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SELECTION CRITERIA OF THE FUNCTION OF EXPLOITATION COSTS

KRYTERIA DOBORU FUNKCJI KOSZTÓW WYDOBYCIA

A method of selecting the function of the costs of rectilinear or curvilinear exploitation is presented. The Cobb-Douglas function of production has been used for this purpose. The function of costs is an inverse function of the Cobb-Douglas function. Next, the equation of the curve of production costs was plotted applying the initial power function. Its diagram was shifted by the vector $[p, q]$ obtaining the curve that does not cross the beginning of the system.

Furthermore, the Newton formula for the development of the binomial to the power n was used. As a result, a polynomial was obtained that must satisfy the criteria mentioned in the paper, thus being a polynomial of the third order.

Key words: costs of exploitation, function of costs, Cobb-Douglas function, criteria of selection, correlation coefficient, correlation ratio

Jednym z podstawowych celów działalności kopalni jest uzyskanie rentowności. W przypadku górnictwa na rentowność ma wpływ wiele czynników, od warunków geologiczno-górniczych oraz właściwej organizacji poczynając, po właściwe zarządzanie finansami. W efekcie końcowym o rentowności decydują poniesione koszty oraz uzyskiwane ceny za wydobytą kopalinę. Ponieważ kopalnia ma ograniczony wpływ na wysokość cen (w wielu przypadkach nie ma żadnego wpływu), realnym sposobem poprawy rentowności jest zatem zmniejszenie kosztów. Można w tym przypadku wykorzystać wszechstronną analizę ponoszonych kosztów, ich wielkości, struktury rodzajowej, związku poszczególnych kosztów z czynnikami mającymi wpływ na ich wielkość. W przypadku takiej analizy wykorzystuje się wewnętrzną strukturę kosztów stałych i zmiennych, oddaje ona bowiem najwierniej istotę samych kosztów, które powinny wzrastać przy wzroście wydobycia i maleć przy jego zmniejszaniu. Badamy wówczas korelację wysokości ponoszonych kosztów przy zmieniającej się skali wydobycia, przy czym związek ten może mieć charakter prostoliniowy bądź krzywoliniowy. W badaniach statystycznych miarą związku zmiennych jest dla regresji prostoliniowej współczynnik korelacji (2), dla regresji krzywoliniowej zaś stosunek korelacyjny (3).

Wartość współczynników określonych wzorami (2) i (3) nie może stanowić kryterium wyboru funkcji określonego typu (prostoliniowego bądź krzywoliniowego), w statystyce zachodzi bowiem

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warunek (4), czyli wartość stosunku korelacyjnego jest równa lub większa od bezwzględnej wartości współczynnika korelacji. Własnym kryterium jest wówczas test T_η i porównanie go z odpowiednią wartością tabelaryczną T, w przypadku gdy $T_\eta > T$ możemy bowiem analizowaną zależność przyjąć jako funkcję krzywoliniową. Jeżeli zachodzi nierówność odwrotna, analizowaną zależność ponoszonych kosztów od wielkości wydobycia należy uznać za prostoliniową.

W przypadku stwierdzenia zależności liniowej, równanie regresji liniowej kosztów wyznaczamy według kryterium minimalnych kwadratów odchyleń, czyli metodą najmniejszych kwadratów.

Jeżeli testowanie wykaże, że miarodajnym obrazem badanej zależności jest krzywoliniowa funkcja kosztów, wówczas funkcja ta powinna spełniać następujące kryteria matematyczno-ekonomiczne:

- wartość zmiennych y oraz x odpowiadają warunkom zapisanym wzorem (12),
- funkcja posiada wyraz wolny, odpowiadający wartości kosztów stałych,
- jest monotonicznie rosnąca,
- posiada punkt przegięcia,
- punkt przegięcia dzieli wykres krzywej na część rosnącą degresywnie i część rosnącą progresywnie.

Tego rodzaju krzywą wyznaczono na podstawie funkcji produkcji Cobba-Douglasa (9), (10), (11), przedstawionej na rysunku 3. Jej funkcja odwrotna (obrócona wokół dwusiecznej pierwszej ćwiartki) spełnia wszystkie wymienione wcześniej kryteria. Następnie wyznaczono równania tej krzywej. Skorzystano w tym celu z wyjściowej funkcji potęgowej (14). Jej wykres przesunięto o wektor $[p, q]$ uzyskując krzywą (16). Wykorzystano dalej wzór Newtona na rozwinięcie dwumianu do n-tej potęgi, w wyniku czego uzyskano wielomian (17). Aby wielomian ten spełniał wymienione kryteria musi być wielomianem trzeciego stopnia (21).

Słowa kluczowe: koszty wydobycia, funkcja kosztów, funkcja Cobba-Douglasa, kryteria doboru, współczynnik korelacyjny, stosunek korelacyjny

1. Introduction

The function of total costs (1) is an example of the function of one variable form:

$$K_C = f(x) \tag{1}$$

where:

K_C — total costs,
x — quantity of output.

Since one does not know in advance what type of function corresponds best to the relation of costs to the quantity of output, one applies empirical values of the observed values of total costs

$$K_{C_0}, K_{C_1}, \dots, K_{C_n}$$

obtained at the corresponding quantities of the output x:

$$x_0, x_1, \dots, x_n$$

The correlation relation of both variables can be of double character, i.e. rectilinear or curvilinear.

The obtained empirical data are related to one another in such a way that on their basis alone it is not possible to determine whether a rectilinear or a curvilinear function of total costs will result.

In the case of rectilinear regression the measure of the relation of variables y and x is the correlation coefficient r defined by the following formula

$$r = \frac{\sigma_{xy}}{\sigma_x \cdot \sigma_y} \tag{2}$$

where:

- y — dependent variable (in this case K_c),
- σ_{xy} — covariance,
- σ_x — standard deviation x,
- σ_y — standard deviation y,

while the measure of the relation of curvilinear regression is the so-called correlation ratio η:

$$\eta = \frac{\sigma_{\bar{y}(x)}}{\sigma_y} \tag{3}$$

where:

- $\sigma_{\bar{y}(x)}$ — standard deviation of the mean values y corresponding to the determined values x.

The choice of the type of function of the examined correlation relation is of subjective character and the following data, among others, ought to be considered:

- high correlation coefficient, even within the range $r = 0.8$–0.9, does not guarantee the correct choice of rectilinear regression because of the general condition:

$$\eta \geq |r| \tag{4}$$

- establishing the correlation relation and the measure of this relation is basically of formal character (mathematical) since in the selection of the one or the other type of correlation function the essence of the realized process ought to be taken into account,
- it happens sometimes that a strong correlation does not confirm the true relationship between the variables.

2. Selection criterion of the type of the function of costs

In the year the tests were made, the following monthly total costs with the corresponding monthly quantities of output x were recorded in mine A:

K_c	15	18	18	20	22	23	25	26	24	24	27	24	26	28	30	27	33	32	35	44	50	50
X	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0	2.25	2.5	2.75	3.0	3.25	3.5	3.75	4.0	4.25	4.5	4.75	5.0	5.25	5.5

Using the least square method the following results were obtained:

Co-variance: $\sigma_{xy} = 12.9147$

variance: $\sigma_y = 8.9564$

$\sigma_x = 1.5860$

thus the correlation coefficient:

$$r = \frac{\sigma_{xy}}{\sigma_y \cdot \sigma_x} = 0.9092$$

The high value of the correlation coefficient may indicate a linear dependence of the total costs on the quantity of production. Taking into account, however, the condition given by formula (4) and the obtained results suggesting a non-linear dependence (Fig. 1), in such a case one ought to determine the value of the correlation ratio.

When statistically tested, it follows from the obtained set of points that the dependence between the variables is not rectilinear. Thus the task is to select a curve that would reflect the course of the phenomenon in a most accurate way.

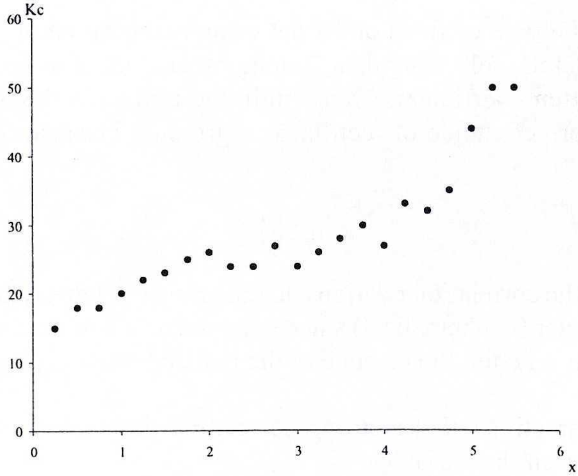

Fig. 1. Dependence of total costs on the quantity output

Rys. 1. Zależność kosztów całkowitych od wielkości wydobycia

From formula (3) it follows that the correlation ratio cannot be calculated from empirical data determining one value y (in this case of the cost K_c) and, the corresponding to it, value x (quantity of the output).

We would then obtain

$$\sigma_{\bar{y}(x)} = \sigma_y \quad (5)\cdot$$

which, subsequently, would mean that the correlation ratio $\eta = 1$.

In order to prevent this, the empirical data are aggregated in the way shown in Table 1 where individual symbols denote:

- $x_1 - x_2$, ..., $x_{r-1} - x_r$, — equal r of the classes size x,
- $y_1 - y_2$, ..., $y_{n-1} - y_n$, — equal n of the classes size y,
- f_{rn} — amount of empirical data in the class $y_j - y_n$ for the corresponding class $x_i - x_r$,
- N_r — amount of data in the class $x_i - x_r$,
- M_j — amount of data in the class $y_j - y_n$,
- $\bar{y}(x)$ — mean value y in the class $x_i - x_r$,
- $\bar{x}(y)$ — mean value x in the class $y_i - y_n$.

TABLE 1

Correlation between costs and the quantity of production

TABLICA 1

Korelacja pomiędzy kosztami a wielkością produkcji

Total cost (y)	Quantity of output (x)			M_{y_j}	$\bar{x}(y)$
	$x_1 - x_2$	...	$x_{r-1} - x_r$		
$y_1 - y_2$ ... ... $y_{n-1} - y_n$	f_{11} f_{1n}		f_{r1} f_{rn}	M_1 M_n	$\bar{x}_1$ $\bar{x}_n$
N_r	N_1		N_r	N	—
$\bar{y}(x)$	$\bar{y}_1$		$\bar{y}_r$	—	—

In this case it important to group the empirical data into suitable classes of output quantity because in the adverse case the calculated value of standard deviation $\sigma_{\bar{y}(x)}$ will not differ from the standard deviation σ_y which will yield, consequently the result $\eta_{yx} = 1$.

An example of aggregation of empirical data connected to the relationship of costs and the quantity of output is presented in Table 2.

TABLE 2

Correlation between costs and the quantity of production

TABLICA 2

Korelacja pomiędzy kosztami a wielkością produkcji

Costs in the interval	Quantity of production in the interval						M_j	$\bar{x}(y)$
	0–1	>1–2	>2–3	>3–4	>4–5	>5–6		
0–20	4						4	0.5
>20–30		4	4	4			12	2.5
30–40					3		3	4.5
40–50					1	2	3	5.16
Nr	4	4	4	4	4	2	22	—
$\bar{y}(x)$	10	25	25	25	37.5	45	—	—

According to the results obtained:

$$\eta_{yx} = 0.9125$$

which confirms condition 4.

Attention should be given to the fact that the coefficient of correlation provides only one value expressing the interrelation of the values x and y. On the other hand, two correlation ratios that differ from each other result. In extreme cases one can be equal to 0 and the other to 1, that is, e.g.:

$$\eta_{yx} = 1; \eta_{xy} = 0$$

The correlation ratio is contained within the limits:

$$0 \leq \eta \leq 1$$

Its high value is evidence of a close connection between the variables and indicates that random factors influence the course of the phenomenon to a minor extent.

In many cases we are not convinced whether a given phenomenon ought to be approximated by a straight line or by a curve although from the character of the correlation ratio there follows the inequality:

$$\eta \geq r$$

The value of the correlation ratio or the correlation coefficient is not, however, the only criterion for the assessment of the dependence. A comparison of the correlation ratio with the correlation coefficient applying the T test (Romanowski 1952; Szulc 1961;

Winkler 1951) is a good method to explain whether the non-linear correlation dependence exists. If one assumes that the tested material is an independent sample derived from a normal general population, the choice is reliable even at low values of N.

For this purpose one calculates

$$T_\eta = \frac{(N-s)\cdot(\eta_{yx}^2 - r^2)}{(s-2)\cdot(1-\eta_{yx}^2)} \tag{6}$$

where:

s — number of partial mean values $\bar{y}$,
N — overall number of observations.

The test T_η is compared with the tabular T for the following degrees of freedom:

$$k_1 = s - 2$$

and

$$k_2 = N - s$$

If $T_\eta > T$, then it points to a significant divergence, i.e. the existence of curvilinear regression.

3. Criterion of the selection of the cost function forms

In practice, in each case of economic activity so-called fixed costs i.e. necessary expenditures, regardless the volume of production, even when the output equals 0, must be taken into consideration (Czopek 2000; Fischer 1985; Leidler, Estrin 1989). Accordingly, this function (1) ought to be presented in the following form:

$$K_C = K_S + g(x) \tag{7}$$

where:

K_S — fixed costs,
$g(x)$ — function of variable costs.

If one determines that a linear form of the function of variable costs ought to be considered, as given at the point 2, then the function (7) reads as follows:

$$K_C = K_S + k_{jz} \cdot x \tag{8}$$

where:

k_{jz} — unit variable cost,

i.e. a so-called directional equation of the straight line for which the values sought, K_s and k_{jz} are determined by the least square method.

In this case of the analysis of variable costs as a curvilinear function when determined, one applies the general theory of costs FVC (Theory of Fixed and Variable Costs).

If the volume of production is denoted by x, then for its implementation suitable costs of labour — L (Labour) and capital input — C (Capital) are essential.

This is a general expression of production costs, i.e. the factors of production requiring consumption of capital (L and C) expressed in terms of money, the function of production taking the form

$$X = f(L, C) \tag{9}$$

The modified function of production has been named the Cobb-Douglas function after the names of its authors and it reads as follows (Leidler, Estrin 1989):

$$X = A \cdot L^{\alpha} \cdot C^{\alpha} \tag{10}$$

where:

- $L + C$ — costs of production,
- A — parameter determining how much production can be obtained at the unit use of both factors L and C,
- α — coefficient of the reaction of the quantity of production to the change of the factor L,
- β — coefficient of the reaction of the quantity of production to the change of the factor C.

The diagram of the Cobb-Douglas function is shown in Fig. 2.

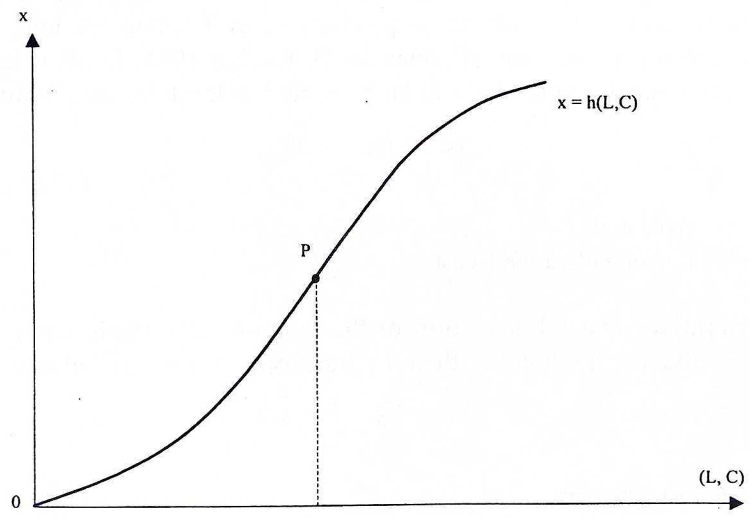

Fig. 2. Curve of production

Rys. 2. Krzywa produkcji

Based on the Cobb-Douglas (Leidler, Estrin 1989), theory and also taking into account the basic principles of economics one can state that:

- increase in production within the interval *(O, P)* causes a progressive increase in production with constant increase in costs *(L, C)*,
- for production at the point P the increase in production is proportional to the increase in costs,
- for production $X > P$ the increase in production is retrogressive (less than proportional to the constant increase of costs).

Taking into account the necessity of covering fixed costs, the developed Cobb--Douglas function is:

$$X = C_v + A \cdot L^{\alpha} \cdot C^{\alpha} \tag{11}$$

where:

C_v — fixed costs,

and is presented in Fig. 3.

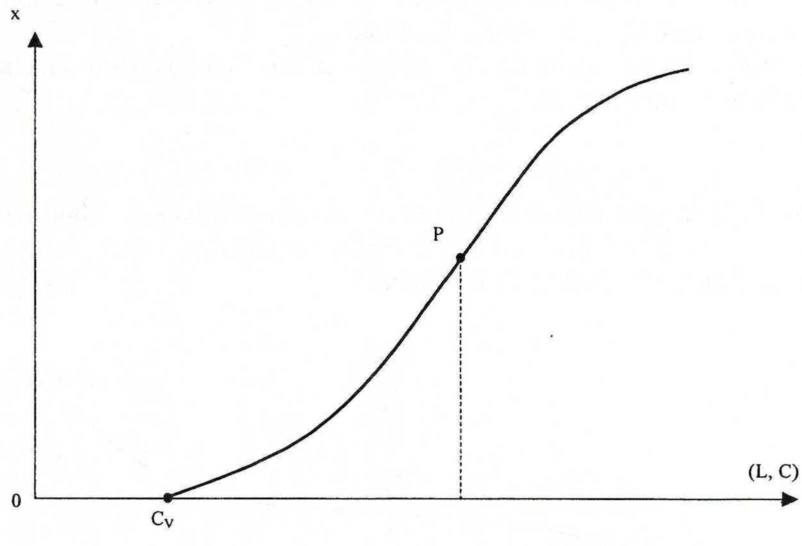

Fig. 3. Curve of production including fixed costs

Rys. 3. Krzywa produkcji z uwzględnieniem kosztów stałych

The Cobb-Douglas function (11) is the function of production whereas we are interested in the function of costs expressed by formula (7). However, it is easy to state that the function (11) can be used to determine the function of costs (7) since the function of costs is the inverse function of the function of production. Moreover, it is known that the diagram of the inverse function is obtained by rotation of the given (original) function by 180° about the bisector of the first and the third quarter.

Since both the costs as well as the quantity of the output cannot be negative, i.e.:

$$\left.\begin{array}{r} L \geq 0 \\ C \geq 0 \\ X \geq 0 \end{array}\right\} \qquad (12)$$

thus the diagram of the function:

$$(L + C) = h(x) \qquad (13)$$

will be obtained by rotation of the function (11) by 180° about the bisector of the first quarter — see Fig. 4.

The fact that the independent variable (costs as the sum $L + C$) includes two variable values L and C is the main drawback while applying the curve of costs presented in practice in Fig. 4. The total costs ought to be considered as a combination of the sum of both variables. This excludes the possibility of selection of the regression curve of the given relation.

While determining the function that has a similar pattern as function (11), one can make use of the rule saying that if the initial function is the power function, then an inverse function must also be a power function.

Let us assume at the beginning that the power function has a similar diagram as function (11) in its final part

$$y = ax^n \qquad (14)$$

Function (14) does not correspond, however, to the conditions 12. Thus to cope with the problem one ought to shift its diagram by the vector $[p,q]$. In this way, a diagram similar to the function (13) (Fig. 5) is obtained.

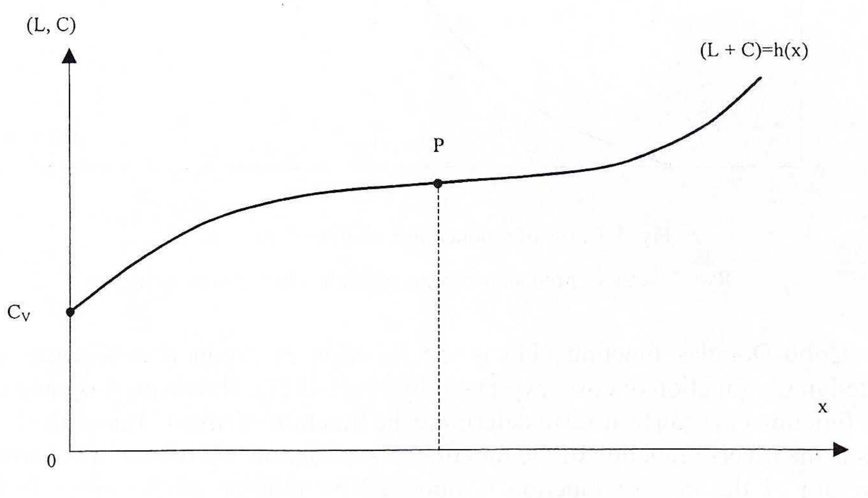

Fig. 4. Inverse function of the curve of production

Rys. 4. Funkcja odwrotna krzywej produkcji

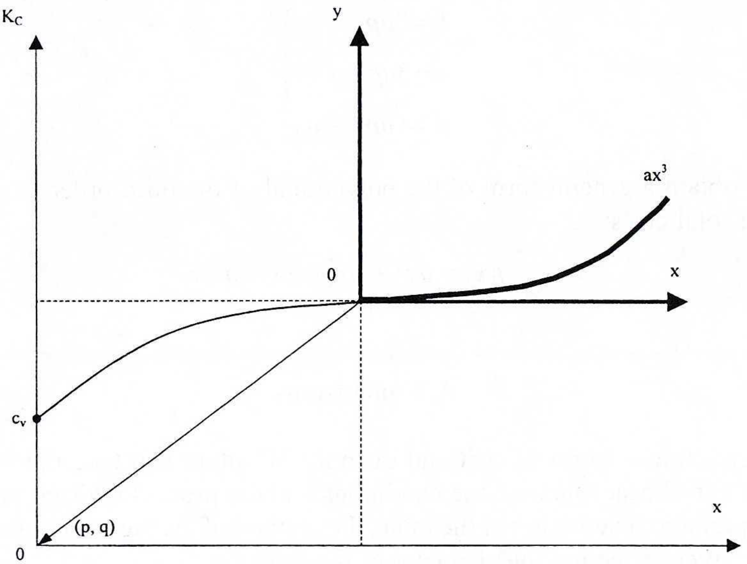

Fig. 5. Power function and the shifted function of total costs

Rys. 5. Funkcja potęgowa i przesunięta funkcja kosztów całkowitych

As a result of shifting the equation (14) by the vector $[p,q]$ we obtain

$$y + q = a(x + p)^n \tag{15}$$

where:

p, q — coordinates of the vector of the shift of the curve 14, i.e.:

$$y = a(x + p)^n - q \tag{16}$$

Using the Newton formula for the development of the binomial to the n-th power, the following equation is obtained

$$y = a\left[x^n + \binom{n}{1}x^{n-1} \cdot p^1 + \binom{n}{2}x^{n-2} \cdot p^2 + \binom{n}{3}x^{n-3} \cdot p^3 + \ldots + \binom{n}{n} \cdot p^n\right] - q \tag{17}$$

One can prove that the curvilinear function of costs presented in Fig. 5 is a polynomial of the third order (Czopek 2001), thus the equation (17) will read as follows:

$$y = a[x^3 + 3x^2p + 3xp^2 + p^3] - q \tag{18}$$

and after transformation:

$$y = ax^3 + 3x^2ap + 3xap^2 + ap^3 - q \tag{19}$$

Introducing, moreover, the following denotations:

$$\left.\begin{aligned} b &= 3ap \\ c &= 3ap^2 \\ d &= (ap^3 - q) \end{aligned}\right\} \tag{20}$$

on will obtain a general form of the polynomial of the third order as a curvilinear function of total costs:

$$f(x) = ax^3 + bx^2 + cx + d \tag{21}$$

4. Conclusions

The correlation relation of cost and quantity of output can take the form of a rectilinear or curvilinear function, the conditions and the process realized in a particular mine , the periodic functioning of the mine, the method of management of costs, among others, decisively affecting the character of the function.

While choosing the final form of representation of the relationship mentioned one ought to take into account the following points:

- a high value of the correlation coefficient does not guarantee the adequacy of approximation by a straight line of regression,
- a curvilinear approximation of the regression function can be considered correct if the difference $\eta^2 - r^2$ is significant (distinct) although this criterion is not decisive,
- in the case of a little difference between the correlation ratio (η) and the correlation coefficient (r), rectilinear regression can be considered as a correct representation of the correlation relation although it is connected with lower assessment of the accuracy of the tested relation,
- a positive value of the test T is a confirmation of the correctness of the acceptance of curvilinear regression
- both the correlation ratio (η) and the correlation coefficient (r) confirm the fact of the existence of a relation of the tested characteristics only, on the other hand, the characteristics of this relation is given by the mathematical form of rectilinear or curvilinear regression,
- in the case of rectilinear regression, the straight line is determined by the least square method,
- in the case of curvilinear regression , the function of total costs in the form of a polynomial of the third order can also be determined by the least square method.

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