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Analysis and comparison of the stability of discrete-time and continuous-time linear systems

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The asymptotic stability of discrete-time and continuous-time linear systems described by the equations $x_{i+1} = \bar{A}^k x_i$ and $\dot{x}(t) = A^k x(t)$ for k being integers and rational numbers is addressed. Necessary and sufficient conditions for the asymptotic stability of the systems are established. It is shown that: 1) the asymptotic stability of discrete-time systems depends only on the modules of the eigenvalues of matrix $\bar{A}^k$ and of the continuous-time systems depends only on phases of the eigenvalues of the matrix A^k, 2) the discrete-time systems are asymptotically stable for all admissible values of the discretization step if and only if the continuous-time systems are asymptotically stable, 3) the upper bound of the discretization step depends on the eigenvalues of the matrix A.

Key words: analysis, comparison, stability, discrete-time, continuous-time, linear system.

1. Introduction

The asymptotic stability is one of the basic notions of the theory of dynamical systems [1, 8, 10, 12]. It has been addressed in many books and papers [1, 3, 6, 10-12]. The approximation of positive standard and fractional stable continuous-time linear systems by suitable discrete-time systems has been analyzed in [3, 4]. Comparison of approximation methods of positive stable continuous-time linear systems by positive stable discrete-time systems has been presented in [5]. The influence of the value of discretization step on the stability of positive and fractional systems has been analyzed in [6]. Inverse systems of linear systems have been investigated in [7].

In this paper the asymptotic stability of discrete-time and continuous-time linear systems described by the equations $x_{i+1} = \bar{A}^k x_i$ and $\dot{x}(t) = A^k x(t)$ for k being integers and rational numbers will be investigated.

The paper is organized as follows. In section 2 the basic definitions and theorems concerning the asymptotic stability of continuous-time and discrete-time systems and theorem on the eigenvalues of the matrix function are recalled. The asymptotic stability

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of the discrete-time linear systems for k being integers and rational numbers are investigated in section 3. Similar problems for continuous-time linear systems are analyzed in section 4. Comparison of the stability of discrete-time and continuous-time linear systems is presented in section 5. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ — the set of real numbers, $\Re^{n\times m}$ — the set of $n\times m$ real matrices, I_n — the $n\times n$ identity matrix, Z_+ — the set of nonnegative integers.

2. Preliminaries

Consider the autonomous continuous-time linear system

$$\dot{x}(t) = Ax(t), \tag{1}$$

where $x(t)\in\Re^n$ is the state vector and $A\in\Re^{n\times n}$. The solution of (1) for the given initial condition has the form [1, 8, 10, 12]

$$x(t) = e^{At}x_0. \tag{2}$$

Definition 1 *The system* (1) *(or equivalently the matrix A) is called asymptotically stable if*

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for all} \quad x_0\in\Re^n \tag{3}$$

Theorem 3 [1, 8, 10, 12] *The system* (1) *(the matrix A) is asymptotically stable if and only if*

$$\operatorname{Re} s_l < 0 \Leftrightarrow \frac{\pi}{2} < \phi < \frac{3\pi}{2} \quad \text{for all} \quad l = 1,\ldots,n, \tag{4}$$

where $s_l = |s_l|e^{j\phi_l}$, $l = 1,\ldots,n$ are the eigenvalues of the matrix A, i.e. the roots of the equation

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 = 0. \tag{5}$$

Similarly, let us consider the autonomous discrete-time linear system [1, 8, 10, 12]

$$x_{i+1} = \bar{A}x_i, \quad i\in Z_+ = \{0,1,\ldots\}, \tag{6}$$

where $x_i\in\Re^n$ is the state vector and $\bar{A}\in\Re^{n\times n}$. The solution of (6) for the given initial condition x_0 has the form [1, 8, 10, 12]

$$x_i = \bar{A}^i 0, \quad i\in Z_+. \tag{7}$$

Definition 2 *The system* 6 *(or equivalently the matrix $\bar{A}$) is called asymptotically stable if*

$$\lim_{i\to\infty} x_i = 0 \quad \text{for all} \quad x_0\in\Re^n. \tag{8}$$

Theorem 4 [1, 8, 10, 12] *The system* (6) *(the matrix $\bar{A}$) is asymptotically stable if and only if*

$$|z_l| < 1 \quad \text{for all} \quad l = 1,\ldots,n, \tag{9}$$

where z_i, $l = 1,\ldots,n$ are the eigenvalues of the matrix $\bar{A}$, i.e. the roots of the equation

$$\det[I_n z - \bar{A}] = z^n + \bar{a}_{n-1}z^{n-1} + \ldots + \bar{a}_1 z + \bar{a}_0 = 0. \tag{10}$$

Theorem 5 *Let s_l, $l = 1,\ldots,n$ be the eigenvalues of the matrix $A \in \Re^{n\times n}$ and $f(s_l)$ be well defined on the spectrum $\sigma_A = \{s_1, s_2, \ldots, s_n\}$ of the matrix A, i.e. $f(s_l)$ are finite for $l = 1,\ldots,n$. Then $f(s_l)$, $l = 1,\ldots,n$ are the eigenvalues of the matrix $f(A)$.*

Proof The proof is given in [2, 9].

For example if s_l, $l = 1,\ldots,n$ are the nonzero eigenvalues (not necessary distinct) of the matrix $A \in \Re^{n\times n}$ then s_l^{-1}, $l = 1,\ldots,n$ are the eigenvalues of the inverse matrix A^{-1}.

3. Discrete-time linear systems

In this section the asymptotic stability of the system

$$x_{i+1} = \bar{A}^k x_i, \quad i \in Z_+ \tag{11}$$

will be investigated for k being integers ($k = \pm1, \pm2, \ldots$) and rational numbers $\left(k = \dfrac{p}{q},\ p,q - \text{integers}\right)$.

For $k = 1,2,\ldots$ we have the following theorem.

Theorem 6 *The linear system* (11) *is asymptotically stable for $k = 1,2,\ldots$ if and only if the linear system* (6) *is asymptotically stable.*

Proof By Theorem 3 if z_l, $l = 1,\ldots,n$ are the eigenvalues of the matrix $\bar{A}$ then the eigenvalues of the matrix $\bar{A}^k$ are z_l, $l = 1,\ldots,n$. Note that $|z_l| < 1$ for and $k = 1,2,\ldots$ if and only if the condition (9) is satisfied. Therefore, by Theorem 4 the system (11) is asymptotically stable if and only if the system (6) is asymptotically stable. □

Example 1 Consider the system (6) with

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} \tag{12}$$

The characteristic polynomial of (12) has the form

$$\det[I_2 z - \bar{A}] = \begin{vmatrix} z & -1 \\ -\frac{1}{6} & z - \frac{1}{6} \end{vmatrix} = z^2 - \frac{1}{6}z - \frac{1}{6} \tag{13}$$

and its zeros are $z_1 = \frac{1}{2}$ and $z_2 = -\frac{1}{3}$.

The eigenvalues of the matrix (12) satisfy the condition (9) and the system is asymptotically stable. By Theorem 6 the system (11) with (12) is also asymptotically stable for $k = 2, 3, \dots$.

For $k = -1, -2, \dots$ we have the following theorem.

Theorem 7 *The linear system* (11) *is asymptotically stable for* $k = -1, -2, \dots$ *if and only if the system* (6) *is unstable, i.e. the eigenvalues of the matrix* $\bar{A}$ *satisfy the condition*

$$|z_j| > 1 \quad \text{for} \quad j = 1, \dots, n. \tag{14}$$

Proof By Theorem 5 if z_j, $j = 1, \dots, n$ are the eigenvalues of the matrix $\bar{A}$ then the eigenvalues of the matrix $\bar{A}^k$ for $k = -1, -2, \dots$ are z_j^k, $k = 1, 2, \dots$. Note that $|z_j|^{-k} < 1$, $k = 1, 2, \dots$ if and only if the condition (14) is satisfied. Therefore, by Theorem 4 the system (11) is asymptotically stable for $k = -1, -2, \dots$ if and only if the system (6) is unstable. □

Example 2 (Continuation of Example 1) The inverse matrix of (12) has the form

$$\bar{A}^{-1} = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix} \tag{15}$$

and its eigenvalues are $\bar{z}_1 = 2$, $\bar{z}_2 = -3$. Therefore, the discrete-time linear system with the matrix (15) is unstable.

Note that for (15) we obtain the matrix

$$\bar{A}^{-2} = (\bar{A}^{-1})^2 = \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 7 & -6 \\ -1 & 6 \end{bmatrix} \tag{16}$$

and its eigenvalues are $\bar{z}_1 = 4$, $\bar{z}_2 = 9$. The linear system (11) for with (16) is unstable. Similar results can be obtained for $k = -3, -4, \dots$.

For $k = \pm\frac{p}{q}$, $p, q \in \{1, 2, \dots\}$ we have the following theorem

Theorem 8 *The linear system* (11) *is asymptotically stable*

1) *for* $k = \frac{p}{q}$, $p, q \in \{1, 2, \dots\}$ *if and only if the linear system* (6) *is asymptotically stable,*

2) *for* $k = -\frac{p}{q}$, $p, q \in \{1, 2, \dots\}$ *if and only if the linear system is unstable.*

Proof By Theorem 5 if z_j, $j = 1,\dots,n$, are the eigenvalues of the matrix $\bar{A}$ then the eigenvalues of the matrix $\bar{A}^{\pm\frac{p}{q}}$ are $z_j^{\pm\frac{p}{q}}$ for $j = 1,\dots,n$ and

$$\ln\left|z_j\right|^{\pm\frac{p}{q}} = \pm\frac{p}{q}\ln\left|z_j\right| \quad \text{for} \quad j = 1,\dots,n. \tag{17}$$

If $\frac{p}{q} > 0$ and $|z_j| < 1$, $j = 1,\dots,n$ then from (17) we have

$$\frac{p}{q}\ln\left|z_j\right| < 0 \quad \text{and} \quad \left|z_j\right|^{\frac{p}{q}} < 1 \quad \text{for} \quad j = 1,\dots,n. \tag{18}$$

Therefore, the system (11) is asymptotically stable for $k = \frac{p}{q} > 0$ if and only if the system (6) is asymptotically stable. Proof in the case 2) is similar. □

Example 3 (Continuation of Example 1) Consider the system (6) with (12) for $p = 3$, $q = 2$. Using (12) we obtain the matrix

$$\bar{A}^3 = \frac{1}{6^2}\begin{bmatrix} 1 & 7 \\ \frac{7}{6} & \frac{13}{6} \end{bmatrix} \tag{19}$$

with the eigenvalues $z_1 = \frac{1}{8}$, $z_2 = -\frac{1}{27}$. The eigenvalues of the matrix

$$\bar{A}^{\frac{3}{2}} = \begin{bmatrix} 0 & 1 \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}^{\frac{3}{2}} \tag{20}$$

are $\hat{z}_1 = \left(\frac{1}{2}\right)^{\frac{3}{2}}$, $\hat{z}_2 = \left(-\frac{1}{3}\right)^{\frac{3}{2}}$ and satisfy the condition (9). Therefore, by Theorem 7 the system (6) with (12) for $p = 3$, $q = 2$ is asymptotically stable.

Remark 1 The asymptotic stability of the discrete-time system (6) depends only on the modules of the eigenvalues of the matrix $\bar{A}$ and it is independent of the phases of the eigenvalues.

Remark 2 The matrix $-\bar{A} \in \Re^{n\times n}$ is asymptotically stable if and only if the matrix $A \in \Re^{n\times n}$ is asymptotically stable since the eigenvalues of the matrices A and $-\bar{A}$ have the same modules.

4. Continuous-time linear systems

In this section the asymptotic stability of the continuous-time linear system

$$\dot{x}(t) = A^k x(t), \quad A \in \Re^{n\times n} \tag{21}$$

will be investigated for k being integers ($k = \pm 1, \pm 2, ...$) and rational numbers $\left(k = \frac{p}{q},\ p,q - \text{integers}\right)$.

Theorem 9 *Let $s_l = |s_l|e^{j\phi_l}$, $l = 1,\dots,n$ be the l-th eigenvalue of the matrix A. The system* (6) *is asymptotically stable if and only if*

$$\frac{\pi}{2} < k\phi_l < \frac{3\pi}{2} \quad \text{for} \quad l = 1,\dots,n. \tag{22}$$

Proof By Theorem 5 if s_l is the l-th eigenvalue of the matrix A then s_l^k, $l = 1,\dots,n$ are the eigenvalues of the matrix A^k and by Theorem 3 the system (21) is asymptotically stable if and only if the condition (22) is satisfied. □

From the condition (22) of Theorem 9 we have the following conclusion.

Conclusion 1 The asymptotic stability of the system (21) for any k depends only on the phases of the eigenvalues s_l, $l = 1,\dots,n$ of the matrix A and it is independent of their modules.

Example 4 Consider the asymptotic stability of the continuous-time linear system (21) with the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \tag{23}$$

for $k = 2,3$ and $k = -1,-2,-3$. The characteristic polynomial of the matrix (4.3) has the form

$$\det[I_2 s - A] = \begin{vmatrix} s & -1 \\ 1 & s+1 \end{vmatrix} = s^2 + s + 1 \tag{24}$$

and its zeros are

$$s_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j\frac{2\pi}{3}}, \quad s_1 = -\frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j\frac{2\pi}{3}}. \tag{25}$$

Therefore, the system (21) with (23) for $k = 1$ is asymptotically stable since (25) satisfy the condition (22).

It is easy to verify that for (23)

$$A^2 = A^{-1} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad A^{-2} = A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \tag{26}$$

and the matrices have the same characteristic polynomial (24) and are asymptotically stable. Note that

$$A^3 = A^{-3} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{27}$$

and the system (21) with (27) is unstable. The same result follows for (27) from the condition (22) since for (25) with $k = \pm 3$ we have the phases $\pm 3\frac{2\pi}{3} = \pm 2\pi$.

Example 5. Consider the asymptotic stability of the system (21) with the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \tag{28}$$

for $k = -1, -2, -3, 2, 3, \frac{1}{2}$. The characteristic polynomial of the matrix (27) has the form

$$\det[I_2 s - A] = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s^2 + 3s + 2 \tag{29}$$

and its zeros are $s_1 = -1$, $s_2 = -2$. Thus, the system for $k = 1$ is asymptotically stable. For $k = -1$ we have

$$A^{-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \tag{30}$$

and

$$\det[I_2 s - A^{-1}] = \begin{vmatrix} s + \frac{3}{2} & \frac{1}{2} \\ -1 & s \end{vmatrix} = s^2 + \frac{3}{2}s + \frac{1}{2} \tag{31}$$

and the eigenvalues of (30) are $s_1 = -1 = e^{j180^\circ}$, $s_2 = -\frac{1}{2} = \frac{1}{2}e^{j180^\circ}$. The system (21) with (30) for is asymptotically stable (the condition (22) is satisfied). For $k = -2$ we obtain the matrix

$$A^{-2} = \begin{bmatrix} \frac{7}{4} & \frac{3}{4} \\ -\frac{3}{2} & -\frac{1}{2} \end{bmatrix} \tag{32}$$

with the eigenvalues $s_1 = 1 = e^{j0^\circ}$, $s_2 = \frac{1}{4} = \frac{1}{4}e^{j0^\circ}$. Therefore, the system (21) with (32) is unstable.

The same result follows from (22) since $k\phi = -2 \cdot 180^0 = 0^0$. For $k = -3$ we obtain the matrix

$$A^{-3} = \begin{bmatrix} -\frac{15}{8} & -\frac{7}{8} \\ \frac{7}{4} & \frac{3}{4} \end{bmatrix} \tag{33}$$

with the eigenvalues $s_1 = -1 = e^{j180^\circ}$, $s_2 = -\frac{1}{8} = \frac{1}{8}e^{j180^\circ}$. Therefore, the system is asymptotically stable. The same result follows from (22). For $k = 2$ we obtain the matrix

$$A^2 = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix} \tag{34}$$

with the eigenvalues $s_1 = 1 = e^{j0^\circ}$, $s_2 = 4 = 4e^{j0^\circ}$. The system (21) with (34) is unstable. For $k = 3$ we have the matrix

$$A^3 = \begin{bmatrix} 6 & 7 \\ -14 & -15 \end{bmatrix} \tag{35}$$

with the eigenvalues $s_1 = -1 = e^{j180^\circ}$, $s_2 = -8 = 8e^{j180^\circ}$. Therefore, the system for $k = 3$ is asymptotically stable. In general case we obtain that the system (21) with (28) is asymptotically stable for $k = \pm(1+2l)$, $l = 0, 1, \ldots$ and unstable for $k = \pm 2l$, $l = 1, 2, \ldots$

Theorem 10 *If the matrix $A \in \Re^{n\times n}$ has at least one real positive eigenvalue then the system* (21) *is unstable for all values of k (integer and rational).*

Proof By Theorem 5 if s_l, $l = 1, \ldots, n$ are the real positive eigenvalues of the matrix A then s_l^k, $l = 1, \ldots, n$, are the real positive eigenvalues of the matrix A^k and the system (21) is unstable. □

Example 6 Consider the system (21) with the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 2 \end{bmatrix} \tag{36}$$

for $k = 2$. The characteristic polynomial of the matrix (36) has the form

$$\det[I_3 s - A] = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & 2 & s-2 \end{vmatrix} = s^3 - 2s^2 + 2s - 1 \tag{37}$$

and its zeros are: $s_1 = 1 = e^{j0^\circ}$, $s_2 = \frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j60^\circ}$, $s_3 = \frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j60^\circ}$. The system (21) with (36) for is unstable. Using (36) we obtain the matrix

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -2 & 2 \\ 2 & -3 & 2 \end{bmatrix}. \tag{38}$$

Characteristic polynomial of (38) has the form

$$\det[I_3 s - A^2] = \begin{bmatrix} s & 0 & -1 \\ -1 & s+2 & -2 \\ -2 & 3 & s-2 \end{bmatrix} \tag{39}$$

and its zeros are $s_1 = 1 = e^{j0^\circ}$, $s_2 = -\frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j120^\circ}$, $s_3 = -\frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j120^\circ}$.

The system (21) with (36) for $k = 2$ is unstable. By Theorem 10 it is unstable for any k. The following example shows that the system (21) can be unstable for $k = 1,2$ and asymptotically stable for $k = 3l$, $l = 1,2,\ldots$.

Example 7 Consider the system (21) with the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \tag{40}$$

for $k = 1,2,3,\ldots$. The characteristic polynomial of (40) has the form

$$\det[I_3 s - A] = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 0 & s \end{vmatrix} = s^3 + 1 \tag{41}$$

and its zeros are: $s_1 = -1 = e^{j180^\circ}$, $s_2 = \frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j60^\circ}$, $s_3 = \frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j60^\circ}$. For $k = 1$ the condition (22) is not satisfied and the system is unstable. For $k = 2$ we have the matrix

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \tag{42}$$

with the eigenvalues $s_1 = 1 = e^{j0^\circ}$, $s_2 = -\frac{1}{2} + j\frac{\sqrt{3}}{2} = e^{j120^\circ}$, $s_3 = -\frac{1}{2} - j\frac{\sqrt{3}}{2} = e^{-j120^\circ}$ and the system is also unstable.

For $k=3$ we obtain the matrix

$$A^3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{43}$$

with the eigenvalues $s_1 = s_2 = s_3 = -1 = e^{j180^\circ}$. Therefore, the system for $k=3$ is asymptotically stable.

It is easy to prove that the system is asymptotically stable for $k = 3l$, $l = 1,2,\dots$.

5. Comparison of the stability of discrete-time and continuous-time linear systems

From the conditions (4) and (9), Remark 1 and Conclusion 1 it follows that the asymptotic stability of the discrete-time linear systems depends only on the modules of the eigenvalues of the matrix $\bar{A}$ and of the continuous-time linear systems only on the phases of the eigenvalues of the matrix A.

To obtain to the continuous-time linear system (1) the corresponding discrete-time linear system (6) we apply the approximation

$$\dot{x}(t) \approx \frac{x(t+h)-x(t)}{h} = \frac{x_{i+1}-x_i}{h} = Ax_i, \quad i \in Z_+ \tag{44}$$

where $x_i = x(t)$, $x_{i+1} = x(t+h)$, $h = \Delta t > 0$. From (44) we have

$$x_{i+1} = \bar{A}x_i \tag{45}$$

where

$$\bar{A} = I_n + hA. \tag{46}$$

From Theorem 5 applied to (46) we obtain

$$z_l = 1 + hs_l, \quad l = 1,\dots,n \tag{47}$$

where z_l are the eigenvalues of the matrix $\bar{A}$ and s_l are the eigenvalues of the matrix A.

Theorem 11 *The discrete-time linear system* (45) *is asymptotically stable for all admissible values of* $h > 0$ *if and only if the continuous-time linear system* (1) *is asymptotically stable.*

Proof From (47) we have

$$s_l = \frac{z_l - 1}{h} = \frac{|z_l|\, e^{j\psi_l} - 1}{h}, \quad l = 1,\dots,n, \tag{48}$$

where $|z_l|$ and ψ_l are the module and phase of z_l and

$$\operatorname{Re} s_l = \frac{|z_l|\cos\psi_l - 1}{h}, \quad l = 1,\dots,n. \tag{49}$$

From (49) it follows that $\operatorname{Re} s_l < 0$ for any admissible $h > 0$ if and only if $|z_l| < 1$, i.e. the discrete-time system is asymptotically stable.

Similarly, from (47) for $s_l = |s_l|e^{j\phi_l}$ we have

$$|z_l|^2 = |1+hs_l|^2 = [1+h|s_l|\cos\phi_l]^2 + [h|s_l|\sin\phi_l]^2 = 1+2h|s_l|\cos\phi + h^2|s_l|^2 < 1 \tag{50}$$

and $|z_l|^2 < 1$ if and only if $\cos\phi < 0$ or equivalently the condition (4) is satisfied. □

Note that the admissible value of $h > 0$ should satisfy the condition (50).

Theorem 12 *The discretization step h of the asymptotically stable systems satisfies the condition*

$$h < \min_{1\leqslant l\leqslant n} \frac{2\alpha_l}{\alpha_l^2+\beta_l^2}, \tag{51}$$

where $s_l = -\alpha_l + j\beta_l$, $l = 1,\dots,n$ *are the eigenvalues of the matrix* A.

Proof From (47) it follows that the discrete-time system (45) is asymptotically stable if and only if

$$|z_l| = |hs_l+1| = |1-h\alpha_l+jh\beta_l| < 1 \quad \text{for} \quad l = 1,\dots,n. \tag{52}$$

From (52) we have

$$(1-h\alpha_l)^2 + (h\beta_l)^2 < 1 \tag{53}$$

and solving (53) with respect to h we obtain (51). □

Example 8 Consider the continuous-time linear system (1) with the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}. \tag{54}$$

The characteristic polynomial of (54) has the form

$$\det[I_2 s - A] = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s^2+3s+2 \tag{55}$$

and the eigenvalues of the matrix (54) are $s_1 = -1$, $s_2 = -2$. The system (1) with (54) is asymptotically stable. The eigenvalues of the corresponding matrix

$$\bar{A} = I_n + hA = \begin{bmatrix} 1 & h \\ -2h & 1-3h \end{bmatrix} \tag{56}$$

of discrete-time system are $z_1 = 1-h$, $z_1 = 1-2h$. The discrete-time system (45) with (56) is asymptotically stable for all $0 < h < 1$.

6. Concluding remarks

The asymptotic stability of discrete-time linear systems (11) and continuous-time linear systems (21) for k integers ($k = \pm 1, \pm 2, \ldots$) and rational $\left(\frac{p}{q},\ p, q-\text{ integers}\right)$ has been investigated. Necessary and sufficient conditions for the asymptotic stability of the systems have been established (Theorems 6, 7, 8, 9, 10). It has been shown that:

1) The asymptotic stability of (11) depends only on the modules of the eigenvalues of the matrix $\bar{A}^k$ and of (21) only on the phases of the eigenvalues of the matrix A^k.

2) The discrete-time systems (11) are asymptotically stable for all admissible values of h if and only if the continuous-time systems (21) are asymptotically stable.

3) The upper bounds of h depends on the eigenvalues of the matrix A.

The considerations have been illustrated by numerical examples of discrete-time and continuous-time linear systems.

The presented considerations can be extended to positive discrete-time and continuous-time linear systems. An open problem is an extension of the considerations to fractional linear systems.

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