(0,0,1)[0, 0, 0] → -BET1, ALFA^(0,0,2)[0, 0, 0] → BET2, ALFA^(0,0,3)[0, 0, 0] → -BET3,
    ALFA^(1,0,0)[0, 0, 0] → -FI11, ALFA^(2,0,0)[0, 0, 0] → FI12, ALFA^(3,0,0)[0, 0, 0] → -FI13}
In[13]:= N7 = N6 /. {ALFA^(1,1,0)[0, 0, 0] → MF11, FI^(1,1)[0, 0] → MF11} // FullSimplify
In[14]:= N8 = N7 /. {FI^(2,1)[0, 0] → -ALFA210, ALFA^(2,1,0)[0, 0, 0] → -ALFA210} // FullSimplify
In[15]:= N9 = N8 /. {ALFA^(0,1,1)[0, 0, 0] → ALFA011, ALFA^(1,0,1)[0, 0, 0] → ALFA101}
In[16]:= N10 =
  N9 /. {ALFA^(1,1,1)[0, 0, 0] → -ALFA111, ALFA^(1,0,2)[0, 0, 0] → -ALFA102, ALFA^(0,1,2)[0, 0, 0] → -ALFA012} //
   FullSimplify
In[17]:=
 N11 = N10 /. a BET1 → ro // FullSimplify
In[18]:= D1 = DENOMINATOR2 /. {s1 → 0, s2 → 0} // FullSimplify
In[19]:= D2 = D1 /. FI[0, 0] → 1 // FullSimplify
In[20]:= D3 = D2 /. BET[0] → 1 // FullSimplify
In[21]:= D4 = D3 /. {FI^(0,1)[0, 0] → -FI21, FI^(1,0)[0, 0] → -FI11, BET′[0] → -BET1} // FullSimplify
In[22]:= D5 = D4 /. a BET1 → ro // FullSimplify
In[23]:= RESULT = (N11 / D5) /. BET1 → ro / a // FullSimplify // Apart
Out[23]= 1/2 a (2 ALFA111 + a ALFA012 FI11 + a ALFA102 FI21) + a^4 BET2^2 FI11 FI21 / (2 (-1 + ro)^2) -
  a^2 (3 a ALFA011 BET2 FI11 + 3 a ALFA101 BET2 FI21 + 2 a BET3 FI11 FI21 + 3 BET2 MF11) / (6 (-1 + ro))
```

Fig. 8. Calculations of δ_{11} for $M/G/1/\infty$ queueing system with random volume customers and sectorized memory containing two sectors in *Mathematica*

C. Calculating multidimensional convolutions. The use of generalized density function

Consider again the most novel models of queueing systems with random volume customers and sectorized memory buffer. But this time indications of total volume are limited (the sectors of memory buffer let store information of size $V_1,\ldots,V_n$, respectively, where n is memory buffer's dimension - see Fig. 9.). For such defined models, we usually want to obtain steady-state characteristics of the number of customers present in the system and formula defining loss probability (in investigated model customer is lost if at least one of his volume vector indi-

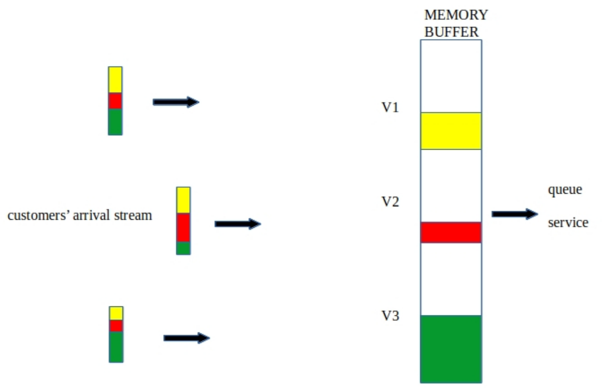


Fig. 9. Scheme of queueing system with random volume customers and limited sectorized memory buffer

cation is too big to be accepted on servicing so the mechanism of rejection of arriving customers is not the same as in classical queueing models where we only take into consideration the number of busy servers and waiting positions). Calculated characteristics contain inconvenient multidimensional Stieltjes convolutions $F_k(x_1,\ldots,x_n)$ that can be defined analogously to (9) but their multidimensional character makes computations are more difficult and obtaining results needs the use of computer algebra systems.

On the other hand, during convolutions calculations we meet two most practical cases: 1) indications of customer's random volume vector are independent (this situation is easy because then multidimensional Stieltjes convolution can be presented as a product of Stieltjes convolutions of its indications); 2) first $n-1$ indications of customer's random volume vector are independent and the last one is proportional to their sum. The second situation (which is often observed in real-life systems) leads to interesting computations in which we additionally may use generalized density function. As an example, we will investigate an interesting model of queueing system of the $M/M/n/(m,V_1,V_2,V_3)$-type. In this model (having n servers and m waiting positions) customers arrive to the system with rate a (time intervals between neighboring moments of customers' arrival are exponentially distributed with parameter a), customers' service time is exponentially distributed with parameter μ and independent of customer's volume vector that is characterized by distribution function $L(x_1,x_2,x_3)$. In addition, memory buffer contains three sectors limited by values V_1,V_2,V_3, respectively. It can be easily proved that formulae defining steady-state number of customers distribution and loss probability have the same form as in (16a-16b) if we substitute instead of one-dimensional convolution $L_k(V)$ its three-dimensional version $L_k(V_1,V_2,V_3)$. Assume now that first two indications ζ_1,ζ_2 (whose distribution functions are $L_1(x)$ and $L_2(x)$, respectively) of three-dimensional non-negative random volume vector $(\zeta_1,\zeta_2,\zeta_3)$ are independent and the last one is proportional to their sum: $\zeta_3=c(\zeta_1+\zeta_2)$, $c>0$. Let us try to obtain the general formula for multidimensional Stieltjes convolution $L_k(x_1,x_2,x_3)$ that defines distribution function of the sum of k independent random vectors whose distribution function is $L(x_1,x_2,x_3)$. Note that even formula defining $L(x_1,x_2,x_3)$ is complicated. Indeed, after some computations, we obtain:

$$L(x_1,x_2,x_3)=\mathbf{P}\{\zeta_1<x_1,\zeta_2<x_2,\zeta_3<x_3\}=$$
$$=\mathbf{P}\{c(\zeta_1+\zeta_2)<x_3,\zeta_1<x_1,\zeta_2<x_2\}=$$
$$=\int_0^{x_1}\int_0^{x_2}\mathbf{P}\{c(\zeta_1+\zeta_2)<x_3,\zeta_1=u_1,\zeta_2=u_2\}\,dL_1(u_1)\,dL_2(u_2)=$$
$$=\int_0^{x_1}\int_0^{x_2}H\left(x_3-c(u_1+u_2)\right)\,dL_1(u_1)\,dL_2(u_2)=$$
$$=\int_0^{x_1}\int_0^{\frac{x_3}{c}-u_1}dL_1(u_1)\,dL_2(u_2)=\int_0^{x_1}L_2\left(\frac{x_3}{c}-u_1\right)\,dL_1(u_1), \tag{18}$$

where $H(x)$ is left-side continuous version of Heaviside's unitstep function. Calculating convolutions of such defined distribution function (based on its definition generalized on multidimensional distribution functions - see again (9)) seems to be a rather impossible exercise to do. But again, we may use very convenient Laplace–Stieltjes transform (its multidimensional version) to obtain results, especially when we additionally consider Dirac-delta distribution $\delta(x)$ as the derivative of Heaviside's function $H(x)$ (in the sense of distribution). Note that from (18) we may also easily obtain

$$dL(x_1,x_2,x_3)=dL_1(x_1)\,dL_2(x_2)\delta\left(x_3-c(x_1+x_2)\right)\,dx_3. \tag{19}$$

If we use multidimensional Laplace–Stieltjes transform to both sides of (19), we finally obtain

$$\alpha(s_1,s_2,s_3)=$$
$$=\int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1x_1-s_2x_2-s_3x_3}\,dL_1(x_1)\,dL_2(x_2)\delta\left(x_3-c(x_1+x_2)\right)\,dx_3=$$
$$=\int_0^\infty\int_0^\infty e^{-s_1x_1-s_2x_2-cs_3(x_1+x_2)}\,dL_1(x_1)\,dL_2(x_2)=$$
$$=\int_0^\infty e^{-x_1(s_1+cs_3)}\,dL_1(x_1)\cdot\int_0^\infty e^{-x_2(s_2+cs_3)}\,dL_2(x_2)=$$
$$=\varphi_1(s_1+cs_3)\varphi_2(s_2+cs_3), \tag{20}$$

where $\varphi_1(s)$ and $\varphi_2(s)$ are one-dimensional Laplace–Stieltjes transforms of functions $L_1(x)$ and $L_2(x)$, respectively. This result can be generalized and give a base to build computational algorithm for calculating multidimensional convolutions of random vectors $(\xi_1,\ldots,\xi_n)$ in which first $n-1$ indications are independent (and we are able to obtain exact forms of their Laplace–Stieltjes transforms) and the last one is proportional to their sum. In this case multidimensional Laplace–Stieltjes transform of this random vector has the form:

$$\alpha(s_1,\ldots,s_n)=\prod_{i=1}^{n-1}\varphi_i(s_i+cs_n), \tag{21}$$

where $\varphi_i(s)$ is Laplace–Stieltjes transform of i-th indication of random vector $(\xi_1,\ldots,\xi_n)$. Now we can obtain the k-fold multidimensional Stieltjes convolution of this vector using the following way (the algorithm is similar to its one-dimensional version presented in Sec. 4.) with the help of *Mathematica* environment:

1. Calculate multidimensional LST of a convolution $F_k(x_1,\ldots,x_n)$ using formula:

$$\alpha_k(s_1,\ldots,s_n)=[\alpha(s_1,\ldots,s_n)]^k=\prod_{i=1}^{n-1}[\varphi_i(s_i+cs_n)]^k;$$

2. Obtain formula for Laplace transform of analyzed convolution:
$$\mathscr{L}_k(s_1,\ldots,s_n)=\frac{\prod_{i=1}^{n-1}\left[\varphi_i(s_i+cs_n)\right]^k}{\prod_{i=1}^{n}s_i};$$
3. Finally, use **InverseLaplaceTransform** command from *Mathematica* environment to obtain exact formula of $F_k(x_1,\ldots,x_n)$ convolution:
$$F_k(x_1,\ldots,x_n)=\mathscr{L}_k^{-1}(s_1,\ldots,s_n).$$

Now we will present application of this method.

Example 5 *Consider random vector (ξ_1,ξ_2,ξ_3) whose first two indications are independent and exponentially distributed with parameters f,g, respectively and $\xi_3=c(\xi_1+\xi_2)$. Then we have : $\varphi_1(s)=\frac{f}{f+s}$, $\varphi_2(s)=\frac{g}{g+s}$. Thus, on the base of (21), we obtain $\alpha(s_1,s_2,s_3)=\frac{fg}{(f+s_1+cs_3)(g+s_2+cs_3)}$ and finally $\mathscr{L}_k(s_1,s_2,s_3)=\frac{(fg)^k}{s_1s_2s_3(f+s_1+cs_3)^k(g+s_2+cs_3)^k}$. Then we may obtain formula for Stieltjes convolution $F_k(x_1,x_2,x_3)$ for the arbitrary k using* **InverseLaplaceTransform** *command from Mathematica environment. The way of getting results for $k=1$ (distribution function $F(x_1,x_2,x_3)$) and $k=2$ (2-fold Stieltjes convolution) is presented in the next Fig. 10. In the same way we may obtain convolutions of higher orders but results are very long and complicated and we will not show them in this paper. Moreover, it is clear that computations may take much more time due to the use of residue method in the case when poles of denominator are of higher orders. In addition, the values of convolutions for some fixed V_1,V_2 and V_3 let obtain number of customers distribution and calculate loss probability values for mentioned earlier queueing system with limited sectorized memory buffer of the $M/M/n/(m,V_1,V_2,V_3)$-type.*

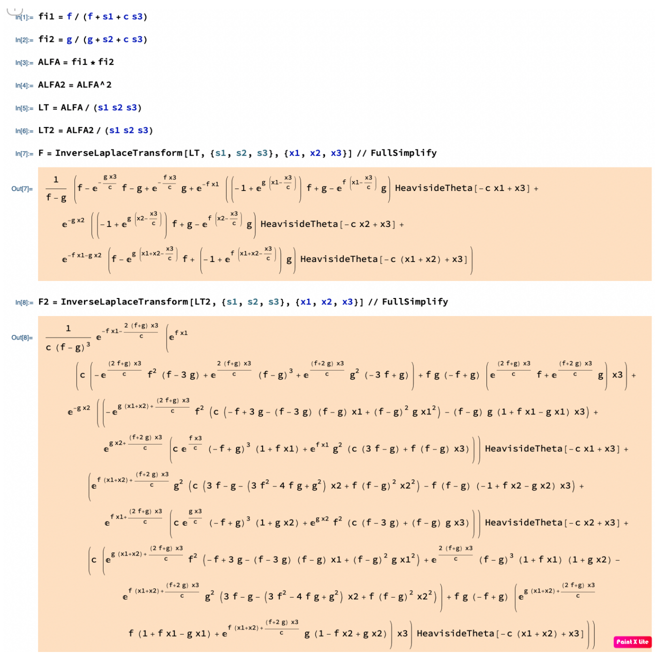


Fig. 10. An example of calculating convolutions of a random vector in *Mathematica*

6. CONCLUSIONS AND FINAL REMARKS

In the present paper, we have confirmed the important role of computer algebra systems in obtaining of main performance characteristics of queueing systems both during analysis of classical queues and their generalizations assuming existence of customers' random volume (that can also be multidimensional). We have shown many examples of effective computational algorithms that, with the help of *Mathematica* environment, let calculate characteristics connected with steady-state number of customers distribution, loss probability (in the case when memory buffer is limited) as well as characteristics of steady-state total volume (in the case when it is one or multidimensional). Moreover, we have investigated possibilities of applications of computer computational systems for obtaining exact forms of derivatives of complicated rational functions of one or many variables consisting undefined symbols for which we have to use classical L'Hospital's rule and its multivariate modification and getting exact formulae of Stieltjes convolutions with an interesting use of chosen integral transforms. In addition, we have discussed the concept of generalized density function and its possible applications in complex calculations from theory of queueing systems area. In the text we have also presented some fragments of *Mathematica* notebooks illustrating the ways of getting the most important results with the help of offered by author computational algorithms. The mentioned in the paper methods can be useful for scientists working with queueing theory models for whom presented techniques give a chance to avoid almost impossible to do manual calculations.

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