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Checking of the positivity of descriptor linear systems by the use of the shuffle algorithm

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Necessary and sufficient conditions for the positivity of descriptor continuous-time and discrete-time linear systems are established. The shuffle algorithm is applied to transform the state equations of the descriptor systems to their equivalent form for which necessary and sufficient conditions for their positivity have been derived. A procedure for checking the positivity of the descriptor systems is proposed and illustrated by numerical examples.

Key words: descriptor, continuous-time, discrete-time, linear system, positivity, shuffle algorithm

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in [6, 9, 10]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc. The Drazin inverse of matrix to analysis of linear algebraic-differential equations has been applied in [4, 5, 8]. Standard descriptor control systems have been addressed in [7, 11]. Positive descriptor linear systems have been analyzed in [1-3, 13].

In this paper the shuffle algorithm will be applied to checking the positivity of descriptor continuous-time and discrete-time linear systems. Necessary and sufficient conditions for the positivity of descriptor linear systems will be established and a procedure for checking the positivity will be proposed.

The paper is organized as follows. In section 2 definitions of the positivity of the descriptor continuous-time and discrete-time linear systems are recalled and sufficient conditions for the positivity of the systems described by equivalent form of the state equations are established. In section 3 the shuffle algorithm is applied to transform the

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state equation to the equivalent forms. Necessary and sufficient conditions for the positivity of the descriptor systems are given in section 3. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, $\Re^{n\times m}(s)$ – the set of $n\times m$ rational matrices in s with real coefficients, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Positivity of the descriptor linear systems

2.1. Descriptor continuous-time linear systems

Consider the descriptor continuous-time linear system

$$E\dot{x}(t) = Ax(t) + Bu(i), \quad x(0) = x_0 \tag{1a}$$

$$y(t) = Cx(t) \tag{1b}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $E,A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$.

The matrix pair (E,A) or the matrix pencil $Es-A$ is called regular if $\det[Es-A]\neq 0$ for some $s\in C$ (the field of complex numbers). Otherwise the pencil is called singular. It is assumed that the pencil is regular.

Definition 1 [2, 3] *The system* (1) *is called (internally) positive if for every consistent nonnegative initial condition $x_0\in\Re_+^n$ and every nonnegative input $u(t)\in\Re_+^m$ such that $u^{(k)}(t) = \frac{d^k u(t)}{dt^k}\in\Re_+^m$ for $k = 1,\dots,q-1$; $x(t)\in\Re_+^n$, $y(t)\in\Re_+^p$ for $t>0$, where q is index of the matrix pair (E,A)* [10].

In the next section it will be show that the system (1) can be described in the following equivalent form by the equations

$$\dot{x}(t) = \bar{A}x(t) + \bar{B}_0 u(t) + \bar{B}_1 u^{(1)}(t) + \dots + \bar{B}_{q-1}u^{(q-1)}(t) \tag{2a}$$

$$y(t) = Cx(t) \tag{2b}$$

where $\bar{A}\in\Re^{n\times n}$, $\bar{B}\in\Re^{n\times n}$, $k = 0,1,\dots,q-1$. Note that the equations (1b) and (2b) have the same form.

Theorem 1 *The descriptor continuous-time linear system* (2) *is positive if and only if*

$$\bar{A}\in M_n,\ \ \bar{B}_k\in\Re_+^{n\times m},\ \ k = 0,1,\dots,q-1,\ \ C\in\Re_+^{p\times n} \tag{3}$$

where M_n is the set of $n \times n$ Metzler matrices.

Proof Assuming $u^{(k)}(t) = 0$ for $k = 0, 1, \dots, q-1$ and $t \geqslant 0$ from (2a) we have

$$\dot{x}(t) = \bar{A}x(t). \tag{4}$$

The solution of (4) has the form

$$x(t) = e^{\bar{A}t}x_0$$

and it is nonnegative $x(t) \in \Re_+^n$ for $t \geqslant 0$ and $x_0 \in \Re_+^n$ if and only if $\bar{A} \in M_n$ [9, 10]. From the equation (2b) it follows that $y(t) \in \Re_+^p$ for every $t \geqslant 0$, if and only if $C \in \Re_+^{p \times n}$. Note that the solution of the equation (2a) for $x_0 = 0$ and every $u^{(k)}(t) \in \Re_+^m$ for $k = 0, 1, \dots, q-1$ if and only if $\bar{B}_k \in \Re_+^{n \times m}$. □

2.2. Descriptor discrete-time linear systems

Consider the descriptor discrete-time linear system

$$Ex_{i+1} = Ax_i + Bu_i, \quad i \in Z_+ = \{0, 1, \dots\} \tag{5a}$$

$$y_i = Cx_i \tag{5b}$$

where $x_i \in \Re^n$, $u_i \in \Re^m$, $y_i \in \Re^p$ are the state, input and output vectors and $E, A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$.

It is assumed that the pencil $Ez - A$ is regular, i.e. $\det[Ez - A] \neq 0$ for some $z \in C$.

Definition 2 *The systems* (5) *is called (internally) positive if for every consistent nonnegative initial condition $x_0 \in \Re_+^n$ and every input $u_i \in \Re_+^m$, $i \in Z_+$, $x_i \in \Re_+^n$ and $y_i \in \Re_+^p$, $i \in Z_+$.*

In the next section it will be show that the system (5) can be described in the following equivalent form by the equations

$$x_{i+1} = \bar{A}x_i + \bar{B}_0 u_i + \bar{B}_1 u_{i+1} + \dots + \bar{B}_{q-1} u_{i+q-1} \tag{6a}$$

$$y_i = Cx_i, \quad i \in Z_+ \tag{6b}$$

where $\bar{A} \in \Re^{n \times n}$, $\bar{B}_k \in \Re^{n \times m}$, $k = 0, 1, \dots, q-1$. Note that the equations (5b) and (6b) have the same form.

Theorem 2 *The descriptor discrete-time linear system* (6) *is positive if and only if*

$$\bar{A} \in \Re_+^{n \times n}, \;\; \bar{B}_k \in \Re_+^{n \times m}, \;\; k = 0, 1, \dots, q-1; \;\; C \in \Re_+^{p \times n}. \tag{7}$$

Proof Sufficiency. The solution of (6a) has the form

$$x_i = \bar{A}^i x_0 + \sum_{k=0}^{i-1} \bar{A}^{i-k-1}(\bar{B}_0 u_k + \bar{B}_1 u_{k+1} + \dots + \bar{B}_{q-1} u_{k+q-1}) \tag{8}$$

From (8) and (6b) it follows that if (7) holds then $x_i \in \Re_+^n$ and $y_i \in \Re_+^p$ for every $x_0 \in \Re_+^n$ and $u_i \in \Re_+^m$, $i \in Z_+$.
Necessity. Let $u_i = 0$ for $i \in Z_+$. Then from (6) for $i = 0$ we have $x_1 = \bar{A}x_0 \in \Re_+^n$ and $y_0 = Cx_0 \in \Re_+^p$. This implies $\bar{A} \in \Re_+^{n\times n}$ and $C \in \Re_+^{p\times n}$ since $x_0 \in \Re_+^n$ is arbitrary. □

3. Transformation of the state equations by the use of the shuffle algorithm

3.1. Continuous-time linear systems

The following elementary row operations will be used:

1. Multiplication of the ith row by a real number c. This operation will be denoted by $L[i \times c]$.
2. Addition to the ith row of the jth row multiplied by a real number c. This operation will be denoted by $L[i + j \times c]$.
3. Interchange of the ith and jth rows. This operation will be denoted by $L[i, j]$.

Using the shuffle algorithm [11] we shall transform the state equation (1a) with $\det E = 0$ and regular pencil $Es - A$ to the equivalent form (2a).

Performing elementary row operations on the array

$$E \ \ A \ \ B \tag{9}$$

or equivalently on (1a) we get

$$\begin{array}{ccc} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{array} \tag{10}$$

and

$$E_1\dot{x}(t) = A_1x(t) + B_1u(t) \tag{11a}$$

$$0 = A_2x(t) + B_2u(t) \tag{11b}$$

where E_1 has full row rank. Differentiation of (11b) with respect to time yields

$$A_2\dot{x}(t) = -B_2\dot{u}(t). \tag{12}$$

The equations (11a) and (12) can be written in the form

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}\dot{x}(t) = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}x(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix}u(t) + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix}\dot{u}(t). \tag{13}$$

The array

$$\begin{array}{cccc} E_1 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & -B_2 \end{array} \tag{14}$$

can be obtained from (10) by performing a shuffle. If matrix

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} \tag{15}$$

is nonsingular then solving (13) we obtain

$$\dot{x}(t) = \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \left(\begin{bmatrix} A_1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} \dot{u}(t) \right).$$

If the matrix (15) is singular then performing elementary row operations on (14) (or equivalently on (13)) we obtain

$$\begin{matrix} E_2 & A_3 & B_3 & C_1 \\ 0 & A_4 & B_4 & C_2 \end{matrix} \tag{16}$$

and

$$E_2\dot{x}(t) = A_3x(t) + B_3u(t) + C_1\dot{u}(t) \tag{17a}$$

$$0 = A_4x(t) + B_4u(t) + C_2\dot{u}(t) \tag{17b}$$

where E_2 has full row rank and $\operatorname{rank} E_2 \geqslant \operatorname{rank} E_1$. Differentiation of (17b) with respect to time yields

$$A_4\dot{x}(t) = -B_4\dot{u}(t) - C_2\ddot{u}(t). \tag{18}$$

The equations (17a) and (18) can be written in the form

$$\begin{bmatrix} E_2 \\ A_4 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} A_3 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} B_3 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} C_1 \\ -B_4 \end{bmatrix} \dot{u}(t) + \begin{bmatrix} 0 \\ -C_2 \end{bmatrix} \ddot{u}(t). \tag{19}$$

The array

$$\begin{matrix} E_2 & A_3 & B_3 & C_1 & 0 \\ A_4 & 0 & 0 & -B_4 & -C_2 \end{matrix} \tag{20}$$

can be obtained from (16) by performing a shuffle. If matrix

$$\begin{bmatrix} E_2 \\ A_4 \end{bmatrix} \tag{21}$$

is nonsingular, we can solve (19) in a similar way to (13). If the matrix (21) is singular, we repeat the procedure for (20). After $q-1$ steps we obtain a nonsingular matrix

$$\begin{bmatrix} E_{q-1} \\ A_{q+1} \end{bmatrix}$$

and

$$\dot{x}(t) = \begin{bmatrix} E_{q-1} \\ A_{q+1} \end{bmatrix}^{-1} \left(\begin{bmatrix} A_q \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} B_q \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} C_q \\ 0 \end{bmatrix} \dot{u}(t) + \cdots + \begin{bmatrix} 0 \\ -H_q \end{bmatrix} u^{(q-1)}(t) \right) \tag{22}$$

where $u^{(k)}(t) = d^k u(t)/dt^k$. From the above considerations, we have the following procedure.

Procedure 1

Step 1 Performing elementary row operations on (9) gives (10) where E_1 has full row rank.

Step 2 Shuffle array (10) to (14). If the matrix

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} \tag{23}$$

is nonsingular, find the desired solution from (13). If the matrix is singular, performing elementary row operations on (14) gives (16). If the pencil is regular, by *Step* 1 and *Step* 2 we finally obtain a regular system (22).

Remark 1 Using the shuffle algorithm we may also find the index q of the pair (E, A). The index is equal to the number of performed shuffles to find (22).

Example 1 Using Procedure 1 transform the equation

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \tag{24}$$

to the form (2a) and find the index q of the pair (E, A).

Step 1 Performing on the array

$$E \quad A \quad B = \begin{array}{cccccccc} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{array}$$

the elementary row operations $L[2+1\times 1]$, $L[2,3]$ we obtain

$$\begin{matrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{matrix} = \begin{matrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \end{matrix}. \tag{25}$$

Step 2 Performing the shuffle on (25) we get

$$\begin{matrix} E_1 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & -B_2 \end{matrix} = \begin{matrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}. \tag{26}$$

The matrix

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

is nonsingular and from (13) we obtain

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \left(\begin{bmatrix} A_1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} \dot{u}(t) \right) \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \dot{u}(t). \end{aligned} \tag{27}$$

In this case we have performed only one shuffle. Therefore, the index is equal $q = 2$.

Theorem 3 *The descriptor continuous-time linear system* (1) *is positive if and only if*

$$\begin{aligned} &\begin{bmatrix} E_{q-1} \\ A_{q+1} \end{bmatrix}^{-1} \begin{bmatrix} A_q \\ 0 \end{bmatrix} \in M_n, \quad \begin{bmatrix} E_{q-1} \\ A_{q+1} \end{bmatrix}^{-1} \begin{bmatrix} B_q \\ 0 \end{bmatrix} \in \Re_+^{n\times m}, \dots \\ &\dots, \begin{bmatrix} E_{q-1} \\ A_{q+1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -H_q \end{bmatrix} \in \Re_+^{n\times m}. \end{aligned} \tag{28}$$

Proof It is well-known [11] that the equations (1a) and (22) have the same solution. By Theorem 2 the equation (22) has a nonnegative solution $x(t) \in \Re_+^n$, $t \geqslant 0$ if and only if the conditions (28) are satisfied. □

Example 2 (continuation of Example 1) The descriptor continuous-time system described by the equation (24) is positive since the matrix

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \begin{bmatrix} A_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is a Metzler matrix and the matrices

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

have nonnegative entries.

3.2. Discrete-time systems

In a similar way as for continuous-time systems using the shuffle algorithm we shall transform the state equation (5a) with $\det E = 0$ and regular pencil $Ez - A$ to the equivalent form (6).

Performing elementary row operations on the array (9) (or equivalently on (5a)) we have (10) and

$$E_1 x_{i+1} = A_1 x_i + B_1 u_i \tag{29a}$$

$$0 = A_2 x_i + B_2 u_i \tag{29b}$$

where E_1 has full row rank. Substituting in (29b) i by $i+1$ we obtain

$$A_2 x_{i+1} = -B_2 u_{i+1}. \tag{30}$$

The equations (29a) and (30) can be written in the form

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix} x_{i+1} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} x_i + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} u_{i+1}. \tag{31}$$

The array

$$\begin{matrix} E_1 & A_1 & B_1 & 0 \\ A_2 & 0 & 0 & -B_2 \end{matrix} \tag{32}$$

can be obtained from (10) by performing a shuffle. If matrix (15) is nonsingular then solving (31) we obtain

$$x_{i+1} = \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \left(\begin{bmatrix} A_1 \\ 0 \end{bmatrix} x_i + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} u_{i+1} \right). \tag{33}$$

If the matrix (15) is singular then performing elementary row operations on (32) we obtain (16) and

$$E_2x_{i+1} = A_3x_i + B_3u_i + C_1u_{i+1} \tag{34a}$$

$$0 = A_4x_i + B_4u_i + C_2u_{i+1} \tag{34b}$$

where E_2 has full row rank and $\operatorname{rank} E_2 \geqslant \operatorname{rank} E_1$. Substituting in (34b) i by $i+1$ we obtain

$$A_4x_{i+1} = -B_4u_{i+1} - C_2u_{i+2}. \tag{35}$$

The equations (34a) and (35) can be written as

$$\begin{bmatrix} E_2 \\ A_4 \end{bmatrix} x_{i+1} = \begin{bmatrix} A_3 \\ 0 \end{bmatrix} x_i + \begin{bmatrix} B_3 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} C_1 \\ -B_4 \end{bmatrix} u_{i+1} + \begin{bmatrix} 0 \\ -C_2 \end{bmatrix} u_{i+2}. \tag{36}$$

The array

$$\begin{matrix} E_2 & A_3 & B_3 & C_1 & 0 \\ A_4 & 0 & 0 & -B_4 & -C_2 \end{matrix} \tag{37}$$

can be obtained from (16) by performing a shuffle. If matrix (21) is nonsingular, we can find x_{i+1} from (36). If the matrix is singular, we repeat the procedure for (37). If the pencil is regular then after $q-1$ steps we obtain a nonsingular matrix

$$\begin{bmatrix} E_{q-1} \\ A_{q+1} \end{bmatrix}$$

and

$$x_{i+1} = \begin{bmatrix} E_{q-1} \\ A_{q+1} \end{bmatrix}^{-1} \left(\begin{bmatrix} A_q \\ 0 \end{bmatrix} x_i + \begin{bmatrix} B_q \\ 0 \end{bmatrix} u_i + \begin{bmatrix} C_q \\ 0 \end{bmatrix} u_{i+1} + \ldots + \begin{bmatrix} 0 \\ -H_q \end{bmatrix} u_{i+q-1} \right). \tag{38}$$

Example 3 Using Procedure 1 transform the equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & 0 \end{bmatrix} x_{i+1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -2 & 3 \end{bmatrix} x_i + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} u_i \tag{39}$$

to the form (6a) and find the index q of the pair (E,A).

Step 1 Performing on the array

$$E \quad A \quad B = \begin{matrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 \\ -2 & -2 & 0 & -2 & -2 & 3 & 2 & 0 \end{matrix}$$

the elementary row operations $L[3+1\times 2]$, $L[3+2\times 1]$ we obtain

$$\begin{matrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{matrix} = \begin{matrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2 & 4 & 1 \end{matrix}. \tag{40}$$

Step 2 Performing the shuffle on (40) we get

$$\begin{matrix} E_1 & A_1 & B_1 & 0 \\ -A_2 & 0 & 0 & B_2 \end{matrix} = \begin{matrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 4 & 1 \end{matrix}. \tag{41}$$

The matrix

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is nonsingular and from (33) we obtain

$$\begin{aligned} x_{i+1} &= \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \left(\begin{bmatrix} A_1 \\ 0 \end{bmatrix} x_i + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} u_{i+1} \right) \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix} u_i + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 0.5 \end{bmatrix} u_{i+1}. \end{aligned} \tag{42}$$

In this case the index is also equal $q = 1$.

Theorem 4 *The descriptor discrete-time linear system* (2) *is positive if and only if*

$$\begin{gathered} \begin{bmatrix} E_{q-1} \\ A_{q+1} \end{bmatrix}^{-1} \begin{bmatrix} A_q \\ 0 \end{bmatrix} \in \Re_+^{n\times n}, \quad \begin{bmatrix} E_{q-1} \\ A_{q+1} \end{bmatrix}^{-1} \begin{bmatrix} B_q \\ 0 \end{bmatrix} \in \Re_+^{n\times m}, \ldots \\ \begin{bmatrix} E_{q-1} \\ A_{q+1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -H_q \end{bmatrix} \in \Re_+^{n\times m}. \end{gathered} \tag{43}$$

Proof is similar to the proof of theorem 3.

Example 4 (continuation of Example 3) The descriptor discrete-time system described by the equation (39) is positive since the matrices

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} A_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 0.5 \end{bmatrix}$$

have nonnegative entries.

4. Concluding remarks

Checking the positivity of descriptor continuous-time and discrete-time linear systems by the use of the shuffle algorithm has been addressed. Necessary and sufficient conditions for the positivity of descriptor linear systems described by the equations (2) and (6) have been established (Theorem 1 and 2). The shuffle algorithm has been applied to transform the equations (1a) and (5a) to their equivalent forms (2a) and (6a). Procedure 1 based on the shuffle algorithm has been proposed and illustrated by numerical examples. Necessary and sufficient conditions for the positivity of the descriptor systems described by the equations (1) and (5) have been given (Theorem 3 and 4). The shuffle algorithm allows us also to find the index q of the pair (E,A). The considerations can be extended for positive fractional linear systems [12].

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