

# Nonnegativity, stability analysis of linear discrete-time positive descriptor systems: an optimization approach

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**Abstract.** This paper discusses an efficient approach to the analysis of positivity and stability of linear discrete-time positive descriptor system. Its main objective is to convert the necessary and sufficient condition of characterizing positivity and stability of positive descriptor systems into an optimization problem, then propose a method to numerically check the positivity and stability of the positive linear discrete-time systems. Comparing with the other methods now available, the approach presented in this paper is less theoretical and easier to implement. Examples are provided in order to validate results.

**Key words:** descriptor systems, positive systems, positivity, stability, optimization.

## 1. Introduction

Descriptor systems arise naturally in many significant applications, for example, in mechanical body motion, chemical processing, power generation, network fluid flow, aircraft guidance and so on [1]. It is common to call these systems descriptor systems as they allow keeping the physical significance of state variables. Since the 1970s, abundant literature has shown the advantages of the generic specificity of descriptor systems. As descriptor systems describe an important class of systems of both theoretical and practical significance, they have been studied for many years since the 1990s. A remarkable number of results have been produced, particularly on some extensions of standard state-space control theory to descriptor systems in fields such as controllability and observability [2], regularity and regularization [3], admissibility and admissibilization [4], linear quadratic optimal control [5],  $H_2$  and  $H_\infty$  analysis and control synthesis [6], observer design [7], stochastic descriptor systems controller design [8, 9], and so on.

Positive descriptor systems (we refer to internal positive systems in this paper, which means the input and state variable are positive) are a class of systems that all the states are nonnegative for any nonnegative initial condition at the nonnegative time instants. Positive descriptor systems can be used to model many related to electric charge, populations, network communication, number of molecules, etc. [10]. They are more complicated than generic descriptor systems due to the special positivity constraints. Positive linear systems have drawn attention of many researchers in recent years, but the results are not satisfying [11–16]. This paper mainly focuses on the nonnegativity and stability analysis of the linear discrete-time positive descriptor systems. Characterization and stability analysis for

positive descriptor systems of continuous-time and discrete-time cases are given in [17, 18]. An algorithm allowing to check the nonnegativity of generic descriptor systems was provided in [19]. A method for checking the positivity of descriptor linear systems with singular matrix pencil was given in [20]. In [21], a necessary and sufficient condition to guarantee the admissibility of positive continuous-time systems was constructed. In [22], authors further extended the results for the positive continuous-time descriptor systems in [17, 23]. Motivated by the above conclusions and by [24], we present a novel numerical method which can be used to check nonnegativity and stability of the linear discrete-time descriptor systems based on equivalent stability conditions. Comparing with the results in [17, 23], our method is less theoretical and is easily implemented.

The paper is organized as follows. In Section 2, some preliminaries are given. The checking methods of positivity of stability are presented in Section 3 and Section 4. Section 5 includes examples validating our methods. The paper ends with conclusions and cited references.

**Notation:**  $C$  denotes the field of complex numbers.  $N$  is the set of nonnegative integers.  $R^n$  is the vector of real numbers, and  $R^{n \times n}$  is the space of  $n \times n$  matrices with real entries.  $I_n$  is the  $n \times n$  identity matrix. For  $v \in R^n$ ,  $v > 0$  ( $\geq 0$ ) means all components of  $v$  are positive (nonnegative). Similarly, for  $A \in R^{n \times n}$ ,  $A > 0$  ( $\geq 0$ ) means all components of  $A$  are positive (nonnegative).

## 2. Preliminaries

Consider the descriptor linear discrete-time system

$$Ex(k+1) = Ax(k), \quad (1)$$

where  $E, A \in R^{n \times n}$  are constant matrices,  $x \in R^n$  is the state variable,  $k \in N$  denotes the discrete-time instant. The systems  $E$

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may be descriptor when matrix  $E$  is singular, while the systems can be transformed as a standard linear system by multiply the inverse of  $E$  on both sides when  $E$  is nonsingular. In this paper, we always assume that  $E$  is singular since we mainly talk about the descriptor systems.

**Definition 1.** Let  $E, A \in R^{n \times n}$ , the matrix pair  $(E, A)$ , or a matrix pencil  $\lambda E - A$ , is called regular if  $\det(\lambda E - A) \neq 0$  for some  $\lambda \in C$ . The regularity of matrix pencil  $\lambda E - A$  ensure the admissibility of systems (1), so we assume the regularity of matrix pencil  $\lambda E - A$  in this paper.

**Theorem 1.** [17] Let  $(E, A)$  be a regular matrix pair. Then, the solution of (1) has the form

$$x(k) = (\hat{E}^D \hat{A})^k \hat{E}^D \hat{E} v, \quad (2)$$

for some  $v \in R^n$ , the matrices  $\hat{E}$  and  $\hat{A}$  are given by  $\hat{E} = (\lambda E - A)^{-1} E$ ,  $\hat{A} = (\lambda E - A)^{-1} A$ , with  $\lambda$  is any complex number. The products  $\hat{E}^D \hat{E}$ ,  $\hat{E}^D \hat{A}$  do not depend on the value of  $\lambda$  in [17].

For any matrix  $A \in R^{n \times n}$ , the Drazin inverse of  $A$ , denoted by  $A^D$ , is the unique solution of the three equations, such that  $AX = XA$ ,  $XAX = A$ ,  $XA^{k+1} = A^k$ , where  $k$  is the smallest non-negative integer such that  $\text{rank}(A^k) = \text{rank}(A^{k+1})$ . Moreover, the Drazin inverse can be computed by means of the Jordan canonical form. It is well known that for any given matrix  $A$ , it can be decomposed as

$$A = T \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} T^{-1}, \quad (3)$$

so the Drazin inverse of matrix  $A$  can be computed by the following (4),

$$A^D = T \begin{pmatrix} M^{-1} & 0 \\ 0 & 0 \end{pmatrix} T^{-1}, \quad (4)$$

where  $M$  is invertible and  $N$  is a nilpotent matrix.

In order to derive our results about the positivity and stability of system (1), we shall make use of the following lemma.

**Lemma 1.** Consider the matrix equations  $AXB = C$ , where  $A, B$  and  $C$  are given matrices and the matrix  $X$  is unknown. We can rewrite this matrix equation into a linear equation systems as the following:

$$(B^T \otimes A) \text{vec}(X) = \text{vec}(AXB) = \text{vec}(C). \quad (5)$$

Here,  $\text{vec}(X)$  denotes the vectorization of the matrix  $X$  formed by stacking the columns of  $X$  into a single column vector,  $\otimes$  is the Kronecker product of two matrices.

**Proof:** lemma 1 can be proved by the Kronecker product of matrices.

Consider the inhomogeneous system of linear equations

$$Ax = b, \quad (6)$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

where  $A$  is the coefficient matrix,  $b$  is a nonzero vector. Taking into account the problem of non-negative solution of linear system (6), we can transform it into an optimization problem, the nonnegative solution of system (6) can be obtained by solving the following (7).

The nonnegative solution of systems (6) can be obtained by solving the following optimization problem (7). The optimal solution of optimization problem (7) is the nonnegative solution of linear equation systems (6). Rewrite matrix  $A = [a_1, a_2, \dots, a_n]$ , where  $a_i, i = 1, 2, \dots, n$  is the column vector of matrix  $A$ . Define function  $f(x) = \sum_{i=1}^n |b_i - a_i^T x|$ , find the nonnegative solution (or approximation solution) of systems (6) can be obtained by finding the solution of the following optimization problem (7).

$$\begin{aligned} \min & f(x) \\ \text{s.t.} & x \geq 0, \quad x \neq 0. \end{aligned} \quad (7)$$

We formulate the conclusion in the following lemma 2.

**Lemma 2.** Find the solution of optimization problem (7) is equivalent to solve the following optimization problem (8).

$$\begin{aligned} \min & g(y) = (u^T, \theta^T) \begin{pmatrix} t \\ x \end{pmatrix} = \sum_{i=1}^n t_i \\ \text{s.t.} & B \begin{pmatrix} t \\ x \end{pmatrix} \geq \begin{pmatrix} b \\ -b \end{pmatrix}, \quad \begin{pmatrix} t \\ x \end{pmatrix} \geq 0 \end{aligned} \quad (8)$$

$$\text{where } B = \begin{pmatrix} I & A \\ I & -A \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}, \quad y = \begin{pmatrix} t \\ x \end{pmatrix},$$

$\theta^T = (0, 0, \dots, 0)$ . The optimal solution of (8) is  $y^0 = \begin{pmatrix} t^0 \\ x^0 \end{pmatrix}$ ,  $x^0$  is the nonnegative solution of optimization (7), moreover,  $f(x^0) = g(y^0) = \sum_{i=1}^n t_i^0$ .

**Proof.** From (8), we have

$$t \geq b - Ax, \quad t \geq -(b - Ax), \quad (9)$$

so we have  $t_i \geq b_i - a_i^T x, t_i \geq -(b_i - a_i^T x)$ , such that

$$t_i \geq |b_i - a_i^T x|, \quad i = 1, 2, \dots, n. \quad (10)$$

Suppose  $y^0 = \begin{pmatrix} t^0 \\ x^0 \end{pmatrix}$  is the solution of (8), then, we shall prove that  $x^0$  is the nonnegative solution of (7) by contradiction.

If not, there must exist a nonnegative vector  $x^*$  such that  $f(x^*) < f(x^0)$ , let

$$t_i^* = |b_i - a_i^T x^*|, \quad i = 1, 2, \dots, n, \quad (11)$$

then we have  $y^* = \begin{pmatrix} t^* \\ x^* \end{pmatrix} \geq 0$ ,

where  $t^* = (t_1^*, t_2^*, \dots, t_n^*)$ ,  $x^* = (x_1^*, x_2^*, \dots, x_n^*)$ ,

$$B y^* = \begin{pmatrix} t^* & A x^* \\ t^* & -A x^* \end{pmatrix} \geq \begin{pmatrix} b \\ -b \end{pmatrix}, \quad (12)$$

which means  $y^*$  satisfies (8) and  $g(y^*) = \sum_{i=1}^n t_i^* = \sum_{i=1}^n |b_i - a_i^T x^*| = f(x^*)$ . Again because  $t^0$  satisfies (10) and  $f(x^*) < f(x^0)$ , we have  $g(y^*) = f(x^*) < f(x^0) = \sum_{i=1}^n |b_i - a_i^T x^0| < \sum_{i=1}^n t_i^0 = g(y^0)$ , which is contradict to the fact that  $y^0$  is the solution of (8).

In the sequel, we shall prove

$$t_i^0 = |b_i - a_i^T x^0|, \quad i = 1, 2, \dots, n. \quad (13)$$

If (13) doesn't hold for all  $i = 1, 2, \dots, n$  from (10), we can deduce that  $g(y^0) = \sum_{i=1}^n t_i^0 = \sum_{i=1}^n |b_i - a_i^T x^0| > f(x^0)$ , then similar to the former proof, we derive a contradiction that  $y^0$  is the solution of (8), then we have (13) holds, i.e.  $g(y^0) = \sum_{i=1}^n t_i^0 = \sum_{i=1}^n |b_i - a_i^T x^0| = f(x^0)$ , which completes the proof.

### 3. Positivity

This section mainly studies the characterization of positivity of system (1).

**Lemma 3** (Equivalent system). Assume matrix  $(\lambda E - A)$  is regular,  $P := \hat{E}^D \hat{E}$  and  $\bar{A} = \hat{E}^D \hat{A}$ ,  $\text{Im}(P)$  is the image of matrix  $P$ , then system (1) is equivalent to the following system (14).

$$\begin{aligned} x(k+1) &= \bar{A}x(k) \\ x(0) &\in \text{Im}(P). \end{aligned} \quad (14)$$

**Proof.** From the definition of the projector matrix  $P = \hat{E}^D \hat{E}$ ,  $\bar{A} = \hat{E}^D \hat{A}$ , it has the following properties: (1)  $P$  is idempotent or a projector (i.e.  $P^2 = P$ ), (2)  $P\bar{A} = \bar{A}P = \bar{A}$ , (3). For any solution  $x(k)$  to systems (1), these properties has been proved in [22] for linear continuous time system, it is the same for the discrete time case.

Multiply matrix  $(\lambda E - A)^{-1}$  to the left on both sides of (1), and then multiply  $\hat{E}^D$  to the left on both sides, so we have  $\hat{E}^D \hat{E}x(k+1) = Px(k) = \hat{E}^D \hat{A}x(k) = \bar{A}x(k)$ , i.e.,  $Px(k+1) = \bar{A}x(k)$ , multiply matrix  $P$  to the left on both sides, from the properties above, and the proof is completed.

**Lemma 4** [18, 25]. A discrete system

$$x(k+1) = Ax(k), \quad k \in N \quad (15)$$

is positive if and only if  $A \geq 0$ .

For a standard linear discrete systems, the positivity of the systems for the given positive initial state  $x(0)$  can be checked by the nonnegativity of state matrix  $A$ . For more details refer to [25].

**Lemma 5** [23]. Let  $M, N \in R^{n \times n}$  be matrices with approximate sizes,  $x \in R^n$ . The following statements are equivalent:

- 1)  $Mx \geq 0$  implies that  $Nx \geq 0$ .
- 2) There exists  $H \geq 0$  satisfying the matrix equation  $N = HM$ .

**Lemma 6.** For systems (1) and (14), the following statements are equivalent.

- 1) Systems (1) (or (14)) is positive for the set of nonnegative admissible initial conditions  $S = \text{Im}(P) \cap R_n^+$ .
- 2) There exists a matrix  $H$  that satisfies the following conditions

$$\begin{cases} H \geq 0 \\ \bar{A} = HP. \end{cases} \quad (16)$$

**Proof.** Proof of Lemma 6 can be obtained from lemma 4 and lemma 5. Also it can be found in [23], we use the same conclusion in [23] to derive our method in this paper.

**Remark 1.** If the set of nonnegative admissible initial conditions  $S = \text{Im}(P) \cap R_n^+$  has only one point  $0 \in R^n$ , statement (2) also holds, but system (1) or (14) will only have a trivial solution. In Section 5, example 4 illustrates this fact. But [23] does not talk about this fact.

**Theorem 2.** Under the assumption that matrix pencil  $\lambda E - A$  is nonsingular, the nonnegative admissible initial set  $S = \text{Im}(P) \cap R_n^+$  is nonempty and nontrivial, the following statements are equivalent.

- 1) System (1) (or (14)) is positive.
- 2) The following optimization problem has at least one non-negative solution  $H_p$ .

$$\begin{aligned} \min g_p(y) &= \sum_{i=1}^{n^2} t_i \\ \text{s.t.} \quad &\begin{cases} B_p \begin{pmatrix} t \\ \text{vec}(H) \end{pmatrix} \geq \begin{pmatrix} \text{vec}(\bar{A}) \\ -\text{vec}(\bar{A}) \end{pmatrix} \\ \begin{pmatrix} t \\ \text{vec}(H) \end{pmatrix} \geq 0 \end{cases} \end{aligned} \quad (17)$$

where

$$B_p = \begin{pmatrix} I & p^T \otimes I \\ I & -p^T \otimes I \end{pmatrix}.$$

**Proof.** According to lemma 3, systems (1) (or (14)) is positive is equivalent to  $\bar{A} \geq 0$  and lemma 4. By the Lemma 5, the Matrix equation of  $\bar{A} = HP$ ,  $H \geq 0$  can be transformed into linear equation

$$[P^T \otimes I] \text{vec}(H) = \text{vec}(\bar{A}). \quad (18)$$

Then, from Lemma 2, we transform the problem of non-negative solution of linear system of equations (18) into an optimization problem. Thus, we obtain the statement (2) of Theorem 2.

#### 4. Stability

**Definition 4.** System (1) is said to be stable if for any initial condition  $x(0) = x_0 > 0$ , we have that  $x(k)$  goes to zero as  $k$  goes to infinity.

**Theorem 3** [22]. Let  $N$  be a nonnegative matrix and consider the following standard linear system

$$x(k+1) = Nx(k). \quad (19)$$

Then the following statements are equivalent.

- 1)  $N$  is Schur, or equivalently, the system (18) is stable for any initial condition.
- 2) There exists  $x(0) = x_0 > 0$  such that

$$\lim_{k \rightarrow \infty} N^k x_0 = 0.$$

- 3)  $(N - I)^{-1}$  exists and  $(N - I)^{-1} < 0$ .

**Theorem 4.** Under the assumption that matrix  $I - A$  is nonsingular, the following statements are equivalent.

- 1) System (1) (or (14)) is stable.
- 2) The following optimization problem has at least one non-negative solution  $H_s$ .

$$\begin{cases} Q \geq 0 \\ (I - \bar{A})Q = I. \end{cases} \quad (20)$$

- 3) There exists a matrix  $Q$  that satisfies the following conditions

$$\begin{aligned} \min g_s(y) &= \sum_{i=1}^{n^2} t_i \\ \text{s.t. } &\begin{cases} B_s \begin{pmatrix} t \\ \text{vec}(Q) \end{pmatrix} \geq \begin{pmatrix} \text{vec}(I) \\ -\text{vec}(I) \end{pmatrix} \\ \begin{pmatrix} t \\ \text{vec}(Q) \end{pmatrix} \geq 0 \end{cases} \end{aligned} \quad (21)$$

where

$$B_s = \begin{pmatrix} I & I \otimes (I - \bar{A}) \\ I - I \otimes (I - \bar{A}) \end{pmatrix}.$$

**Proof.** From Theorem 3(3), matrix  $I - \bar{A}$  is invertible is equivalent to the fact that there exists a matrix  $Q$  such that  $(I - \bar{A})Q = I$ , and  $Q \geq 0$ , so the proof of (2) is finished, proof of (3) can be finished from Theorem 2.

#### 5. Examples

In order to illustrate the proposed methods, we use examples in the references.

**Example 1** (Example 1 in [26]). Let (1) be given by

$$E = \begin{pmatrix} 1.875 & 0 & -1.625 \\ -0.250 & 0 & 0.750 \\ -0.875 & 0 & 0.625 \end{pmatrix}, \quad A = \begin{pmatrix} 6.75 & -1 & -6.25 \\ -0.5 & 0 & 1.5 \\ -3.75 & 1 & 3.25 \end{pmatrix}.$$

We choose  $\lambda = 0$  (regularity can be satisfied). By some calculation, we have

$$\hat{A} = \begin{pmatrix} -1.0000 & 0 & 0 \\ 0 & -1.0000 & 0 \\ 0 & 0 & -1.0000 \end{pmatrix},$$

$$\hat{E} = \begin{pmatrix} -0.2500 & 0 & -1.2500 \\ -0.3333 & 0 & 0.3333 \\ 0.0833 & 0 & 0.5833 \end{pmatrix}, \quad \hat{E}^D = \begin{pmatrix} -3.5 & 0 & 1.5 \\ -3 & 0 & 3 \\ -0.5 & 0 & 1.5 \end{pmatrix},$$

$$P = \begin{pmatrix} 1.0000 & 0 & 0 \\ 1.0000 & 0 & -1.0000 \\ 0.0000 & 0 & 1.0000 \end{pmatrix},$$

$$\bar{A} = \begin{pmatrix} 3.5000 & 0 & -1.5000 \\ 3.0000 & 0 & -3.0000 \\ 0.5000 & 0 & 1.5000 \end{pmatrix}.$$

Utilizing toolbox in Matlab, the optimization problem can be solved efficiently. We obtain  $g_p(y) = 1.0942 \times 10^{-12}$ , and

$$H_p = \begin{pmatrix} 0.9421 & 2.5579 & 1.0579 \\ 0.0000 & 3.0000 & 0.0000 \\ 0.2410 & 0.2590 & 1.7590 \end{pmatrix} \geq 0.$$

By Theorem 2, it is easy to see that the system is positive. By Theorem 4, The stability of this systems can be checked by finding the solution matrix of the following optimization problem, we obtained

$$Q_s = \begin{pmatrix} -0.2500 & -0.0000 & -0.7500 \\ -1.5000 & 1.0000 & 1.5000 \\ 0.2500 & -0.0000 & -1.2500 \end{pmatrix}.$$

It is easy to determine that this systems is unstable.

**Example 2** (Example 4.8 in [23]). Let Leontief model [5, Example 1] with no inputs (i.e.,  $d(k) = 0$ ) be given by the matrices

$$C = \begin{pmatrix} 0.3 & 0.4 & 0.45 \\ 0 & 0 & 0 \\ 0.6 & 0.8 & 0.9 \end{pmatrix}, \quad L = \begin{pmatrix} 0.3 & 0.3 & 0.3 \\ 0.4 & 0.1 & 0.5 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}.$$

This model is suitable for our framework by considering system (1) with  $E = C$  and  $A = C - L + I$ . Since  $A$  is invertible, we can select  $\lambda = 0$ . By some calculation, we have  $\hat{A} = -I$ ,

$$\hat{E}^D = \hat{C}^D = -\begin{pmatrix} 0.2347 & 0.3129 & 0.3520 \\ 0.2526 & 0.3368 & 0.3789 \\ 0.2669 & 0.3559 & 0.4044 \end{pmatrix},$$

$$\bar{A} = -\hat{C}^D = \begin{pmatrix} 0.2485 & 0.3313 & 0.3727 \\ 0.2674 & 0.3566 & 0.4011 \\ 0.2826 & 0.3768 & 0.423 \end{pmatrix} > 0.$$

From Theorem 2, we obtained that

$$H_p = \begin{pmatrix} 0.3362 & 0.3186 & 0.3074 \\ 0.0020 & 0.3432 & 0.6471 \\ 0.3854 & 0.3624 & 0.3471 \end{pmatrix} \geq 0.$$

By Theorem 2, we know that this systems are positive. By Theorem 4, we obtained  $g_s(y) = 1.2609$ ,

$$Q_s = \begin{pmatrix} -7.6032 & -11.4709 & -12.9048 \\ -9.2599 & -11.3465 & -13.8899 \\ -9.7853 & -13.0471 & -13.6779 \end{pmatrix} < 0.$$

By Theorem 4, it is easy to see that the system is unstable.

**Example 3** (Example 2 in [26]). Let (1) be given by

$$E = \begin{pmatrix} 1.5 & 0.15 & -0.75 \\ 0 & 1 & 0 \\ 1 & 0.3 & -0.5 \end{pmatrix}, A = \begin{pmatrix} 0.1 & -0.09 & 0.45 \\ 0 & 0.3 & 0 \\ -0.6 & -0.2 & 1.3 \end{pmatrix}.$$

$$H_s = \begin{pmatrix} 0.4000 & 0.0000 & 0.0000 \\ 0.0000 & 0.3000 & 0.0000 \\ 0.1963 & 0.0389 & 0.1056 \end{pmatrix} \geq 0.$$

We choose  $\lambda = 0$ . By some calculation, we have

$$\hat{A} = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{E} = \hat{E}^D = \begin{pmatrix} -3.75 & -0.375 & 1.8750 \\ 0 & -3.3333 & 0 \\ -2.5 & -0.9167 & 1.25 \end{pmatrix},$$

$$I - \bar{A} = \begin{pmatrix} -2.75 & -0.3750 & 1.8760 \\ 0 & -2.3333 & 0 \\ -2.5 & -0.9167 & 2.2500 \end{pmatrix}.$$

From Theorem 2, we obtained

$$H_p = \begin{pmatrix} 0.4000 & 0.0000 & 0.0000 \\ 0.0000 & 0.3000 & 0.0000 \\ 0.1963 & 0.0389 & 0.1056 \end{pmatrix} \geq 0,$$

So this system is positive. From Theorem 4, We obtained

$$Q_s = \begin{pmatrix} 2.0000 & 0.1000 & -0.5000 \\ 0 & 1.4286 & 0 \\ 0.6667 & 0.1524 & 0.6667 \end{pmatrix}.$$

By Theorem 4, the system is unstable. But using the other methods, we know that this system is stable. So our method is invalid. The reason is that the eigenvalues of  $\bar{A}$  are not all real number, so under this case, our method is invalid.

**Example 4** (Example 3.4 in [23]). Let (1) be given by

$$E = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We choose  $\lambda = 0$  (regularity can be satisfied). By some calculation, we have

$$E = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix},$$

$$\hat{E} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \hat{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \hat{E}^D = \begin{pmatrix} 0.25 & -0.25 \\ -0.25 & 0.25 \end{pmatrix},$$

$$P = \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{pmatrix}, \hat{A} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, by Theorem 2, the positivity of this systems can be checked by solving the following optimization problem

$$\min g_p(y) = \sum_{i=1}^4 t_i$$

$$s.t. \begin{cases} B_p = \begin{pmatrix} t \\ \text{vec}(H) \end{pmatrix} \geq \begin{pmatrix} \text{vec}(\bar{A}) \\ -\text{vec}(\bar{A}) \end{pmatrix} \\ \begin{pmatrix} t \\ \text{vec}(H) \end{pmatrix} \geq 0 \end{cases} \quad (22)$$

where

$$B_p = \begin{pmatrix} I & P^T \otimes I \\ I & -P^T \otimes I \end{pmatrix}.$$

We obtained that  $g_p(y) = 8.2399 \times 10^{13}$ , and

$$H_p = \begin{pmatrix} 100.6132 & 101.1132 \\ 101.1132 & 100.6132 \end{pmatrix}.$$

As  $H_p$  is nonnegative, then it is easy to see that the system is positive by Theorem 2, this conclusion is the same as that in [23].

By Theorem 4, The stability of this systems can be checked by finding the solution matrix of the following optimization problem,

$$\begin{aligned} \min g_s(y) &= \sum_{i=1}^4 t_i \\ \text{s.t. } \begin{cases} B_s \begin{pmatrix} t \\ \text{vec}(Q) \end{pmatrix} \geq \begin{pmatrix} \text{vec}(I) \\ -\text{vec}(I) \end{pmatrix} \\ \begin{pmatrix} t \\ \text{vec}(Q) \end{pmatrix} \geq 0 \end{cases} \end{aligned} \quad (23)$$

where

$$B_s = \begin{pmatrix} I & I \otimes (I - \bar{A}) \\ I & -I \otimes (I - \bar{A}) \end{pmatrix}.$$

We obtained the matrix  $Q$  by using any mathematical toolbox such as Matlab, in this problem, matrix

$$Q_s = \begin{pmatrix} 0.8333 & 0.1667 \\ 0.1667 & 0.8333 \end{pmatrix} > 0,$$

from Theorem 4, we know that this descriptor positive systems are stable which is that same conclusion as that obtained by other methods.

**Remark 2.** In this example, the set of nonnegative admissible initial conditions  $S = \text{Im}(P) \cap R_n^+$  has only one point  $0 \in R^n$ , even statement (2) in Lemma 6 holds, the system in this example has only 0 solution, i.e. the trajectory of this system is degenerate.

## 6. Conclusion

We have presented an efficient approach to analysis the positivity and stability of discrete descriptor system in this paper. Our main project is to convert the necessary and sufficient condition of characterizing positivity and stability into an optimization problem, the positivity and the stability of the descriptor systems can be checked by solving the corresponding optimization problem. Compared with the traditional methods, the approach we proposed in this paper is less theoretical and easily checked in the reason that the optimization problem is a linear programming. Numerical examples are given to validate our results. The weakness of our approach is that when the Drazin inverse of the matrix  $E$  is a complex matrix, Lemma 5 and Lemma 6 do not hold any more, so our approach will not be suitable, similarly to that [23]. However, this could not have been concluded from [23].

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