

## On the Empirical Importance of Periodicity in the Volatility of Financial Returns - Time Varying GARCH as a Second Order APC(2) Process

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### Abstract

We discuss the empirical importance of long term cyclical effects in the volatility of financial returns. Following Amado and Teräsvirta (2009), Čížek and Spokoiny (2009) and others, we consider a general conditionally heteroscedastic process with stationarity property distorted by a deterministic function that governs the possible time variability of the unconditional variance. The function proposed in this paper can be interpreted as a finite Fourier approximation of an Almost Periodic (AP) function as defined by Corduneanu (1989). The resulting model has a particular form of a GARCH process with time varying parameters, intensively discussed in the recent literature.

In the empirical analyses we apply a generalisation of the Bayesian AR(1)-GARCH model for daily returns of S&P500, covering the period of sixty years of US postwar economy, including the recently observed global financial crisis. The results of a formal Bayesian model comparison clearly indicate the existence of significant long term cyclical patterns in volatility with a strongly supported periodic component corresponding to a 14 year cycle. Our main results are invariant with respect to the changes of the conditional distribution from Normal to Student- $t$  and to the changes of the volatility equation from regular GARCH to the Asymmetric GARCH.

**Keywords:** GARCH models, Bayesian inference, periodically correlated stochastic processes, volatility, unconditional variance

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## 1 Introduction

Starting from seminal works by Clark (1973), Engle (1982) and Bollerslev (1986) stochastic processes used to describe observed properties of the volatility of financial time series have been tailored to identify short term features. In particular, the resurgence of stochastic volatility (SV) models in the 90's relied on the assumption that there exists a stochastic factor independent of the past of the process, which influences volatility in the short term. The resulting literature concerning GARCH and SV models, its properties and practical importance is enormous, however empirical analyses of the dynamic behaviour of the volatility in the long term has not been fully explored so far.

Recently, some attempts to model long term features of volatility have been made. Since empirical analyses of long time series of financial returns clearly indicated that parameters of volatility models may vary over time, it is obvious that models applied so far may not capture properties of volatility which are important in the long term. At the beginning of the 90's the GARCH-type models became a very popular tool of volatility modelling. But parallelly some problems were identified with their applications to long time series of financial returns. For example, Lamoreux and Lastrapes (1990) and Engle and Mustafa (1992) suggested that parameters of GARCH-type processes are very strongly identified, because while in econometric applications their estimates are statistically highly significant, they are not stable over time. Consequently, the constancy of parameters initially imposed in GARCH-type processes was subject to criticism that prompted new studies concerning generalisations. In particular Mikosh and Stărică (2004) indicate that the IGARCH effect is often spuriously supported by data, because in the case of long time series variability of parameters is natural. Hence the regular GARCH(1,1) structure is unable to capture nonlinearity and possible complex stochastic properties of the observed process. Teräsvirta (2009) points out a more formal motivation in favour of time variability of parameters in a parametric GARCH scheme, suggesting that *constancy of parameters can be a testable restriction and if it is rejected, the model should be generalised*. Several approaches have been proposed imposing time variability of parameters in volatility models. We see two basic fundamental approaches applied in this respect, the first one relates to variability governed by a random process, and the second relies on deterministic framework. Within the first approach, Hamilton and Susmel (1994) conducted research on the empirical importance of the assumption that stock returns are characterised by different ARCH processes at different points in time, with the shifts between processes mediated by a Markov chain. This straightforward approach opened new topics in financial econometrics based on the application of Markov switching mechanisms in volatility modelling. A possible variability of parameters described by a deterministic function was also subject to analysis. Teräsvirta (2009) modified the smooth transition GARCH model by imposing a transition function of the form that guarantees variability of parameters for a process observed in finite time interval. The transition

function depends on the length of the observed time series. Čížek and Spokoiny (2009) present a review of literature concluding that relaxing time homogeneity of the process is a promising approach but causes serious problems with proper estimation methods. For instance, when some or all model parameters will vary over time, a more subtle treatment of testing structural breaks in financial returns may be obtained; see Fan and Zhang (1999), Cai, Fan, Li (2000), Fan, Yao, Cai (2003). An approach to the specification of time varying GARCH models was developed in the field of nonparametric statistics. Under very general conditions concerning the regularity of parameters treated as functions of time, nonparametric methods of estimation were proposed; see Härdle, Herwatz, Spokoiny (2003), Mercurio and Spokoiny (2004), Spokoiny and Chen (2007) and Čížek and Spokoiny (2009).

The main purpose of this paper is to propose a simple generalisation of the GARCH model which would enable to model long term features of volatility. Our construct is strictly related to the literature studying the properties of GARCH processes with time varying parameters and is based on the parametric approach; see Teräsvirta (2009), Amado and Teräsvirta (2008) and (2012). The variability of unconditional moments is governed by a class of Almost Periodic (AP) functions, proposed by Corduneanu (1989) as a generalisation of the class of periodic functions. Since in our approach the unconditional second moment exhibits almost periodic variability, the process can be also interpreted as a second order Almost Periodically Correlated (APC) stochastic process, discussed from the theoretical point of view by Hurd and Miamee (2007). During the last half century the APC class of processes was broadly applied in telecommunication (Gardner (1986), Napolitano and Spooner (2001)), climatology (Bloomfield, Hurd, Lund (1994)) and many other fields. Application of APC class in business cycle analysis has been also considered. Recently Lenart (2013) investigated properties of subsampling estimator of frequencies defining time varying unconditional expectation of APC process. For an exhaustive review of possible applications see Gardner, Napolitano, Paura (2006).

We make a formal statistical inference, from the Bayesian viewpoint, about the cyclicity of volatility changes and present evidence in favour of the empirical importance of such an effect. On the basis of very intuitive explanation of almost periodicity, we provide an economic interpretation of time variability of unconditional moments supported by data. The illustration is conducted on the basis of daily returns of the S&P500 index covering the period from 18 January 1950 till 7 February 2012.

## 2 A simple nonstationary process obtained from the GARCH(1,1) model

We start from a general definition of a conditionally heteroscedastic model which nests many ARCH-type volatility models developed during more than the last three decades in the field of financial econometrics.

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**Definition 1** *Discrete, real valued, stochastic process  $\{\xi_t, t \in Z\}$  is called conditionally heteroscedastic if:*

$$\xi_t = \sqrt{h_t(\omega, \Psi_{t-1})} z_t, \quad z_t \sim iid(0, 1),$$

where  $h_t(\omega, \Psi_{t-1})$  describes volatility equation and is defined as a parametric function of the information set  $\Psi_{t-1} = (\dots, \xi_{t-2}, \xi_{t-1})$ , i.e. the history of the process  $\{\xi_t, t \in Z\}$ , with parameters  $\omega$ . For  $z_t$ ,  $D(0, 1)$  denotes a distribution with zero mean and unit variance.

Any conditionally heteroskedastic GARCH-type model, defined in the literature, starting from the ARCH( $p$ ) model proposed by Engle (1982) and the GARCH( $p, q$ ), proposed by Bollerslev (1986), can be obtained by imposing some particular functional form of  $h_t(\omega, \Psi_{t-1})$ .

For further analysis let us consider the discrete and real valued stochastic process  $\{\varepsilon_t, t \in Z\}$  of the form:

$$\varepsilon_t = \sqrt{g(t, \gamma)} \xi_t, \quad (1)$$

where  $\{\xi_t, t \in Z\}$  follows Definition 1, and  $g(\cdot, \gamma)$  is a positive real valued function of time domain  $Z$ , parameterised by  $\gamma$ . The form of the process  $\{\varepsilon_t, t \in Z\}$  is related to the general specification considered by Amado and Teräsvirta (2012). The aim of our study is a proper specification of function  $g$ , so that it has an economic interpretation and is empirically important. For a process  $\{\varepsilon_t, t \in Z\}$  in (1), where  $\{\xi_t, t \in Z\}$  is given by Definition 1 and for a bounded function  $g(\cdot, \gamma)$  we have the following equivalences:

1. For each  $n \in N$ ,  $E(\varepsilon_t^n)$  exists and  $E(\varepsilon_t^n) = g(t, \gamma)^{\frac{n}{2}} E(\xi_t^n)$  if and only if  $E(\xi_t^n)$  exists
2. For each  $n \in N$ ,  $E(\varepsilon_t^n | \Psi_{t-1})$  exists and:  
 $E(\varepsilon_t^n | \Psi_{t-1}) = g(t, \gamma)^{\frac{n}{2}} h_t(\theta, \Psi_{t-1})^{\frac{n}{2}} E(z_t^n)$  if and only if  $E(z_t^n)$  exists.

As an example of the process in Definition 1 let us consider the seminal Generalised Autoregressive Conditional Heteroskedastic (GARCH) process, initially defined by Bollerslev (1986). Formally Bollerslev (1986) defined the GARCH( $p, q$ ) process for any natural  $p$  and  $q$ . However, just like in predominant papers, both, theoretically and empirically driven, we focus our attention on the case with  $p = 1$  and  $q = 1$ . Consequently, let consider the case with  $h_t$  of the form:

$$h_t = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \beta_1 h_{t-1}, \quad (2)$$

for  $\alpha_0 > 0$ ,  $\alpha_1 \geq 0$ ,  $\beta_1 \geq 0$ . When analysing stochastic properties of the process  $\xi_t$  with  $h_t$  given by (2), it is crucial to pay attention on the restriction  $\alpha_1 + \beta_1 < 1$ . It ensures moment existence up to the second order, its stability over time and, consequently, covariance stationarity of the process. The GARCH(1,1) process with

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$\alpha_1 + \beta_1 = 1$  is called IGARCH. This case still represents the process stationary in the strict sense, but covariance stationarity is no longer fulfilled. Bauwens, Lubrano and Richard (1999) listed the properties of the GARCH(1,1) process. Given restriction  $\alpha_1 + \beta_1 < 1$ , process  $\{\xi_t, t \in Z\}$  is covariance stationary with unconditional zero mean and finite unconditional variance  $V(\xi_t) = E(\xi_t^2) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$ .

Now let consider the process  $\{\varepsilon_t, t \in Z\}$  defined by (1), generated by the GARCH(1,1) process  $\{\xi_t, t \in Z\}$ . Automatically we obtain the following properties:

1.  $E(\varepsilon_t | \Psi_{t-1}) = 0$
2.  $V(\varepsilon_t | \Psi_{t-1}) = g(t, \gamma) h_t$
3.  $E(\varepsilon_t) = 0$
4.  $V(\varepsilon_t) = E(\varepsilon_t^2) = g(t, \gamma) \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$ , if additionally  $\alpha_1 + \beta_1 < 1$

It is clear that process  $\{\varepsilon_t, t \in Z\}$  is nonstationary in the strict sense and also covariance nonstationary. Function  $g(\cdot, \gamma)$  assures variability of unconditional variance. Also variability over time of conditional variance of  $y_t$  is decomposed into GARCH(1,1) effect and deterministic component, that changes dispersion of the conditional distribution according to the form of function  $g$ .

Another interesting feature of  $\{\varepsilon_t, t \in Z\}$  can be observed if we rewrite the equation for conditional variance in the GARCH-type form. If the process  $\{\varepsilon_t, t \in Z\}$  is defined by equation (1) and  $\{\xi_t, t \in Z\}$  in (1) is GARCH(1,1), we have:

$$E(\varepsilon_t^2 | \Psi_{t-1}) = g(t, \gamma) h_t = \alpha_{0,t} + \alpha_{1,t} \xi_{t-1}^2 + \beta_{1,t} h_{t-1}, \quad (3)$$

where  $\alpha_{0,t} = g(t, \gamma) \alpha_0$ ,  $\alpha_{1,t} = g(t, \gamma) \alpha_1$  and  $\beta_{1,t} = g(t, \gamma) \beta_1$ . Hence, the process  $\{\varepsilon_t, t \in Z\}$  can be also interpreted as a GARCH(1,1) model with time varying parameters.

Equation 3 involves very similar idea to the construct proposed by Baillie and Morana (2009). In our approach we focus on a simpler GARCH-type process and do not generalise equation for  $h_t$  to the fractionally integrated GARCH, considered in Baillie and Morana (2009). But we allow time variability of each parameter in equation for conditional variance of the process  $\{\varepsilon_t, t \in Z\}$ .

Just like for the GARCH(1,1) process, one may consider properties of the process  $\{\varepsilon_t, t \in Z\}$ , when different functional forms of  $h_t$  are assumed. In particular, in the empirical part of the paper we confront a simple GARCH(1,1) model with Asymmetric-GARCH specification, proposed by Glosten, Jagannathan, Runkle (1993), with  $h_t$  of the following form:

$$h_t = \alpha_0 + \alpha_1^+ \xi_{t-1}^2 I(\xi_{t-1} \geq 0) + \alpha_1^- \xi_{t-1}^2 I(\xi_{t-1} < 0) + \beta_1 h_{t-1},$$

with  $\alpha_0 > 0$ ,  $\alpha_1^+ \geq 0$ ,  $\alpha_1^- \geq 0$ ,  $\beta_1 \geq 0$ . We will denote this specification by GJR(1,1). Analogously to the case with GARCH(1,1), the conditional second moment of  $\varepsilon_t$  is

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given by the form:

$$E(\varepsilon_t^2 | \Psi_{t-1}) = \alpha_{0,t} + \alpha_{1,t}^+ \xi_{t-1}^2 I(\xi_{t-1} \geq 0) + \alpha_{1,t}^- \xi_{t-1}^2 I(\xi_{t-1} < 0) + \beta_{1,t} h_{t-1} \quad (4)$$

where  $\alpha_{0,t} = g(t, \gamma)\alpha_0$ ,  $\alpha_{1,t}^+ = g(t, \gamma)\alpha_1^+$ ,  $\alpha_{1,t}^- = g(t, \gamma)\alpha_1^-$  and  $\beta_{1,t} = g(t, \gamma)\beta_1$ . This leads us to the Asymmetric-GARCH model with time varying parameters.

Explicit formulae of conditional and unconditional moments in case of GARCH( $p, q$ ), for  $p > 1$  and  $q > 1$  and also in case of GJR model can be found in Bollerslev (1986), Chiangli (1997) and Chiangli and Teräsvirta (1999).

### 3 A model for periodic volatility

The main purpose of the paper is such a definition of function  $g$  in (1) that would enable to test the variability of parameters in (3) but would also provide an economic interpretation of such an effect. The vast literature concerning time-varying GARCH models does not seem to explore this aspect in detail, focusing only on the statistical properties of estimation methods, given very general assumptions about the variability of parameters.

Some attempts to interpret time heterogeneity of processes describing volatility have been made. One of them was adopted by Hamilton (1989). In this seminal paper formal statistical representation of the old idea that expansion and contraction constitute two distinct economic phases was considered. Hamilton proposed to model real output growth by two autoregressions, depending on whether the economy is expanding or contracting. Possible changes between those autoregressions were subordinated to a Markov chain. The main contribution of Hamilton (1989) consisted of a very intuitive economic interpretation of a purely random construct as a factor governing changes between states of different intensity of economic activity. This idea was easily instilled in modelling financial time series, where Markov switching ARCH and GARCH models were specifically developed for volatility modelling; see Hamilton and Susmel (1994), Susmel (2000), Haas, Mittnik, Paoletta (2004), Li and Lin (2004). Changes in conditional volatility according to Markov chain were considered jointly with changes in the conditional mean; see for example Berkes, Gombay, Horváth, Kokoszka (2004). Recently, Markov switching Stochastic Volatility models have also been considered; see Kwiatkowski (2010) for empirical analyses for Polish financial market. Markov switching volatility models are able to distinguish phases of low and high volatility, or - in the case of many regimes - many different levels of risk intensity. However, as concluded by Langa and Rahbek (2009), in spite of the fact that Markov switching volatility models have recently received much interest in applications, a sufficiently complete theory of these models is still missing.

Analogously to modelling economic activity of the real sector, the volatility of financial time series also seems to have phases of expansion and contraction in the long term. An analysis of financial returns in the span of decades shows that changes between

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states are much closer to continuous rather than discrete. Since those phases alternate in cases of boom and bust on the market, volatility observed over decades should also exhibit cyclical behaviour. In order to test for such an effect, a stochastic process with an approximately periodic structure of unconditional moments should be considered. For a process  $\{\varepsilon_t, t \in Z\}$  defined by equation (1), where  $\{\xi_t, t \in Z\}$  in (1) is GARCH(1,1) process, it can easily be done on the basis of an appropriately defined function  $g(\cdot, \gamma)$ , which describes the variability of moments. In general, we follow the idea of generalisation of periodicity of real valued functions proposed by Corduneanu (1989).

**Definition 2** *A real-valued function  $f : Z \rightarrow \mathbb{R}$  of an integer variable is called almost periodic (AP in short), if for any  $\epsilon > 0$  there exists an integer  $L_\epsilon > 0$ , such that among any  $L_\epsilon$  consecutive integers, there is an integer  $p_\epsilon$  with the property*

$$\sup_{t \in Z} |f(t + p_\epsilon) - f(t)| < \epsilon.$$

Any periodic function is also almost periodic. Conditions from Definition 3 constitute a class of almost periodically correlated (APC) stochastic processes as a generalisation of periodically correlated (PC) stochastic processes. In the case of APC processes, an almost periodic function, and in the case of PC processes, a periodic function, determines the cyclical variability of conditional and unconditional moments. Therefore PC stochastic processes are also called cyclostationary.

The main properties of the APC class was presented by Corduneanu (1989). In particular, any almost periodic function from Definition 3 has its unique Fourier expansion of the form:

$$f(t) = \sum_{i=1}^{\infty} (g_{si} \sin(h_i t) + g_{ci} \cos(h_i t)), \quad (5)$$

with the series of coefficients  $(g_{si})_{i=1}^{\infty}$ ,  $(g_{ci})_{i=1}^{\infty}$  and  $(h_i)_{i=1}^{\infty}$  that express amplitude and frequency of each individual cyclical component in (5).

For further research concerning cyclical behavior of volatility, we consider the following function  $g(\cdot, \gamma)$  in (1):

$$g(t, \gamma) = e^{f(t, \gamma)}, \quad (6)$$

where

$$f(t, \gamma) = \sum_{i=1}^F (\gamma_{si} \sin(\phi_i t) + \gamma_{ci} \cos(\phi_i t)), \quad (7)$$

with  $\gamma = (\gamma_{s1}, \dots, \gamma_{sF}, \gamma_{c1}, \dots, \gamma_{cF}, \phi_1, \dots, \phi_F)$ . Function  $f(\cdot, \gamma)$  is defined as a sum of periodic functions, with parameters  $\phi_i$  determining frequencies, while  $\gamma_{si}$  and  $\gamma_{ci}$  control amplitudes. Since we limit the infinite series to its finite substitute, formula (7) yields finite approximation of order  $F$  of the almost periodic function, that governs moment variability of the process.

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The case  $\gamma_{si} = 0$  and  $\gamma_{ci} = 0$  for all  $i = 1, \dots, F$ , in (7), determines constant function  $g(\cdot, \gamma) \equiv 1$ . According to Hurd and Miamee (2007), the process  $\{\varepsilon_t, t \in Z\}$  defined by equation (1), where  $\{\xi_t, t \in Z\}$  in (1) follows GARCH(1,1) process and  $g(t, \gamma)$  is defined by (6) and (7), is also Almost Periodically Correlated. The function  $g(\cdot, \gamma)$  in (7) enables to capture cyclicity in the conditional and unconditional variance of the process. This property will be subject to formal statistical inference in the empirical part of the paper.

It is clear that defining function  $f$  in (7) we follow idea of application of the Flexible Fourier Form (FFT). This form is known in econometrics for decades, and in 70's and 80's was intensively applied on the field of microeconometrics. In particular the expenditure system on the basis of FFT was proposed by Gallant (1981), who focused on the empirical importance of a specific flexible function in demand system analysis.

## 4 Basic model framework and posterior inference

We model logarithmic returns on the financial instrument with price  $x_t$  at time  $t$ . Suppose, we observe time series of logarithmic returns given by the form:

$$y_t = 100 \ln \frac{x_t}{x_{t-1}}, t = -1, 0, 1, \dots, T.$$

Denote by  $y = (y_1, \dots, y_T)$  the vector of modelled observations. Daily returns  $y_{-1}$  and  $y_0$  are used as initial values.

As a point of departure of modelling the dynamics of financial returns we assume an AR(1) process; see for example Bauwens, Lubrano and Richard (1999) for univariate case or Osiewalski and Pipień (2004) for multivariate setting. We generalise the approach, by considering nonstationary disturbances in the following equation:

$$y_t = \mu_t + \varepsilon_t, \quad t = 1, \dots, T, \quad (8)$$

where  $\mu_t = \delta + \rho(y_{t-1} - \delta)$  and  $\varepsilon_t$  is a process defined as follows:

$$\varepsilon_t = \sqrt{g(t, \gamma)} \xi_t, \quad t = 1, \dots, T.$$

For a process  $\{\xi_t, t \in Z\}$  we consider two alternative specifications, namely GARCH(1,1) and GJR(1,1), while the function  $g$  is given by (6) and (7).

Generally we assume that random variables  $z_t$  are independent and follow Student- $t$  distribution with zero mean, unit variance and  $\nu > 4$  degrees of freedom. In the literature the conditional Student- $t$  distribution in GARCH models is considered with standard restriction  $\nu > 2$ . This restriction assures existence of conditional variance and is necessary for existence of unconditional variance. However, as it is shown in Weiss (1993), Lumsdaine (1995) and Gouriéroux (1997) *Maximum Likelihood Estimator* (MLE) is consistent and asymptotically Normal in GARCH-type models provided the existence of the fourth conditional moment. This sufficient condition

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is sometimes too strong in empirical applications, but assures, that MLE with small sample approximation of the Fisher's information matrix can be interpreted in a standard way. Several attempts to weaken this condition have been made, indicating that the properties of the asymptotic likelihood inference methods in GARCH-type models are still questionable. Additionally, as suggest Hall and Yao (2003) the distribution of MLE estimator in case of GARCH-type models with very heavy tailed conditional distribution (but still with finite conditional variance) may be very irregular. Hence, in spite of the fact that empirical analysis bases fully on Bayesian approach to inference, for the sake of possible future comparison of results with the researchers that apply MLE, we assume that the fourth moment of the conditional distribution of the error term in (8) exists.

The density of  $z_t$  is given by the formula:

$$f_s(z_t|0, 1, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi(\nu-2)}} \left[1 + \frac{z_t^2}{\nu-2}\right]^{-\frac{\nu+1}{2}} = c(\nu) \left[1 + \frac{z_t^2}{\nu-2}\right]^{-\frac{\nu+1}{2}},$$

where  $f_s(z_t|m, s^2, \nu)$  denotes the density of the Student- $t$  distribution with mean  $m$ , variance  $s^2$  and  $\nu > 4$  degrees of freedom.

The conditional distributions of  $\xi_t$  and  $\varepsilon_t$  are Student- $t$  distributions with zero mean,  $\nu$  degrees of freedom and variances  $h_t$  and  $g(t, \gamma)h_t$ , respectively:

$$p(\xi_t|\Psi_{t-1}) = f_s(\xi_t|0, h_t, \nu) = \frac{c(\nu)}{\sqrt{h_t}} \left[1 + \frac{\xi_t^2}{(\nu-2)h_t}\right]^{-\frac{\nu+1}{2}},$$

$$p(\varepsilon_t|\Psi_{t-1}) = f_s(\varepsilon_t|0, g(t, \gamma)h_t, \nu) = \frac{c(\nu)}{\sqrt{g(t, \gamma)h_t}} \left[1 + \frac{\varepsilon_t^2}{(\nu-2)g(t, \gamma)h_t}\right]^{-\frac{\nu+1}{2}},$$

where  $\Psi_0 = (h_0, y_{-1}, y_0)$ , and  $\Psi_{t-1} = (\Psi_0, y_1, \dots, y_{t-1}) = (\Psi_0, y^{(t-1)})$ . Consequently, the conditional distribution of daily return in (8) is Student- $t$  distribution with mean  $\mu_t = \delta + \rho(y_{t-1} - \delta)$ , variance  $g(t, \gamma)h_t$  and  $\nu > 4$  degrees of freedom:

$$p(y_t|\Psi_{t-1}) = f_s(y_t|\mu_t, g(t, \gamma)h_t, \nu) = \frac{c(\nu)}{\sqrt{g(t, \gamma)h_t}} \left[1 + \frac{(y_t - \mu_t)^2}{(\nu-2)g(t, \gamma)h_t}\right]^{-\frac{\nu+1}{2}}. \quad (9)$$

Let  $\theta$  denote the vector that contains all model parameters. We assume that  $\theta = (\mu', \sigma'^2, \nu, \gamma')$  where vectors  $\mu$ ,  $\sigma^2$  and  $\gamma$  collect parameters of the conditional mean of  $y_t$ , the conditional variance of  $\xi_t$  and the function  $g$ , respectively. In particular,  $\mu = (\delta, \rho)'$  and  $\gamma = (\gamma_{s1}, \dots, \gamma_{sF}, \gamma_{c1}, \dots, \gamma_{cF}, \phi_1, \dots, \phi_F)$ . If  $h_t$  follow the GARCH(1,1) process, then  $\sigma^2 = (\alpha_0, \alpha_1, \beta_1)'$ , while in the case of the GJR(1,1) process  $\sigma^2 = (\alpha_0, \alpha_1^+, \alpha_1^-, \beta_1)'$ .

According to (9) we rewrite the sampling model for a vector  $y$  in the following way:

$$p(y|\theta) = \prod_{t=1}^T p(y_t|\Psi_{t-1}) = \prod_{t=1}^T f_s(y_t|\mu_t, g(t, \gamma)h_t, \nu).$$

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The Bayesian model, i.e. the joint distribution of observables and parameters, requires specification of the prior distribution  $p(\theta)$ . In the case of the GARCH(1,1) process for  $\xi_t$  we assume the following prior independence:

$$p(y, \theta) = p(y|\theta)p(\theta) = p(y|\theta)p(\delta)p(\rho)p(\alpha_0)p(\alpha_1, \beta_1)p(\nu)p(\gamma),$$

while in the case of the GJR(1,1) process we consider the following model:

$$p(y, \theta) = p(y|\theta)p(\theta) = p(y|\theta)p(\delta)p(\rho)p(\alpha_0)p(\alpha_1^+, \alpha_1^-, \beta_1)p(\nu)p(\gamma).$$

Additionally we assume that  $p(\delta)$  is the standard Normal distribution,  $p(\rho)$  is uniform over  $(-1,1)$ ,  $p(\alpha_0)$  is the exponential distribution with unit mean,  $p(\nu)$  is the exponential distribution with mean 10 truncated at  $\nu > 4$ ,  $p(\gamma)$  is multivariate normal,  $p(\phi)$  is multivariate uniform over the set obtained by identification restrictions that eliminate label-switching effect in (7). We assume, that  $L < \phi_1 < \dots < \phi_F < U$ , for appropriately chosen  $L$  and  $U$ , which eliminates frequencies of length shorter than a quarter and longer than the time interval covering the observed time series. In the case of the GARCH(1,1) model  $p(\alpha_1, \beta_1)$  is the bivariate uniform distribution on the unit square  $[0, 1]^2$ , while in case of GJR(1,1)  $p(\alpha_1^+, \alpha_1^-, \beta_1)$  is trivariate uniform over  $[0, 1]^3$ .

Alternatively, we consider for  $z_t$  standard Normal distribution, with the density:

$$f_N(z_t|0, 1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z_t^2}{2}\right),$$

where  $f_N(z_t|m, s^2)$  denotes the density of Normal distribution with mean  $m$  and variance  $s^2$ .

In this case the conditional distributions of  $\xi_t$  and  $\varepsilon_t$  are Normal with zero mean and variance  $h_t$  and  $g(t, \gamma)h_t$  respectively:

$$p(\xi_t|\Psi_{t-1}) = f_N(\xi_t|0, h_t) = \frac{1}{\sqrt{2\pi h_t}} \exp\left(-\frac{\xi_t^2}{2h_t}\right),$$

$$p(\varepsilon_t|\Psi_{t-1}) = f_N(\varepsilon_t|0, g(t, \gamma)h_t) = \frac{1}{\sqrt{2\pi g(t, \gamma)h_t}} \exp\left(-\frac{\varepsilon_t^2}{2g(t, \gamma)h_t}\right).$$

The conditional distribution of daily return  $y_t$  is Normal with mean  $\mu_t$  and variance  $g(t, \gamma)h_t$ :

$$p(y_t|\Psi_{t-1}) = f_N(y_t|\mu_t, g(t, \gamma)h_t) = \frac{1}{\sqrt{2\pi g(t, \gamma)h_t}} \exp\left(-\frac{(y_t - \mu_t)^2}{2g(t, \gamma)h_t}\right).$$

In the case of conditional normality, the vector of model parameters  $\theta_{-\nu}$  differs from  $\theta$  only by degrees of freedom parameter  $\nu$ . The sampling model is given as follows:

$$p(y|\theta_{-\nu}) = \prod_{t=1}^T p(y_t|\Psi_{t-1}) = \prod_{t=1}^T f_N(y_t|\mu_t, g(t, \gamma)h_t).$$

When specifying the Bayesian model in this case, we formulate the prior distribution of elements of the vector  $\theta_{-\nu}$  in the same way as in the case of the conditional Student- $t$  distribution.

## 5 Empirical analysis

In this section we present the empirical analysis and make formal Bayesian inference about the empirical importance of the cyclical component in the volatility of daily returns of one of the most important US Stock Market indices. Our dataset consists of  $T = 15615$  observations of daily logarithmic returns on the S&P500 index, covering the period starting from the postwar era of the US economy till the beginning of 2012. The time series starts on 18 January 1950 and ends on 7 February 2012.

Table 1: Decimal logarithms of the marginal data densities for the set of competing specifications and posterior probabilities in case of prior model probabilities defined as proportional to  $6^{-3F}$ , where  $F = 0, 1, 2, 3, 4$  denotes the number of frequencies in function  $g$ . Posterior model probabilities are calculated separately within each class of models.

Logarithms of the marginal data density				
Number of frequencies	N-GARCH	N-GJR	$t$ -GARCH	$t$ -GJR
0	-163.64	-121.03	0.30	37.84
1	-158.76	-110.44	5.25	42.00
2	-152.94	-100.95	7.59	48.24
3	-150.65	-99.18	8.52	48.98
4	-149.77	-96.35	9.89	49.95
Posterior model probabilities within each class of models				
Number of frequencies	N-GARCH	N-GJR	$t$ -GARCH	$t$ -GJR
0	0.00	0.00	0.00	0.00
1	0.00	0.00	0.49	0.00
2	0.52	0.47	0.49	0.97
3	0.47	0.13	0.02	0.025
4	0.02	0.40	0.00	0.00

In Table 1 we present results of Bayesian model comparison, conducted for four subsets of competing specifications. We considered alternatively Normal or Student- $t$  conditional distribution, and also for the volatility  $h_t$  the GARCH(1,1) or GJR(1,1) equation. Given a particular type of conditional distribution and functional form imposed on  $h_t$ , we consider a pure GARCH-type model, namely with constant

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parameters, versus APC(F)-GARCH(1,1) models with  $F = 1, 2, 3, 4$  individual periodic components. This gives us four subsets of models, denoted by N-GARCH,  $t$ -GARCH, N-GJR and  $t$ -GJR respectively. Within each class of models we considered five competing specifications, as  $F = 0, 1, 2, 3$  and 4. In Table 1 we collect the marginal data density values for each model approximated by the Newton and Raftery (1994) estimator. We see that, in the case of models with constant parameters the GJR(1,1) specification with the conditional Student- $t$  distribution receives the greatest data support, completely outperforming the GARCH(1,1) model and conditional Normality. It seems, that stronger rejection is attached to the conditional Normality, since, given a particular form of the  $h_t$ , the marginal data density value of the conditional Student- $t$  models is about a hundred orders of magnitudes greater, as compared to the case with the conditional Normal distribution. On the other hand, given a particular conditional distribution, the GJR(1,1) form of  $h_t$  yields the marginal data density value greater by a dozens of orders of magnitude, than in the case of GARCH(1,1) model. This effect is also present in the case of time varying GARCH-type models. We see the same regularity for each case of  $F = 1, 2, 3, 4$ , making the conditional Student- $t$  distribution and the GJR(1,1) equation for  $h_t$  decisively important in the view of the data.

Analysing results presented in Table 1 we see that the dataset strongly supports time variability of parameters, and consequently nonstationarity in the strict sense. This effect is invariant with respect to the type of  $h_t$  and to the type of conditional distribution. For each of subsets of models, the case with  $F = 0$ , representing constancy of parameters, is rejected by the data, as the marginal data density in all four cases is smaller by at least a couple of orders of magnitude compared to marginal data density for any of generalisation with time varying parameters. The constancy of parameters receives the strongest rejection in the case of conditional Normality and the GJR(1,1), where marginal data density value for the GJR(1,1) model is about nine of orders of magnitude smaller than the worst case with time varying parameters. In all remaining cases constancy of parameters yields marginal data density value about five orders of magnitude smaller than its worst (in the view of the data) generalisation that assures variability of parameters in time.

Slightly disappointing, the dataset does not determine decisively the form the of function which assures time variability of parameters in  $h_t$ . In both cases of the conditional distribution, and also for both types of volatility equation, the data leave a considerably great uncertainty about the number of individual periodic components in function  $g$ . The existence of the single periodic component ( $F = 1$ ) is decisively more probable a posteriori than GARCH-type models with constant parameters. The values of the marginal data density increase as  $F$  increases. However, the greater value of  $F$  one may take, the less relative support receives model with  $F + 1$  individual periodic elements in function  $g$  against the model with  $F$  components. We did not check the data support of models for  $F > 4$ , because those specifications are not parsimoniously parameterised. In order to adjudicate which case receives the greatest

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Table 2: Posterior modes ( $Mod(\cdot|y)$ ), means  $E(\cdot|y)$  and standard deviations ( $D(\cdot|y)$ ) of parameters in APC( $F$ ), for  $F = 0$  (GARCH(1,1) and GJR(1,1)) cases and  $F = 2$  (the best models within subsets with the same type of conditional distribution and functional form imposed on  $h_t$ )

Parameters	$\delta$	$\rho$	$\nu$	$\alpha_0$	$\beta_1$	$\alpha_1$	$\alpha_1^+$	$\alpha_1^-$	$\gamma_{s1}$	$\gamma_{c1}$	$\gamma_{s2}$	$\gamma_{c2}$	$\phi_1$	$\phi_2$
<b>N-GARCH</b>														
$Mod(\cdot y)$	0.046	0.160		0.005	0.930	0.063								
$E(\cdot y)$	0.046	0.159		0.005	0.930	0.063								
$D(\cdot y)$	0.006	0.008		0.001	0.004	0.004								
<b>APC(2)-N-GARCH</b>														
$Mod(\cdot y)$	0.047	0.160		0.009	0.921	0.065			-0.121	0.347	-0.174	-0.314	0.000744	0.001769
$E(\cdot y)$	0.046	0.160		0.009	0.921	0.065			-0.131	0.321	-0.152	-0.310	0.000731	0.001764
$D(\cdot y)$	0.006	0.008		0.001	0.005	0.004			0.106	0.074	0.101	0.076	0.000055	0.000042
<b>N-GJR</b>														
$Mod(\cdot y)$	0.030	0.161		0.006	0.931	0.026	0.026	0.095						
$E(\cdot y)$	0.030	0.161		0.006	0.931	0.026	0.026	0.095						
$D(\cdot y)$	0.006	0.008		0.001	0.004	0.003	0.003	0.006						
<b>APC(2)-N-GJR</b>														
$Mod(\cdot y)$	0.029	0.162		0.010	0.921	0.022	0.022	0.105	-0.047	0.413	-0.106	0.383	0.000734	0.004194
$E(\cdot y)$	0.029	0.160		0.010	0.922	0.022	0.022	0.104	-0.124	0.375	-0.142	0.088	0.000700	0.003180
$D(\cdot y)$	0.006	0.008		0.001	0.005	0.003	0.003	0.006	0.121	0.067	0.087	0.341	0.000045	0.001190
<b>t-GARCH</b>														
$Mod(\cdot y)$	0.049	0.159	5.932	0.004	0.937	0.059								
$E(\cdot y)$	0.049	0.159	5.912	0.004	0.937	0.059								
$D(\cdot y)$	0.006	0.008	0.285	0.001	0.004	0.004								
<b>APC(2)-t-GARCH</b>														
$Mod(\cdot y)$	0.049	0.160	5.963	0.008	0.930	0.059			-0.038	0.373	-0.331	-0.272	0.000899	0.001836
$E(\cdot y)$	0.049	0.159	5.928	0.006	0.933	0.059			1.353	0.538	-1.686	-0.681	0.001260	0.002220
$D(\cdot y)$	0.006	0.008	0.286	0.001	0.005	0.004			5.412	2.383	5.447	2.291	0.000559	0.002890
<b>t-GJR</b>														
$Mod(\cdot y)$	0.038	0.162	6.227	0.005	0.934	0.020	0.020	0.099						
$E(\cdot y)$	0.038	0.162	6.227	0.005	0.934	0.020	0.020	0.099						
$D(\cdot y)$	0.006	0.008	0.308	0.001	0.005	0.004	0.004	0.007						
<b>APC(2)-t-GJR</b>														
$Mod(\cdot y)$	0.037	0.162	6.322	0.010	0.925	0.016	0.016	0.105	-0.284	0.274	-0.244	-0.264	0.000698	0.001814
$E(\cdot y)$	0.038	0.162	6.304	0.009	0.927	0.016	0.016	0.105	-0.275	0.238	-0.234	-0.256	0.000692	0.001817
$D(\cdot y)$	0.006	0.008	0.321	0.001	0.006	0.004	0.004	0.008	0.120	0.088	0.102	0.096	0.000067	0.000043

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posterior probability, we imposed an extra information, penalising the number of parameters determining the form of function  $g$ . Intuitively, the strongest periodic effect in the volatility of financial returns can be related with the cyclical component of rather long period. One may not reject immediately possibility that the volatility cycles of the short period (like monthly or quarterly) are also important, but long term changes in the volatility occur in the foreground. Additionally, the long term cycle in volatility is economically interpretable as an empirical consequence of existence of market crashes and booms. Consequently, we believe that the model with limited number of individual periodic components in function  $g$  is sufficient to capture the effect of long term fluctuations – the strongest data feature observed in case of long financial time series. Those beliefs are reflected in our prior model probabilities. In case of model with  $F$  individual periodic elements, the vector  $\gamma'$  contains  $3F$  free parameters. Hence, separately for each subset of models, N-GARCH,  $t$ -GARCH, N-GJR,  $t$ -GJR, we consider prior model probabilities proportional to  $6^{-3F}$ , strongly penalising models with too much expanded form of function  $g$  and strongly supporting (a priori) models with constant parameters.

The idea of imposing prior model probabilities of such a form was discussed by Osiewalski and Steel (1993) and applied in Bayesian comparison of GARCH models by Osiewalski and Pipień (2003). The resulting posterior model probabilities, calculated separately for each subset of models, are presented in Table 1. When penalty on the number of periodic components is imposed, the model with  $F = 2$  individual cycles receives the greatest posterior probability. It seems, that the APC(2) specification is a reasonable choice among competing time varying GARCH-type processes. It receives substantial data support and imposes time variability of parameters in a parsimonious way. This result is invariant with respect to the changes in the conditional distribution and also is observed in both cases of the functional form of  $h_t$ . Only in the case of GARCH(1,1) model with conditional Student- $t$  distribution the model with  $F = 2$  individual periodic components has the same posterior probability as the model with  $F = 1$ . Additionally, constancy of parameters receives almost zero posterior probability despite of very high prior probability attached to this case.

In Table 2 we present the results of Bayesian estimation of parameters within a selected subset of models. We consider four basic pure GARCH-type models with constant parameters, namely the conditionally Normal GARCH(1,1) model (N-GARCH), the conditionally Student- $t$  GARCH(1,1) model ( $t$ -GARCH), the conditionally Normal GJR(1,1) model (N-GJR) and the conditionally Student- $t$  GJR(1,1) model ( $t$ -GJR). Additionally we consider appropriate APC( $F$ ) generalisations of those models with the greatest posterior probability, discussed previously. We report the following posterior summaries: the mean ( $E(.|y)$ ), the modal value ( $Mod(.|y)$ ) and the standard deviation ( $D(.|y)$ ) of model parameters. Posterior inference about the autocorrelation parameter  $\rho$  remains unchanged. The marginal posterior distribution for  $\rho$  is very strongly concentrated around value  $\rho = 0.16$  in all specifications. Posterior inference about  $\delta$  is also qualitatively the same

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in each of models, however the localisation and spread of its posterior distribution changes slightly. It seems, that in the case of the models with the GARCH(1,1) equation for  $h_t$ , the posterior distribution of  $\delta$  is localised around value  $\delta = 0.047$ , while in the case of models with the GJR(1,1) equation defined for  $h_t$ , we may report a slight change of location. The posterior inference on the degrees of freedom parameter  $\nu$  in the conditionally Student- $t$  models clearly confirms the results of model comparison. For each specification the marginal posterior distribution of  $\nu$  is strongly concentrated around value  $\nu = 6$ , with rather small posterior standard deviation. The data support moment existence at least up the fifth order, and consequently, the assumption  $\nu > 4$ , initially imposed, is also empirically valid.

Another interesting empirical feature can be observed when analysing posterior summaries for parameters describing  $h_t$ . In the case of the GJR(1,1) specifications, we see strong support in favour of asymmetric reaction of the conditional variance of  $\xi_t$  to the news. According to the relative differences between the posterior expectations of the parameters  $\alpha_1^+$  and  $\alpha_1^-$ , the dataset supports a news impact curve with asymmetric shape. It seems that the volatility increases three to six times faster as the *bad news* from the past of the process is observed, compared to the case, when the *good news* are considered. The asymmetric effect seems to be strengthened in the case of the models

with time varying parameters. The ratio  $\frac{\alpha_1^-}{\alpha_1^+}$ , calculated for the values representing posterior expectations, increases from 3.65 to 4.77 in the case of the N-GJR model and increases from 4.95 to 6.56 in the case of the  $t$ -GJR model.

In contrast, the posterior distributions of parameters describing time variability in  $h_t$  are irregular, as the expectation and modal value in these cases may be located in different areas of the parameter space. We see relative strong dispersion of the parameters controlling amplitudes in (7), namely for  $\gamma_{s1}$ ,  $\gamma_{s2}$ ,  $\gamma_{c1}$  and  $\gamma_{c2}$  in the case of the conditionally Student- $t$  APC(2) specification. For the frequency parameters  $\phi_1$  and  $\phi_2$  a more regular posterior distribution was obtained.

Figure 1 shows the results of posterior inference about the length of the period  $p_i$  of a single cyclical component in (2), induced by the posterior distribution of the frequency parameters  $\phi_i$  for  $i = 1, 2$  in the case of the APC(2) models. We show histograms of the length in years, according to the formula:

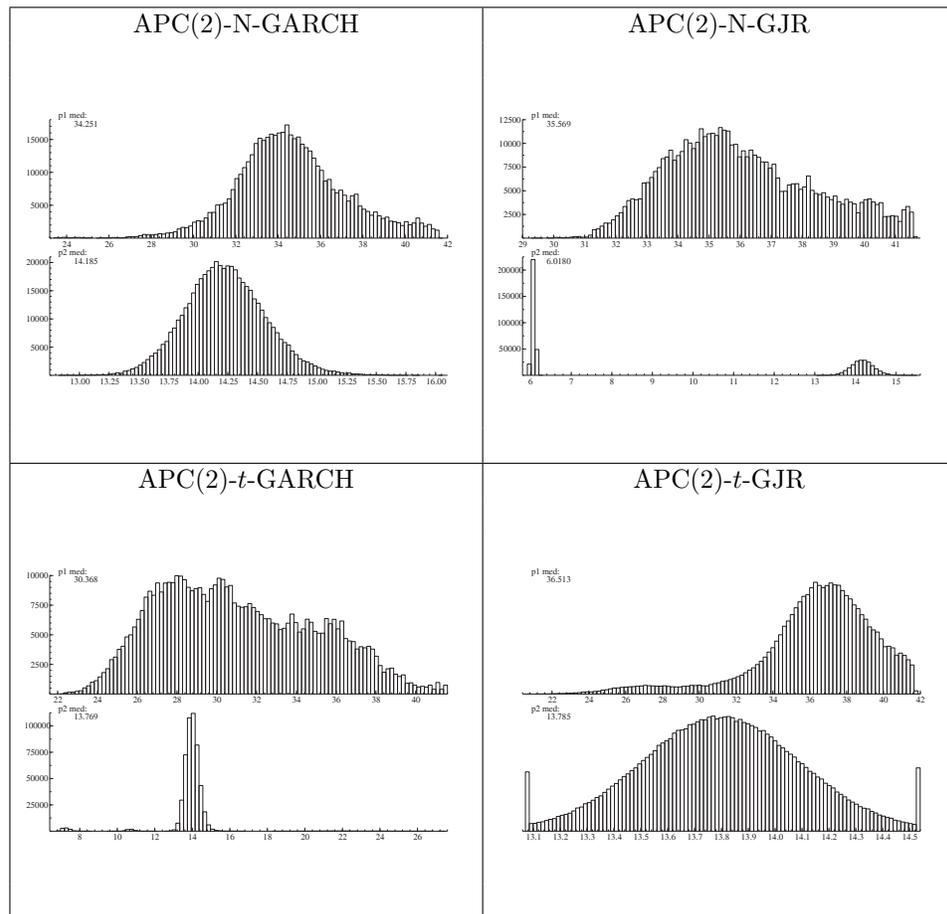
$$p_i = \frac{2\pi}{\phi_i 250}, i = 1, 2,$$

assuming 250 trading days per year. We compare histograms in cases where function  $f$  is defined as a sum of two different cyclical components. Those models receives the greatest posterior probability, as i was discussed above. Besides the conditionally Normal model with the GJR(1,1) structure in  $h_t$ , all specifications describe long term periodicity in the volatility in the qualitatively the same manner. The posterior distributions of the length in years are located around the values 31-47 for  $p_1$  and 14-15 for  $p_2$ , clearly indicating the existence of two cycles in the volatility, the first one of length about 14 years and the second one of length more than 34 years. This long

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term cycle is also supported in the case of the N-GJR model, however multimodality of the marginal distribution of  $p_2$  leaves great uncertainty about the remaining cyclical component of this model.

Figure 1: Posterior inference about the length of the period (in years)  $p_i$  induced by frequency parameters  $\phi_i$  according to the transformation  $p_i = \frac{2\pi}{250\phi_i}$ ,  $i = 1, 2$ .

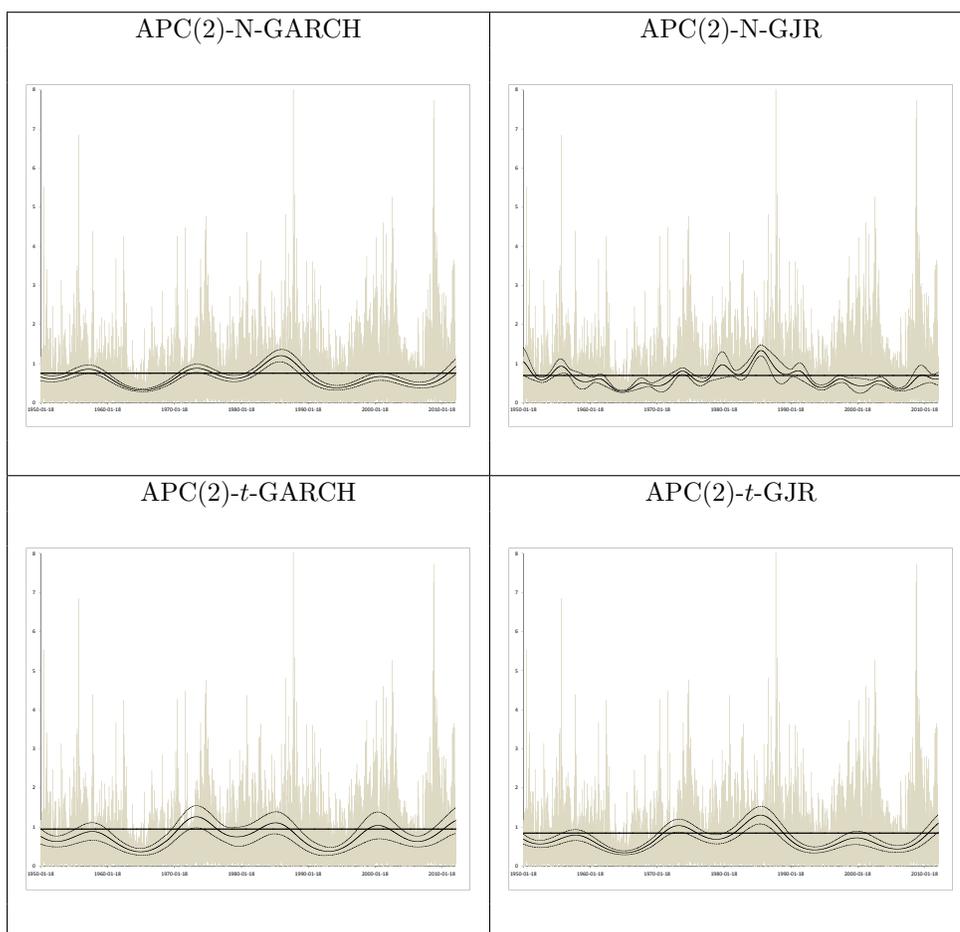


The effect of cyclicity for the unconditional variance of the error term in (8) and its empirical importance is depicted in Figure 2. We present the plot of the series of the absolute returns and the posterior means of the unconditional variance calculated for each data point, with bounds covering the range of two posterior standard deviations. We also plotted the posterior mean of the unconditional variance obtained on the

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basis of the corresponding GARCH-type model with constant parameters. Again, the data clearly support the variability of the unconditional variance. The constancy of parameters is precluded since the changes of variance in time are substantial. The plots of unconditional variance obtained in each model are characterised by fluctuations, strongly associated with long term changes in volatility in line with boom and bust periods on the US Stock Exchange. The similarity of the dynamic pattern of the unconditional variance in the case of the GARCH models and the GJR model with the conditional Student- $t$  distribution is clear. As an exception, the conditionally Normal GJR model exhibits some irregularities, due to rather nonstandard shape of the marginal posterior distribution of one of the frequencies.

Figure 2: Posterior inference about unconditional variance of the error term



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An interesting aspect of the posterior inference about the unconditional variance is connected with changes in spread of the posterior distributions of  $V(\varepsilon_t)$ . We see that it strongly declines in periods characterised by low volatility, but it becomes much greater in the case of periods when volatility intensifies. This leaves much greater uncertainty about the possible deterministic profile of the cyclical component in the volatility equation in periods associated with crises when abnormal volatility is observed.

In the vast literature concerning the empirical properties of the US business cycle, an interesting analysis was conducted by Chauvet and Potter (2001). According to the Bayesian analysis presented in that paper, the posterior expectation of time to wait until the next recession (if we are in recession currently) is equal approximately 14 years. This fully corresponds to our posterior inference about frequencies  $\phi_i$  and length in years  $p_i$  discussed above. However, it is difficult to find a linkage between long term volatility cycles and the business cycle. Only in the case of the recession in the mid 70's and during the dotcom crisis at the beginning of the 21st century, a visible increase in unconditional variance accompanies economic slowdown. Also some short recessions in the 50's coexist with a long-term but relatively small increase in unconditional volatility.

## 6 Conclusions

The main purpose of this paper was to investigate properties of a simple generalisation of the GARCH process that would enable to model long term features of volatility. Variability of unconditional moments was described by a class of Almost Periodic (AP) functions, proposed by Corduneanu (1989). Since in our approach the unconditional second moment exhibits an almost periodic variability, the process can also be interpreted as a second order Almost Periodically Correlated (APC) stochastic process; see Hurd and Miamee (2007).

We make formal Bayesian statistical inference on the cyclicity of volatility changes and we present evidence in favour of the empirical importance of such an effect. The illustration was conducted on the basis of daily returns of the S&P500 index covering the period from the 18 January 1950 till the 7 February 2012.

According to Bayesian model comparison, the cyclical behaviour of unconditional variance was strongly supported, making GARCH-type specification with constant parameters improbable. This result was invariant with respect to the conditional distribution (Normal or Student- $t$ ) and the type of the volatility equation (pure GARCH(1,1) or asymmetric GJR(1,1)). Among competing specifications, the greatest posterior probability received models where the time variability of the unconditional variance is described by a combination of two different cycles, with periods equal about 14 and 30 years. Those cycles were attributed to relatively different amplitudes, making the dynamic pattern of the unconditional volatility rather complex. The posterior probabilities, reported in the empirical analyses

were obtained by imposing very informative prior model probabilities that strongly penalised unparsimonious specifications of the function, that enabled variability of the unconditional variance. Models with two individual cyclical components were chosen as a reasonable compromise linking substantial data support with parsimonious parameterisation. However, according to the marginal data density values, even more flexible, but at the same time very expanded, functional forms may be necessary to adequately describe deterministic time variability of the unconditional variance of the error term in competing APC( $F$ )-GARCH specifications.

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