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# Second-order maximum principle controlled weakly singular Volterra integral equations

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This work studies a class of singular Volterra integral equations that are (controlled) and can be applied to memory-related problems. For optimum controls, we prove a second-order Pontryagin type maximal principle.

**Key words:** second-order maximum principle, singular Volterra integral equation, optimal control, Pontryagin's maximum principle

## 1. Introduction

Studying optimal control problems described by singular integral equations offers a unique and intellectually stimulating challenge that has important implications for both theoretical research and practical applications in many fields. Singular integral equations, which involve integrals that may have singularities (i.e., integrals that become unbounded or undefined at certain points), arise in a variety of complex systems, and addressing these problems opens the door to solving real-world optimization issues in diverse scientific and engineering domains. The study of optimal control problems described by singular integral equations is not just a niche but a powerful and essential area of research with vast implications in multiple fields. From engineering design and material science to robotics and fluid dynamics, singular integral equations capture the essential behaviors of systems with singular interactions, and optimal control provides the framework to optimize these systems for better performance, efficiency, and stability.

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This area offers a rich, interdisciplinary research environment where innovations in mathematics, physics, and engineering can come together to solve real-world problems. By tackling the challenges posed by singular integral equations, you are contributing to the development of both theoretical and computational methods that will have far-reaching practical applications, advancing not only control theory but also our ability to model and optimize complex systems in nature and technology.

In this paper, we investigate the controlled Volterra integral equation given below:

$$y(t) = \eta(t) + \int_0^t \frac{f(t,s,y(s),u(s))}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad a.e. \quad t \in [0,T]. \tag{1}$$

The above representations of $\eta(\cdot)$ and $f(\cdot,\cdot,\cdot,\cdot)$ are maps that are referred to as the generator and the free term of the state equation, respectively; $y(\cdot)$ represents the state trajectory and takes values in the Euclidean space $\mathbb{R}^n$; $u(\cdot)$ represents the control and takes values in convex subset $U \subset \mathbb{R}^n$, constant $\alpha \in (0,1)$. We present the cost functional performance metric to gauge the control's effectiveness.

$$J(u(\cdot)) = \int_0^T g(t,y(t),u(t))\mathrm{d}t + \sum_{i=1}^m h^i(y(t_i)), \tag{2}$$

The running cost and the prespecified instant costs ($at \quad 0 \leqslant t_1 < t_2 < \cdots < t_m \leqslant T$), are represented by the two terms on the right hand, respectively.

The Pontryagin Maximum Principle can indeed be extended to optimal control problems described by singular integral equations, but this extension involves overcoming significant challenges due to the singularities present in the system dynamics and the adjoint equations. The singular kernels complicate both the theoretical derivation of the adjoint equations and the practical process of maximizing the Hamiltonian.

To address these challenges, the first-order necessary conditions (Pontryagin's type maximum principle) for optimal control problems involving singular integral equations were established by Lin and Yong in [1]. Additionally, in [6], it was shown that a Pontryagin maximum principle can be applied to terminal state-constrained optimal control problems involving Volterra integral equations with singular kernels. This result extends the applicability of Pontryagin Maximum Principle to more complex problems where the system dynamics exhibit singular behavior.

On the other hand, the second-order necessary conditions for optimal control problems are important for ensuring that a candidate solution is not only a critical point (satisfying the first-order conditions) but also a local minimum of the cost functional. These conditions involve the second variation of the cost functional and the second derivative of the Hamiltonian. They help refine our understanding of the optimal control problem by providing a more thorough test for optimality and are essential for distinguishing between minima, maxima, and saddle points in the optimization process. For the integral necessary condition of optimality of the second order for control problems given by volterra integral equations and a system of integro-differential equations, we refer to [2–4], and for singular controls for systems with fractional derivatives, and dynamic systems, see [5,7,8].

Motivated by the previous works, this research seeks to explore the second-order necessary conditions for optimal control problems of the form (1), which are governed by singular Volterra integral equations. The aim is to develop a second-order Pontryagin maximum principle specifically tailored to the optimal control problem in this context.

## 2. Main Result

We analyze the state equation (1) alongside the cost functional (2). Let $U$ be a nonempty bounded or unbounded convex subset of $\mathbb{R}^n$. For any $p \geqslant 1$, we define the set $\mathcal{U}^p[0,T]$ as follows:

$$\mathcal{U}^p[0,T] = \{u : [0,T] \to U \mid u(\cdot) \in L^p(0,T;\mathbb{R}^m)\}.$$

Also, we define

$$L^{p+}(0,T;\mathbb{R}^n) = \bigcup_{q>p} L^q(0,T;\mathbb{R}^n), \quad p \in [1,\infty).$$

If $n = 1$, we use notation $L^{p+}(0,T) = L^{p+}(0,T;\mathbb{R})$.

Problem(P) Find a $u^*(\cdot) \in \mathcal{U}^p[0,T]$ such that

$$J(u^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}^p[0,T]} J(u(\cdot)). \tag{3}$$

Any $u^*(\cdot)$ satisfying (3) is called an optimal control of Problem(P), the corresponding state $y^*(\cdot)$ ius called an optimal state and $(y^*(\cdot), u^*(\cdot))$ is called an optimal pair.

• (A1): Let $f : \Delta\times\mathbb{R}^n\times U \to \mathbb{R}^n$ be a transformation with $(t,s) \mapsto f(t,s,y,u)$ being measurable, $(y,u) \mapsto f(t,s,y,u)$ being continuously differentiable up to order-1. There are nonnegative functions $L_0(\cdot), L(\cdot)$ with

$$L_0(\cdot) \in L^{\frac{1}{\alpha}+}(0,T), \quad L(\cdot) \in L^{\frac{p}{p\alpha-1}+}(0,T),$$

for some $p > \dfrac{1}{\alpha}$, $u_0 \in U$.

$$|f(t,s,0,u_0)| \leqslant L_0(s), \quad (t,s) \in \Delta,$$
$$|f(t,s,x,u) - f(t,s,x',u')| \leqslant L(s)[|x-x'| + |u-u'|],$$
$$(t,s) \in \Delta, x, x' \in \mathbb{R}^n, u, u' \in U.$$

We denote $\Delta = \{(t,s) \in [0,T] \times [0,T] | \quad 0 \leqslant s < t \leqslant T\}$.

• (A2): Let $\eta(\cdot)$ be continuous at $t_i, i = 1,2,\dots,m$. Let $h^i : \mathbb{R}^n \to \mathbb{R}, i = 1,2,\dots,m$ be continuously differentiable up to order-2, and $g : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ be a transformation with $t \mapsto g(t,y,u)$ being measurable, $(y,u) \mapsto g(t,y,u)$ being continuously differentiable up to order-1. There is a constant $L > 0$ such that

$$|g(t,0,u)| + |g_x(t,x,u)| + |g_u(t,x,u)| \leqslant L, \quad (t,y,u) \in [0,T] \times \mathbb{R}^n \times U.$$

**Theorem 1.** *Let* $(A1)$ *and* $(A2)$ *hold, and* $\eta(\cdot) \in L^p(0,T;\mathbb{R}^n)$. *Suppose* $(y^*(\cdot), u^*(\cdot))$ *is an optimal pair of* (1) – (2). *Then there a solution* $\psi(\cdot) \in L^{\frac{p}{p-1}}(0,T;\mathbb{R}^n)$ *of the following adjoint equation*

$$\psi(t) = \int_t^T \frac{f_y^\top(s,t,y^*(t),u^*(t))}{(s-t)^{1-\alpha}} \psi(s) \mathrm{d}s - g_y(t,y^*(t),u^*(t))$$
$$- \sum_{i=1}^m 1_{[0,t_i)}(t) \frac{f_y^\top(t_i,t,y^*(t),u^*(t))}{(t_i-t)^{1-\alpha}} h_y^i(y^*(t_i)), \quad a.e. \quad t \in [0,T], \tag{4}$$

*such that following estimation holds:*

$$\int_0^T H_u(t) v(t) \mathrm{d}t \leqslant 0,$$

*where*

$$H(s,y,u,\psi) = \int_s^T \psi(t) \frac{f(t,s,y^*(s),u(s))}{(t-s)^{1-\alpha}} \mathrm{d}t - g(s,y^*(s),u(s))$$
$$- \sum_{i=1}^m 1_{[0,t_i)}(s) \frac{f(t_i,s,y^*(s),u(s))}{(t_i-s)^{1-\alpha}} h_y^i(y^*(t_i)). \tag{5}$$

• (B1): Let $f : \Delta \times \mathbb{R}^n \times U \to \mathbb{R}^n$ be a transformation with $(t,s) \mapsto f(t,s,y,u)$ being measurable, $(y,u) \mapsto f(t,s,y,u)$ being continuously differentiable up to order-2. There are nonnegative functions $L_0(\cdot), L(\cdot)$ with

$$L_0(\cdot) \in L^{\frac{1}{\alpha}+}(0,T), \quad L(\cdot) \in L^{\frac{p}{p\alpha-1}+}(0,T),$$

for some $p > \frac{1}{\alpha}$, $u_0 \in U$.

$$|f(t,s,0,u_0)| \leqslant L_0(s), \quad (t,s) \in \Delta,$$
$$|f(t,s,x,u) - f(t,s,x',u')| \leqslant L(s)[|x-x'| + |u-u'|],$$
$$(t,s) \in \Delta, x, x' \in \mathbb{R}^n, u, u' \in U.$$

- (B2): Let $\eta(\cdot)$ be continuous at $t_i, i = 1,2,\ldots,m$. Let $h^i : \mathbb{R}^n \to \mathbb{R}, i = 1,2,\ldots,m$ be continuously differentiable up to order-2, and $g : [0,T]\times\mathbb{R}^n\times U \to \mathbb{R}$ be a transformation with $t \mapsto g(t,y,u)$ being measurable, $(y,u) \mapsto g(t,y,u)$ being continuously differentiable up to order-2. There is a constant $L > 0$ such that

$$|g(t,0,u)| + |g_x(t,x,u)| + |g_u(t,x,u)| \leqslant L,$$
$$|g_{xx}(t,x,u)| + |g_{xu}(t,x,u)| + |g_{uu}(t,x,u)| \leqslant L, \quad (t,y,u) \in [0,T] \times \mathbb{R}^n \times U.$$

**Definition 1.** *An admissible control $u(t)$ is considered to be singular according to the Pontryagin maximum principle if, within the process $\{u(t), y(t)\}$, it implies that*

$$H_u(t) = 0, \quad t \in [0,T]. \tag{6}$$

**Theorem 2.** *Let $(B1)$ and $(B2)$ hold, and $\eta(\cdot) \in L^p(0,T;\mathbb{R}^n)$. Suppose $(y^*(\cdot), u^*(\cdot))$ is an optimal pair of (1) – (2). Then there a solution $\psi(\cdot) \in L^{\frac{p}{p-1}}(0,T;\mathbb{R}^n)$ of the following adjoint equation*

$$\psi(t) = \int_t^T \frac{f_y^\top(s,t,y^*(t),u^*(t))}{(s-t)^{1-\alpha}}\psi(s)\,\mathrm{d}s - g_y(t,y^*(t),u^*(t))$$
$$- \sum_{i=1}^m 1_{[0,t_i)}(t)\frac{f_y^\top(t_i,t,y^*(t),u^*(t))}{(t_i-t)^{1-\alpha}} h_y^i(y^*(t_i)), \quad a.e. \quad t \in [0,T], \tag{7}$$

*such that following estimation holds:*

$$\int_0^T H_{uu}(t)v^2(t)\,\mathrm{d}t + \Bigg[\int_0^T\int_0^T v(\tau)M(\tau,s)v(s)\,\mathrm{d}s\,\mathrm{d}\tau$$
$$+2\int_0^T\Bigg[\int_0^t v(s)H_{yu}(t)Q(t,s)v(t)\,\mathrm{d}t\Bigg]\,\mathrm{d}s\Bigg] \leqslant 0,$$

*where*

$$Q(t,s) = \frac{f_u(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} + \int_s^t \frac{\Phi(t,\tau) f_u(\tau,s,y^*(s),u^*(s))}{(\tau-s)^{1-\alpha}} \mathrm{d}\tau,$$

$$M(\tau,s) = \int_{\max\{\tau,s\}}^{T} Q(t,s) H_{yy}(t) Q(t,\tau) \mathrm{d}t$$
$$- \sum_{i=1}^{m} 1_{[0,t_i)}(s) 1_{[0,t_i)}(\tau) Q(t_i,s) h^i_{yy}(y^*(t_i)) Q(t_i,\tau), \quad \tau,s \in [0,T].$$

An extended Gronwall's inequality with a singular kernel is given in the following lemma.

**Lemma 1.** *[1] Let $\alpha \in (0,1)$ and $q > \frac{1}{\alpha}$. Let $L(\cdot), a(\cdot), y(\cdot)$ be nonnegative functions with $L(\cdot) \in L^q(0,T)$ and $a(\cdot), y(\cdot) \in L^{\frac{q}{q-1}}(0,T)$. Suppose*

$$y(t) \leqslant a(t) + \int_0^t \frac{L(s) y(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad a.e. \quad t \in [0,T].$$

*Then there exists a constant $K > 0$ such that*

$$y(t) \leqslant a(t) + K \int_0^t \frac{L(s) a(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad a.e. \quad t \in [0,T].$$

Let $p \geqslant 0$ and cosider the following linear integral equation:

$$y(t) = \eta(t) + \int_0^t \frac{A(t,s) y(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad a.e. \quad t \in [0,T]. \tag{8}$$

where $\alpha \in (0,1), \eta(\cdot) \in L^p(0,T;\mathbb{R}^n)$, and $A : \Delta \to \mathbb{R}^{n\times n}$ is measurable and satisfies

$$|A(t,s)| \leqslant L(s), \quad (t,s) \in \Delta,$$

for some measurable function $L(\cdot) \in L^{(\frac{1}{\alpha} \vee \frac{p}{p-1})+}(0,T)$.

**Lemma 2.** *[1] Let $1 \leqslant p < \dfrac{1}{1-\alpha}$, for any $s \in [0,T)$*

$$\Phi(t,s) = \frac{A(t,s)}{(t-s)^{1-\alpha}} + \int_s^t \frac{A(t,\tau)\Phi(\tau,s)}{(t-\tau)^{1-\alpha}} \mathrm{d}\tau, \quad a.e. \quad t \in (s,T].$$

*Then*

$$y(t) = \eta(t) + \int_0^t \Phi(t,s)\eta(s)\,\mathrm{d}s, \quad a.e. \quad \in [0,T]$$

*the expression gives a representation for the solution to the linear equation (8).*

### 2.1. Proof of the Theorem 2

Let $(y^*(\cdot), u^*(\cdot))$ be an optimal pair of (1) – (2) and fix any $u(\cdot) \in \mathcal{U}^p[0,T]$. Denote

$$u^\delta(\cdot) = u^*(\cdot) + \delta v(\cdot) \quad \text{where} \quad v(\cdot) = u(\cdot) - u^*(\cdot). \tag{9}$$

Clearly, $u^\delta(\cdot) \in \mathcal{U}^p[0,T]$. Let $y^\delta(\cdot) = y(\cdot, \eta(\cdot), u^\delta(\cdot))$ be the corresponding state.

It follows that

$$\begin{aligned} y^\delta(t) - y^*(t) &= \int_0^t \frac{f(t,s,y^\delta(s),u^\delta(s)) - f(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}}\,\mathrm{d}s \\ &= \int_0^t \frac{f_y^\delta(t,s)}{(t-s)^{1-\alpha}}(y^\delta(t) - y^*(t))\,\mathrm{d}s \\ &\quad + \int_0^t \frac{f_u^\delta(t,s)}{(t-s)^{1-\alpha}}(u^\delta(t) - u^*(t))\,\mathrm{d}s, \end{aligned} \tag{10}$$

where

$$\begin{aligned} f_y^\delta(t,s) &= \int_0^1 f_y(t,s,y^*(s) + \tau[y^\delta(t) - y^*(t)], u^\delta(s))\,\mathrm{d}\tau, \quad (t,s) \in \Delta, \\ f_u^\delta(t,s) &= \int_0^1 f_u(t,s,y^\delta(s), u^*(s) + \tau[u^\delta(t) - u^*(t)])\,\mathrm{d}\tau, \quad (t,s) \in \Delta. \end{aligned} \tag{11}$$

(A1) provides with

$$\left|f_y^\delta(t,s)\right| \leqslant L(s), \quad |f_u^\delta(t,s)| \leqslant L(s), \quad (t,s) \in \Delta. \tag{12}$$

Clearly, $L(s) \in L^q(0,T)$ for some $q \in \left(\frac{1}{\alpha}, p\right)$. That being so

$$|y^\delta(t) - y^*(t)| = \int_0^t \frac{L(s)}{(t-s)^{1-\alpha}} |y^\delta(t) - y^*(t)| \mathrm{d}s + \int_0^t \frac{L(s)}{(t-s)^{1-\alpha}} |u^\delta(t) - u^*(t)| \mathrm{d}s, \quad t \in [0,T]. \tag{13}$$

By the extended Gronwall's inequality Lemma 1 and (9), choosing $q' \in \left(\frac{1}{\alpha}, q\right)$ (see, [1]),

$$|y^\delta(t) - y^*(t)| \leqslant K\delta^{\frac{q-q'}{q'q}} \longrightarrow 0, \quad \delta \to 0 \quad \text{uniformly in} \quad t \in [0,T]. \tag{14}$$

- let $Y_1(\cdot)$ is the solution of the following first-order variational equation:

$$Y_1(t) = \int_0^t \frac{f_y(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} Y_1(s) \mathrm{d}s + \int_0^t \frac{f_u(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} v(s) \mathrm{d}s \quad t \in [0,T]. \tag{15}$$

- let $Y_2(\cdot)$ is the solution of the following second-order variational equation:

$$Y_2(t) = \int_0^t \frac{f_y(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} Y_2(s) \mathrm{d}s + \int_0^t \frac{f_{yy}(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} Y_1^2(s) \mathrm{d}s + \int_0^t \frac{2f_{yu}(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} Y_1(s) v(s) \mathrm{d}s + \int_0^t \frac{f_{uu}(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} v^2(s) \mathrm{d}s, \quad t \in [0,T]. \tag{16}$$

As a consequence of those,

$$
\begin{aligned}
&\frac{y^\delta(t) - y^*(t)}{\delta} - Y_1(t) \\
&= \int_0^t \frac{f_y^\delta(t,s)}{(t-s)^{1-\alpha}} \left( \frac{y^\delta(t) - y^*(t)}{\delta} \right) \mathrm{d}s + \int_0^t \frac{f_u^\delta(t,s)}{(t-s)^{1-\alpha}} \left( \frac{u^\delta(t) - u^*(t)}{\delta} \right) \mathrm{d}s \\
&= \int_0^t \frac{f_y(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} Y_1(s)\,\mathrm{d}s + \int_0^t \frac{f_u(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} v(s)\,\mathrm{d}s \\
&= \int_0^t \frac{f_y^\delta(t,s)}{(t-s)^{1-\alpha}} \left( \frac{y^\delta(t) - y^*(t)}{\delta} - Y_1(s) \right) \mathrm{d}s \\
&\quad + \int_0^t \frac{f_u^\delta(t,s)}{(t-s)^{1-\alpha}} \left( \frac{u^\delta(t) - u^*(t)}{\delta} - v(s) \right) \mathrm{d}s \\
&\quad + \int_0^t \frac{f_y^\delta(t,s) - f_y(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} Y_1(s)\,\mathrm{d}s \\
&\quad + \int_0^t \frac{f_u^\delta(t,s) - f_u(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} v(s)\,\mathrm{d}s \\
&= \int_0^t \frac{f_y^\delta(t,s)}{(t-s)^{1-\alpha}} \left( \frac{y^\delta(t) - y^*(t)}{\delta} - Y_1(s) \right) \mathrm{d}s \\
&\quad + a_1^\delta(t) + a_2^\delta(t) + a_3^\delta(t) \quad t \in [0,T],
\end{aligned} \tag{17}
$$

where

$$
a_1^\delta(t) = \int_0^t \frac{f_u^\delta(t,s)}{(t-s)^{1-\alpha}} \left( \frac{u^\delta(t) - u^*(t)}{\delta} - v(s) \right) \mathrm{d}s,
$$

$$
a_2^\delta(t) = \int_0^t \frac{f_y^\delta(t,s) - f_y(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} Y_1(s)\,\mathrm{d}s,
$$

$$
a_3^\delta(t) = \int_0^t \frac{f_u^\delta(t,s) - f_u(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} v(s)\,\mathrm{d}s.
$$

The dominated convergence theorem (like [1]), and (9) supplies

$$\lim_{\delta\to 0}|a_1^\delta(t)|=0,\qquad \lim_{\delta\to 0}|a_2^\delta(t)|=0,\qquad \lim_{\delta\to 0}|a_3^\delta(t)|=0.$$

The dominated convergence theorem and the extended Gronwall's inequality Lemma 1(like [1]) produces

$$\lim_{\delta\to 0}\left|\frac{y^\delta(t)-y^*(t)}{\delta}-Y_1(t)\right|=0. \tag{18}$$

Sequentially,

$$\begin{aligned}
&\frac{y^\delta(t)-y^*(t)}{\delta}-Y_1(t)-\frac{\delta}{2}Y_2(t)\\
&=\int_0^t \frac{f_y(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}}\left(\frac{y^\delta(s)-y^*(s)}{\delta}-Y_1(s)-\frac{\delta}{2}Y_2(s)\right)\mathrm{d}s\\
&\quad+\frac{f_u(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}}\left(\frac{u^\delta(s)-u^*(s)}{\delta}-v(s)\right)\mathrm{d}s\\
&\quad+\frac{1}{2\delta}\int_0^t \frac{f_{yy}(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}}(y^\delta(s)-y^*(s))^2\mathrm{d}s\\
&\quad-\frac{\delta}{2}\int_0^t \frac{f_{yy}(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}}Y_1^2(s)\mathrm{d}s\\
&\quad+\frac{1}{\delta}\int_0^t \frac{f_{yu}(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}}(y^\delta(s)-y^*(s))(u^\delta(s)-u^*(s))\mathrm{d}s\\
&\quad-\frac{\delta}{2}\int_0^t \frac{f_{yu}(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}}Y_1(s)v(s)\mathrm{d}s\\
&\quad+\frac{1}{2\delta}\int_0^t \frac{f_{uu}(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}}(u^\delta(s)-u^*(s))^2\mathrm{d}s\\
&\quad-\frac{\delta}{2}\int_0^t \frac{f_{uu}(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}}v^2(s)\mathrm{d}s \quad t\in[0,T].
\end{aligned}$$

The extended Gronwall's inequality Lemma 1, (18), and (9) yields

$$\lim_{\delta\to 0}\left|\frac{y^{\delta}(t)-y^{*}(t)}{\delta}-Y_1(t)-\frac{\delta}{2}Y_2(t)\right|=0. \tag{19}$$

Also, by the optimality of $(y^{*}(\cdot),u^{*}(\cdot))$, one has

$$0\leqslant J(u^{\delta}(\cdot))-J(u^{*}(\cdot))=\int_0^T\left[g(t,y^{\delta}(t),u^{\delta}(t))-g(t,y^{*}(t),u^{*}(t))\right]\mathrm{d}t$$

$$=\int_0^T g_y(t,y^{*}(t),u^{*}(t))(y^{\delta}(t)-y^{*}(t))\,\mathrm{d}t+\int_0^T g_u(t,y^{*}(t),u^{*}(t))(u^{\delta}(t)-u^{*}(t))\mathrm{d}t$$

$$+\frac{1}{2}\int_0^T g_{yy}(t,y^{*}(t),u^{*}(t))(y^{\delta}(t)-y^{*}(t))^2\mathrm{d}t$$

$$+\int_0^T g_{yu}(t,y^{*}(t),u^{*}(t))(y^{\delta}(t)-y^{*}(t))(u^{\delta}(t)-u^{*}(t))\,\mathrm{d}t$$

$$+\frac{1}{2}\int_0^T g_{uu}(t,y^{*}(t),u^{*}(t))(u^{\delta}(t)-u^{*}(t))^2\mathrm{d}t+\sum_{i=1}^{m}h_y^i(y^{*}(t_i))(y^{\delta}(t)-y^{*}(t))$$

$$+\frac{1}{2}\sum_{i=1}^{m}h_{yy}^i(y^{*}(t_i))(y^{\delta}(t)-y^{*}(t))^2. \tag{20}$$

Further, from (19), (18), and (14), we get

$$0\leqslant J(u^{\delta}(\cdot))-J(u^{*}(\cdot))=\delta\int_0^T g_y(t,y^{*}(t),u^{*}(t))Y_1(t)\,\mathrm{d}t$$

$$+\frac{\delta^2}{2}\int_0^T g_y(t,y^{*}(t),u^{*}(t))Y_2(t)\,\mathrm{d}t+\delta\int_0^T g_u(t,y^{*}(t),u^{*}(t))v(t)\,\mathrm{d}t$$

$$+\frac{\delta^2}{2}\int_0^T g_{yy}(t,y^{*}(t),u^{*}(t))Y_1^2(t)\,\mathrm{d}t+\delta^2\int_0^T g_{yu}(t,y^{*}(t),u^{*}(t))Y_1(t)v(t)\,\mathrm{d}t$$

$$+\frac{\delta^2}{2}\int_0^T g_{uu}(t,y^*(t),u^*(t))v(t)^2\mathrm{d}t+\delta\sum_{i=1}^m h_y^i(y^*(t_i))Y_1(t_i)$$

$$+\frac{\delta^2}{2}\sum_{i=1}^m h_y^i(y^*(t_i))Y_2(t_i)+\frac{\delta^2}{2}\sum_{i=1}^m h_{yy}^i(y^*(t_i))Y_1^2(t_i)+o(\delta^2)$$

$$=\delta\int_0^T g_y(t,y^*(t),u^*(t))Y_1(t)\mathrm{d}t+\frac{\delta^2}{2}\int_0^T g_y(t,y^*(t),u^*(t))Y_2(t)\mathrm{d}t$$

$$+\delta\int_0^T g_u(t,y^*(t),u^*(t))v(t)\mathrm{d}t+\frac{\delta^2}{2}\int_0^T g_{yy}(t,y^*(t),u^*(t))Y_1^2(t)\mathrm{d}t$$

$$+\delta^2\int_0^T g_{yu}(t,y^*(t),u^*(t))Y_1(t)v(t)\mathrm{d}t+\frac{\delta^2}{2}\int_0^T g_{uu}(t,y^*(t),u^*(t))v(t)^2\mathrm{d}t$$

$$+\delta\int_0^T\sum_{i=1}^m h_y^i(y^*(t_i))1_{[0,t_j)}(s)\Bigg(\frac{f_y(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}Y_1(s)$$

$$+\frac{f_u(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}v(s)\mathrm{d}s\Bigg)\mathrm{d}s$$

$$+\frac{\delta^2}{2}\int_0^T\sum_{i=1}^m h_y^i(y^*(t_i))1_{[0,t_j)}(s)\Bigg(\frac{f_y(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}Y_2(s)\mathrm{d}s$$

$$+\frac{f_{yy}(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}Y_1^2(s)+\frac{2f_{yu}(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}Y_1(s)v(s)$$

$$+\frac{f_{uu}(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}v^2(s)\Bigg)\mathrm{d}s+\frac{\delta^2}{2}\sum_{i=1}^m h_{yy}^i(y^*(t_i))Y_1^2(t_i)+o(\delta^2)$$

$$=\delta\int_0^T\Bigg(g_y(t,y^*(t),u^*(t))+\sum_{i=1}^m h_y^i(y^*(t_i))1_{[0,t_j)}(s)\frac{f_y(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}\Bigg)Y_1(t)\mathrm{d}t$$

$$+\frac{\delta^2}{2}\int_0^T\Bigg(g_y(t,y^*(t),u^*(t))+\sum_{i=1}^m h_y^i(y^*(t_i))1_{[0,t_j)}(s)\frac{f_y(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}\Bigg)Y_2(t)\mathrm{d}t$$

$$+\delta\int_0^T g_u(t,y^*(t),u^*(t))v(t)\,\mathrm{d}t+\frac{\delta^2}{2}\int_0^T g_{yy}(t,y^*(t),u^*(t))Y_1^2(t)\,\mathrm{d}t$$

$$+\delta^2\int_0^T g_{yu}(t,y^*(t),u^*(t))Y_1(t)v(t)\,\mathrm{d}t+\frac{\delta^2}{2}\int_0^T g_{uu}(t,y^*(t),u^*(t))v(t)^2\,\mathrm{d}t$$

$$+\delta\int_0^T\sum_{i=1}^m h_y^i(y^*(t_i))\frac{f_u(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}v(s)\,\mathrm{d}s$$

$$+\frac{\delta^2}{2}\int_0^T\Bigg(\frac{f_{yy}(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}Y_1^2(s)+\frac{2f_{yu}(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}Y_1(s)v(s)$$

$$+\frac{f_{uu}(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}v^2(s)\Bigg)\mathrm{d}s$$

$$+\frac{\delta^2}{2}\sum_{i=1}^m h_{yy}^i(y^*(t_i))Y_1^2(t_i)+o(\delta^2),\quad \delta\to 0. \tag{21}$$

Applying (7), (15), and (16)

$$0\leqslant J(u^\delta(\cdot))-J(u^*(\cdot))$$

$$=\delta\int_0^T\left(-\psi(t)+\int_t^T\frac{f_y(s,t,y^*(t),u^*(t))}{(s-t)^{1-\alpha}}\psi(s)\,\mathrm{d}s\right)Y_1(t)\,\mathrm{d}t$$

$$+\frac{\delta^2}{2}\int_0^T\left(-\psi(t)+\int_t^T\frac{f_y(s,t,y^*(t),u^*(t))}{(s-t)^{1-\alpha}}\psi(s)\,\mathrm{d}s\right)Y_2(t)\,\mathrm{d}t$$

$$+\delta\int_0^T g_u(t,y^*(t),u^*(t))v(t)\,\mathrm{d}t+\frac{\delta^2}{2}\int_0^T g_{yy}(t,y^*(t),u^*(t))Y_1^2(t)\,\mathrm{d}t$$

$$+\delta^2\int_0^T g_{yu}(t,y^*(t),u^*(t))Y_1(t)v(t)\,\mathrm{d}t+\frac{\delta^2}{2}\int_0^T g_{uu}(t,y^*(t),u^*(t))v(t)^2\,\mathrm{d}t$$

$$+\delta\int_0^T\sum_{i=1}^m h_y^i(y^*(t_i))\frac{f_u(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}v(s)\,\mathrm{d}s$$

$$+\frac{\delta^2}{2}\int_0^T\left(\frac{f_{yy}(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}Y_1^2(s)+\frac{2f_{yu}(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}Y_1(s)v(s)\right.$$
$$\left.+\frac{f_{uu}(t_i,s,y^*(s),u^*(s))}{(t_i-s)^{1-\alpha}}v^2(s)\right)\mathrm{d}s+\frac{\delta^2}{2}\sum_{i=1}^m h_{yy}^i(y^*(t_i))Y_1^2(t_i)+o(\delta^2),\ \ \delta\to 0. \tag{22}$$

By using again (5),(15),(16) and Fubini′s theorem,

$$0\leqslant J(u^\delta(\cdot))-J(u^*(\cdot))=-\delta\int_0^T H_u(t)v(t)\mathrm{d}t$$
$$-\frac{\delta^2}{2}\int_0^T H_{yy}(t)Y_1^2(t)\mathrm{d}t-\delta^2\int_0^T H_{yu}(t)Y_1(t)v(t)\mathrm{d}t-\frac{\delta^2}{2}\int_0^T H_{uu}(t)v^2(t)\mathrm{d}t$$
$$+\frac{\delta^2}{2}\sum_{i=1}^m h_{yy}^i(y^*(t_i))Y_1^2(t_i)+o(\delta^2)\quad(\delta\to 0). \tag{23}$$

Lemma 2 provides following expressions

$$\Phi(t,s)=\frac{f_y(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}}+\int_s^t\frac{f_y(t,\tau,y^*(\tau),u^*(\tau))\Phi(\tau,s)}{(t-s)^{1-\alpha}}\mathrm{d}\tau,$$

and

$$Y_1(t)=\int_0^t\frac{f_u(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}}v(s)\mathrm{d}s$$
$$+\int_0^t\Phi(t,s)\left(\int_0^s\frac{f_u(s,\tau,y^*(\tau),u^*(\tau))}{(s-\tau)^{1-\alpha}}v(\tau)\mathrm{d}\tau\right)\mathrm{d}s$$
$$=\int_0^t\left[\frac{f_u(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}}+\int_s^t\frac{\Phi(t,\tau)f_u(\tau,s,y^*(s),u^*(s))}{(\tau-s)^{1-\alpha}}\mathrm{d}\tau\right]v(s)\mathrm{d}s.$$

As a result, we have

$$Y_1(t)=\int_0^t Q(t,s)v(s)\mathrm{d}s, \tag{24}$$

where

$$Q(t,s) = \frac{f_u(t,s,y^*(s),u^*(s))}{(t-s)^{1-\alpha}} + \int_s^t \frac{\Phi(t,\tau)f_u(\tau,s,y^*(s),u^*(s))}{(\tau-s)^{1-\alpha}}\mathrm{d}\tau$$

Substitute (24) into (23), and Fubini's theorem

$$\begin{aligned}\int_0^T H_{yy}(t)Y_1^2(t)\mathrm{d}t &= \int_0^T H_{yy}(t)\left(\int_0^t Q(t,s)v(s)\mathrm{d}s\right)\left(\int_0^t Q(t,\tau)v(\tau)\mathrm{d}\tau\right)\mathrm{d}t\\ &= \int_0^T\int_0^T v(s)\left[\int_{\max\{\tau,s\}}^T Q(t,s)H_{yy}(t)Q(t,\tau)dt\right]v(\tau)\mathrm{d}s\mathrm{d}\tau\\ &= \int_0^T\int_0^T v(s)M(\tau,s)v(\tau)\mathrm{d}s\mathrm{d}\tau,\end{aligned}$$

and

$$\begin{aligned}&\sum_{i=1}^m h_{yy}^i(y^*(t_i))Y_1^2(t_i)\\ &\quad= \int_0^T\int_0^T v(s)\left[\sum_{i=1}^m 1_{[0,t_i)}(s)1_{[0,t_i)}(\tau)Q(t_i,s)h_{yy}^i(y^*(t_i))Q(t_i,\tau)\right]v(\tau)\mathrm{d}\tau\mathrm{d}s.\end{aligned}$$

Therefore,

$$\begin{aligned}0 \leqslant J(u^\delta(\cdot)) - J(u^*(\cdot)) &= -\delta\int_0^T H_u(t)v(t)\mathrm{d}t - \frac{\delta^2}{2}\int_0^T H_{uu}(t)v^2(t)\mathrm{d}t\\ &\quad - \frac{\delta^2}{2}\left[\int_0^T\int_0^T v(\tau)M(\tau,s)v(s)\mathrm{d}s\mathrm{d}\tau\right.\\ &\quad\left. + 2\int_0^T\left[\int_0^t v(s)H_{yu}(t)Q(t,s)v(t)\mathrm{d}t\right]\mathrm{d}s\right] + o(\delta^2) \quad (\delta\to 0), \qquad (25)\end{aligned}$$

where

$$M(\tau,s)=\int\limits_{\max\{\tau,s\}}^{T} Q(t,s)H_{yy}(t)Q(t,\tau)\,\mathrm{d}t$$
$$-\sum_{i=1}^{m} 1_{[0,t_i)}(s)1_{[0,t_i)}(\tau)Q(t_i,s)h^i_{yy}(y^*(t_i))Q(t_i,\tau),\quad \tau,s\in[0,T].$$

We can now prove our theorem, which is based on the cost functional estimation mentioned previously.

The theorem is obtained by dividing the right side of (25) by $\delta^2$ and allowing $\delta$ to approach zero, while taking into account $H_u(t)=0$ in expression (25).

The Theorem 2 was proved.

### 2.2. Proof of the Theorem 1

By the optimality of $(y^*(\cdot),u^*(\cdot))$, we have

$$\begin{aligned}0\leqslant J(u^\delta(\cdot))-J(u^*(\cdot)) &= \int\limits_0^T [g(t,y^\delta(t),u^\delta(t))-g(t,y^*(t),u^*(t))]\,\mathrm{d}t\\
&= \int\limits_0^T g_y(t,y^*(t),u^*(t))(y^\delta(t)-y^*(t))\,\mathrm{d}t\\
&\quad+\int\limits_0^T g_u(t,y^*(t),u^*(t))(u^\delta(t)-u^*(t))\,\mathrm{d}t\\
&\quad+\sum_{i=1}^{m} h^i_y(y^*(t_i))(y^\delta(t)-y^*(t)).\end{aligned}\tag{26}$$

Morever, from (18), and (14), it follows

$$0 \leqslant J(u^{\delta}(\cdot)) - J(u^{*}(\cdot)) = \delta \int_{0}^{T} g_{y}(t, y^{*}(t), u^{*}(t))Y_{1}(t)\mathrm{d}t$$
$$+ \delta \int_{0}^{T} g_{v}(t, y^{*}(t), u^{*}(t))v(t)\mathrm{d}t$$
$$+ \delta \sum_{i=1}^{m} h_{y}^{i}(y^{*}(t_{i}))Y_{1}(t_{i}) + o(\delta), \quad (\delta \to 0). \tag{27}$$

Substituting (7), (15), and (5) into (27), we obtain theexpression below

$$0 \leqslant J(u^{\delta}(\cdot)) - J(u^{*}(\cdot)) = -\delta \int_{0}^{T} H_{u}(t)v(t)\mathrm{d}t + o(\delta), \quad (\delta \to 0). \tag{28}$$

The Theorem 1 was proved.

**Example 1.** Consider the problem

$$y(t) = 1 + t\sqrt{t} + \int_{0}^{t} \frac{ty(s)u(s)}{(t-s)^{\frac{1}{2}}}, \quad a.e \quad t \in [0, 1],$$

$$J(u) = y(1) + \int_{0}^{1} y(s)u(s)\mathrm{d}s \longrightarrow \min, \quad |u| \leqslant 1.$$

We are evaluating the efficiency of the control input $u(t) = 1$ and analyzing its optimality. This particular selection of control corresponds to the solution $1 + t\sqrt{t}$ for the integral equation. Throughout the course of the process represented by $(1 + t\sqrt{t}, 0)$, we have noted the following outcome.

$$\psi(t) = 1, \quad H = 0.$$

Therefore, the control $u(t) = 0$ is identified as a singular control. Clearly, employing the control $u(t) = 0$ yields a performance measure value of $J(u) = 2$. Now, let's investigate if there is an alternative control function that leads to functional values less than 2. We will compute the value of $J$ for the admissible control $u(t) = -\dfrac{1}{2}$.

$$y(t) = 1.$$

Then, we have

$$\frac{1}{2} = J\left(-\frac{1}{2}\right) \leqslant J(0) = 2.$$

This indicates that opting for the control $u(t) = 0$ within the interval $t \in [0, 1]$ is not an optimal.

## References


[1] P. Lin and J. Yong: Controlled singular Volterra integral equations and Pontryagin maximum principle. *SIAM Journal on Control and Optimization*, 58(1), (2020), 136–164. DOI: 10.1137/19M124602X

[2] J.J.Gasimov, J.A. Asadzade and N.I. Mahmudov: Pontryagin maximum principle for fractional delay differential equations and controlled weakly singular Volterra delay integral equations. *arXiv preprint*, (2023). DOI: 10.48550/arXiv.2309.14007

[3] M.J. Mardanov, K.B. Mansimov and N.H. Abdullayeva: Integral necessary condition of optimality of the second order for control problems described by system of integro-differential equations with delay. *Journal of Samara State Technical University*, Ser. Physical and Mathematical Sciences, **22**(2), (2018), 254–268. DOI: 10.14498/vsgtu1597

[4] A.A. Abdullayev and K.B. Mansimov: Multipoint necessary optimality conditions for singular controls in processes described by the system of volterra integral equations. *Cybernetics and Systems Analysis*, **49**(6), (2013), 845–851.

[5] M.J. Mardanov and T.K. Melikov: On the theory of singular optimal controls in dynamic systems with control delay. *Computational Mathematics and Mathematical Physics*. **57**(5), (2017), 749–69. DOI: 10.1134/S0965542517050086

[6] J. Moon: A Pontryagin maximum principle for terminal state-constrained optimal control problems of Volterra integral equations with singular kernels. *AIMS Mathematics*, **8**(10), (2023), 22924–22943. DOI: 10.3934/math.20231166

[7] S.S. Susubov and E.N. Mahmudov: Necessary optimality conditions for quasi-singular controls for systems with Caputo fractional derivatives. *Archives of Control Sciences*, **33**(3), (2023), 463–496. DOI: 10.24425/acs.2023.146955

[8] S.S. Yusubov and E.N. Mahmudov: Optimality conditions of singular controls for systems with Caputo fractional derivatives. *Journal of Industrial and Management Optimization*, **19**(1), (2023). DOI: 10.3934/jimo.2021182