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Minimum energy control of fractional positive continuous-time linear systems using Caputo-Fabrizio definition

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Abstract. The Caputo-Fabrizio definition of the fractional derivative is applied to minimum energy control of fractional positive continuous-time linear systems with bounded inputs. Conditions for the reachability of standard and positive fractional linear continuous-time systems are established. The minimum energy control problem for the fractional positive linear systems with bounded inputs is formulated and solved.

Key words: fractional positive continuous-time system, minimum energy control, bounded inputs.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [1, 2]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

Mathematical fundamentals of the fractional calculus are given in the monographs [3–5]. The positive fractional linear systems have been investigated in [6–9]. Stability of fractional linear continuous-time systems has been investigated in [9, 10]. The notion of practical stability of positive fractional discrete-time linear systems has been introduced in [11]. Descriptor fractional discrete-time linear systems with different orders have been addressed in [12]. The positivity and stability of fractional discrete-time nonlinear systems have been analyzed in [13], the Drazin inverse matrix method for analysis descriptor fractional discrete-time linear systems has been proposed [14]. Some recent interesting results in fractional systems theory and its applications can be found in [5, 15–17].

The minimum energy control problem for standard linear systems has been formulated and solved by J. Klamka in [18–20] and for 2D linear systems with variable coefficients in [21]. The controllability and minimum energy control problem of fractional discrete-time linear systems has been investigated by Klamka in [22]. The minimum energy control of positive continuous-time linear systems has been addressed in [23–25] and for positive discrete-time linear systems in [26, 27]. The minimum energy control problem for positive fractional electrical circuits has been investigated in [24] and for positive fractional linear systems with two different fractional orders in [28]. Robust stability and stabilization of the continuous-time fractional positive systems has been considered in [29, 30] and continuous-time fractional positive systems with bounded states in [31].

Recently a new definition of the fractional derivative without singular kernel has been proposed in [32, 33].

In this paper, the Caputo-Fabrizio definition of the fractional derivative will be applied to the minimum energy control problem for fractional positive continuous-time linear systems with bounded inputs.

The paper is organized as follows. In Section 2 the conditions for the reachability of the standard and positive fractional linear continuous-time systems will be given. The minimum energy control problem for the fractional positive continuous-time linear systems with bounded inputs is formulated and solved in Section 3. Procedure for computation of the optimal input that steers the state of the system from zero initial state to the desired final state is given and illustrated by example of positive fractional electrical circuit in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices, I_n – the $n\times n$ identity matrix.

2. Reachability of standard fractional systems

The Caputo-Fabrizio definition of fractional derivative of order α of the function $f(t)$ for $0<\alpha<1$ has the form [32, 33]

$$ {}^{CF}D^\alpha f(t) = \frac{1}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)\dot{f}(\tau)d\tau\,, \qquad \dot{f}(\tau) = \frac{df(\tau)}{d\tau}\,,\ t\geq 0\,. \tag{1}$$

Consider the fractional differential state equations

$$ {}^{CF}D^\alpha x(t) = \frac{d^\alpha x(t)}{dt^\alpha} = Ax(t)+Bu(t)\,,\ 0<\alpha<1 \tag{2a}$$

$$ y(t) = Cx(t)+Du(t), \tag{2b}$$

where $u(t)\in\Re^n$, $u(t)\in\Re^m$, $y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times n}$.

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Theorem 1. The solution $x(t)$ of the equation (2a) for a given initial condition $x(0) = x_0$ and input $u(t)$ has the form

$$x(t) = e^{\hat{A}t}(\hat{x}_0 + \hat{B}u_0) + \int_0^t e^{\hat{A}(t-\tau)}\hat{B}[\beta u(\tau) + \dot{u}(\tau)]d\tau, \quad (3a)$$

where

$$\begin{aligned} &\hat{A} = \alpha[I_n - (1-\alpha)A]^{-1}A, \\ &\hat{B} = [I_n - (1-\alpha)A]^{-1}(1-\alpha)B, \\ &\beta = \frac{\alpha}{1-\alpha}, \hat{x}_0 = [I_n - (1-\alpha)A]^{-1}x_0, \\ &e^{\hat{A}t} = \mathcal{L}^{-1}\{[I_n s - \hat{A}]^{-1}\}, \ \dot{u}(\tau) = \frac{du(\tau)}{d\tau}, \ u(0) = u_0. \end{aligned} \quad (3b)$$

Proof. The proof is given in [13] under the assumption that the matrix $[I_n - (1 - \alpha)A]$ is non-singular.

Definition 1. A state $x_f \in \Re^n$ of the standard system (2) is called reachable in time $t \in [0, t_f]$ if there exists an input $u(t) \in \Re^m$ for $t \in [0, t_f]$ which steers the state of the system from zero initial condition $x_0 = 0$ to the final state $x_f \in x(t_f)$. If every state $x_f \in \Re^n$ is reachable in time $t \in [0, t_f]$ then the system is called reachable in time $t \in [0, t_f]$. The system (2) is called reachable if for every $x_f \in \Re^n$ there exists t_f and an input $u(t) \in \Re^m$ for $t \in [0, t_f]$ which steers the state of the system from $x_0 = 0$ to x_f.

Theorem 2. The standard fractional system (2) is reachable in time $t \in [0, t_f]$ if and only if the matrix

$$R_f = R(t_f) = \int_0^{t_f} e^{\hat{A}t}\hat{B}\hat{B}^T e^{\hat{A}^T t}dt \quad (4)$$

is invertible.
The input which steers the state of the system from $x_0 = 0$ to x_f is given by

$$u(t) = \int_0^t e^{-\beta(t-\tau)}\hat{B}^T e^{\hat{A}^T(t_f-\tau)}d\tau R_f^{-1}x_f, \quad t \in [0, t_f] \text{ and } u_0 = u(0) = 0. \quad (5)$$

Proof. Substituting

$$\overline{u}(t) = \beta u(t) + \dot{u}(t) \quad (6)$$

into (3a) for $x_0 = 0$, $u_0 = 0$ we obtain

$$x(t) = \int_0^t e^{\hat{A}(t-\tau)}\hat{B}\overline{u}(\tau)d\tau. \quad (7)$$

The solution of the differential equation (6) for $u_0 = u(0) = 0$ has the form

$$u(t) = \int_0^t e^{-\beta(t-\tau)}\overline{u}(\tau)d\tau. \quad (8)$$

To show that the input

$$\overline{u}(t) = \hat{B}^T e^{\hat{A}(t_f-t)}R_f^{-1}x_f, \ t \in [0, t_f] \quad (9)$$

steers the state from $x_0 = 0$ to x_f in time $t \in [0, t_f]$ we substitute (9) into (7) and we obtain

$$\begin{aligned} x(t_f) &= \int_0^{t_f} e^{\hat{A}(t_f-\tau)}\hat{B}\hat{B}^T e^{\hat{A}^T(t_f-\tau)}d\tau R_f^{-1}x_f = \\ &= R_f R_f^{-1}x_f = x_f. \end{aligned} \quad (10)$$

Substituting (9) into (8) we obtain (5). □
From Theorem 1 and its proof follows the corollary.

Corollary 1. The fractional system (2) is reachable in time $t \in [0, t_f]$ if and only if the fractional system

$$\frac{d^\alpha x(t)}{dt^\alpha} = \hat{A}x(t) + \hat{B}u(t) \quad (11)$$

is reachable in time $t \in [0, t_f]$.

The input $\overline{u}(t)$ steers the state $x(t)$ from $x_0 = 0$ to x_f in time $t \in [0, t_f]$ of the system (11) if and only if the input (8) steers the state from $x_0 = 0$ to x_f in time $t \in [0, t_f]$ of the system (2a).

Definition 2. The fractional system (11) is called (internally) positive if $x(t) \in \Re_+^n$, $t \geq 0$ for every $x_0 \in \Re_+^n$ and all $u(t) \in \Re_+^m$, $t \geq 0$.

Theorem 3. The fractional system (11) is positive if and only if $\hat{A} \in M_n$ and $\hat{B} \in \Re_+^{n\times m}$.

Proof is similar to the one given in [9].

Definition 3. A state $x_f \in \Re_+^n$ of the positive system (2) is called reachable in time $t \in [0, t_f]$ if there exists an input $u(t) \in \Re_+^m$ for $t \in [0, t_f]$ which steers the state of the system from zero initial condition $x_0 = 0$ to the final state $x_f \in \Re_+^n$. If every state $x_f \in \Re_+^n$ is reachable in time $t \in [0, t_f]$ then the system is called reachable in time $t \in [0, t_f]$. The positive system (2) is called reachable if for every $x_f \in \Re_+^n$ there exists t_f and an input $u(t) \in \Re_+^m$ for $t \in [0, t_f]$ which steers the state of the system from $x_0 = 0$ to x_f.

Definition 4. A matrix $A \in \Re^{n\times n}$ is called monomial if in each row and in each column only one entry is positive and the remaining entries are zero.

Theorem 4. The positive fractional system (2) is reachable in time $t \in [0, t_f]$ if the matrix

$$R_f = R(t_f) = \int_0^{t_f} e^{\hat{A}t}\hat{B}\hat{B}^T e^{\hat{A}^T t}dt \quad (12)$$

is monomial.

The input which steers the state of the system from $x_0 = 0$ to x_f is given by

$$u(t) = \int_0^t e^{-\beta(t-\tau)} \hat{B}^T e^{\hat{A}^T(t_f-\tau)} d\tau R_f^{-1} x_f. \tag{13}$$

Proof. It is well-known [2] that $R_f^{-1} \in \Re_+^{n\times n}$ if and only if the matrix $R_f \in \Re_+^{n\times n}$ is monomial. In a similar way as in proof of Theorem 1 it can be shown that the input (13) steers the state of positive system from $x_0 = 0$ to $x_f \in \Re_+^n$ in time $t \in [0, t_f]$. From (13) it follows that $u(t) \in \Re_+^m$ since $e^{-\beta t} > 0$ for $\beta = \frac{\alpha}{1-\alpha} > 0$, $0 < \alpha < 1$, $\hat{B}^T e^{\hat{A}(t_f-\tau)} \in \Re_+^{m\times n}$ and $R_f^{-1} x_f \in \Re_+^n$. □

3. Problem formulation and its solution

Consider the fractional positive system (2) with $A \in M_n$ and $B \in \Re_+^{n\times m}$ monomial. If the system is reachable in time $t \in [0, t_f]$, then usually there are many different inputs $u(t) \in \Re_+^n$ steering the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$. Among these inputs we are looking for input $u(t) \in \Re_+^n$, $t \in [0, t_f]$ satisfying the condition

$$u(t) \le U \in \Re_+^n, \ t \in [0, t_f] \tag{14}$$

that minimizes the performance index

$$I(u) = \int_0^{t_f} u^T(\tau) Q u(\tau) d\tau, \tag{15}$$

where $Q \in \Re_+^{n\times n}$ is a symmetric positive definite matrix and $Q^{-1} \in \Re_+^{n\times n}$.

The performance index (15) is a measure of the energy used for steering the state of the systems from $x_0 = 0$ to x_f.

The minimum energy control problem for the fractional positive system (2) can be stated as follows.

Given the matrices $A \in M_n$ monomial, $B \in \Re_+^{n\times m}$, α, $U \in \Re_+^n$ and $Q \in \Re_+^{n\times n}$ of the performance matrix (15), $x_f \in \Re_+^n$ and $t > 0$, find an input $u(t) \in \Re_+^n$ for $t \in [0, t_f]$ satisfying (14) that steers the state vector of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (15).

To solve the problem we define the matrix

$$W(t_f) = \int_0^{t_f} e^{\hat{A}(t_f-\tau)} \hat{B} Q^{-1} \hat{B}^T e^{\hat{A}^T(t_f-\tau)} d\tau. \tag{16}$$

From (16) and Theorem 4, it follows that the matrix (16) is monomial if and only if the fractional positive system (2) is reachable in time $t \in [0, t_f]$. In this case we may define the input

$$\hat{u}(t) = Q^{-1} \hat{B}^T e^{\hat{A}^T(t_f-t)} W^{-1}(t_f) x_f \quad \text{for } t \in [0, t_f]. \tag{17}$$

Note that the input (17) satisfies the condition $u(t) \in \Re_+^n$ for $t \in [0, t_f]$ if

$$Q^{-1} \in \Re_+^{n\times n} \text{ and } W^{-1}(t_f) \in \Re_+^{n\times n}. \tag{18}$$

Theorem 5. Let $\tilde{u}(t) \in \Re_+^n$ for $t \in [0, t_f]$ be an input satisfying (14) that steers the state of the fractional positive system (2) from $x_0 = 0$ to $x_f \in \Re_+^n$. Then the input (17) satisfying (14) also steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (15), i.e. $I(\hat{u}) \le I(\tilde{u})$.

The minimal value of the performance index (15) is equal to

$$I(\hat{u}) = x_f^T W^{-1}(t_f) x_f. \tag{19}$$

Proof. If the conditions (18) are met then the input (17) is well defined and $\hat{u}(t) \in \Re_+^n$ for $t \in [0, t_f]$. We shall show that the input steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$. Substitution of (17) into (3a) for $x_0 = 0$, $u_0 = 0$ and $t = t_f$ yields

$$\begin{aligned} x(t_f) &= \int_0^{t_f} e^{\hat{A}(t_f-\tau)} \hat{B} \hat{u}(\tau) d\tau = \\ &= \int_0^{t_f} e^{\hat{A}(t_f-\tau)} \hat{B} Q^{-1} \hat{B}^T e^{\hat{A}^T(t_f-\tau)} d\tau W_f^{-1} x_f = x_f \end{aligned} \tag{20}$$

since (16) holds.

To find $t \in [0, t_f]$ for which $\hat{u}(t) \in \Re_+^n$ reaches its maximal value using (17) we compute the derivative

$$\frac{d\hat{u}(t)}{dt} = Q^{-1} B^T \Psi(t) W^{-1}(t_f) x_f, \ t \in [0, t_f] \tag{21}$$

where

$$\Psi(t) = \frac{d}{dt} e^{\hat{A}^T(t_f-t)}. \tag{22}$$

Knowing $\Psi(t)$ and using the equality

$$\Psi(t) W^{-1}(t_f) x_f = 0 \tag{23}$$

we can find $t \in [0, t_f]$ for which $\hat{u}(t)$ reaches its maximal value and we check if the condition (14) is satisfied. If not, we increase the value of t_f so that the condition is satisfied.

Note that if the system is asymptotically stable $\lim_{t\to\infty} e^{\hat{A}t} = 0$ then $u(t)$ reaches its maximal value for $t = t_f$ and if it is unstable then for $t = 0$.

By assumption the inputs $\tilde{u}(t)$ and $\hat{u}(t)$, $t \in [0, t_f]$ steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$, i.e.

$$x_f = \int_0^{t_f} e^{\hat{A}(t_f-\tau)} \hat{B} \tilde{u}(\tau) d\tau = \int_0^{t_f} e^{\hat{A}(t_f-\tau)} \hat{B} \hat{u}(\tau) d\tau \tag{24a}$$

and

$$\int_0^{t_f} e^{\hat{A}(t_f-\tau)} \hat{B} [\tilde{u}(\tau) - \hat{u}(\tau)] d\tau = 0. \tag{24b}$$

By transposition of (24b) and postmultiplication by $W^{-1}(t_f)x_f$ we obtain

$$\int_0^{t_f} [\tilde{u}(\tau) - \hat{u}(\tau)]^T \hat{B}^T e^{\hat{A}^T(t_f-\tau)} d\tau W^{-1}(t_f)x_f = 0. \quad (25)$$

Substitution of (17) into (25) yields

$$\begin{aligned} &\int_0^{t_f} [\tilde{u}(\tau) - \hat{u}(\tau)]^T \hat{B}^T e^{\hat{A}^T(t_f-\tau)} d\tau W^{-1}x_f = \\ &= \int_0^{t_f} [\tilde{u}(\tau) - \hat{u}(\tau)]^T Q\hat{u}(\tau)d\tau = 0. \end{aligned} \quad (26)$$

Using (26) it is easy to verify that

$$\begin{aligned} &\int_0^{t_f} [\tilde{u}(\tau) - \hat{u}(\tau)]^T \hat{B}^T e^{\hat{A}^T(t_f-\tau)} d\tau W^{-1}x_f = \\ &= \int_0^{t_f} [\tilde{u}(\tau) - \hat{u}(\tau)]^T Q\hat{u}(\tau)d\tau = 0. \end{aligned} \quad (27)$$

From (27) it follows that $I(\hat{u}) < I(\tilde{u})$ since the second term in the right-hand side of the inequality is nonnegative. To find the minimal value of the performance index (15) we substitute (17) into (15) and we obtain

$$\begin{aligned} I(\hat{u}) &= \int_0^{t_f} \hat{u}^T(\tau)Qu(\hat{u})d\tau \\ &= x_f^T W^{-1} \int_0^{t_f} e^{\hat{A}(t_f-\tau)} \hat{B}Q^{-1}\hat{B}^T e^{\hat{A}^T(t_f-\tau)} d\tau W^{-1}x_f \\ &= x_f^T W^{-1} x_f \end{aligned} \quad (28)$$

since (26) holds. □

4. Procedure and example

From the considerations given in Section 3, we have the following procedure for computation of the optimal inputs satisfying the condition (14) that steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (15).

Procedure 1.

Step 1. Knowing $A \in M_n$, $B \in \Re_+^{n\times m}$ and using (3b) compute $\hat{A}$, $\hat{B}$, $e^{\hat{A}t}$.

Step 2. Using (16) compute the matrix W_f for given $\hat{A}$, $\hat{B}$, Q, α and some t_f.

Step 3. Using (17) and (23) find t_f for which $\hat{u}(t)$ satisfying (14) reaches its maximal value and the desired $\hat{u}(t)$ for given $U \in \Re_+^n$ and $x_f \in \Re_+^n$.

Step 4. Using (19) compute the maximal value of the performance index.

Example 1. Consider the fractional electrical circuit shown in Fig. 1 with given resistances R_1, R_2, R_3, fractional inductances L_1, L_2 and source voltages e_1, e_2.

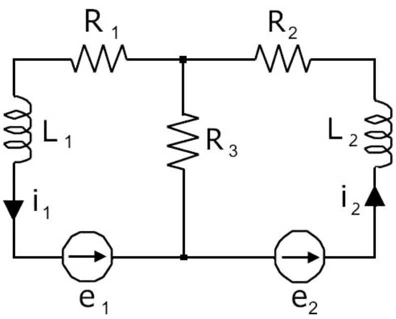

Fig. 1. Electrical circuit

Using the Kirchhoff's laws we can write the equations

$$e_1 = R_3(i_1 - i_2) + R_1 i_1 + L_1 \frac{d^\alpha i_1}{dt^\alpha}, \quad (29a)$$

$$e_2 = R_3(i_2 - i_1) + R_2 i_2 + L_2 \frac{d^\alpha i_2}{dt^\alpha} \quad (29b)$$

where $0 < \alpha < 1$, which can be written in the form

$$\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad (30a)$$

where

$$A = \begin{bmatrix} -\frac{R_1+R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \quad (30b)$$

The fractional electrical circuit is positive since the matrix A is Metzler matrix and the matrix B has nonnegative entries.

Note that the standard (full-order) electrical circuit is reachable for all positive values of the resistances and inductances since $B = \frac{1}{L_1 L_2} \neq 0$.

Using (3b) and (30b) we obtain

$$\hat{A}=[I_n-(1-\alpha)A]^{-1}\alpha A=$$
$$=\begin{bmatrix} 1+(1-\alpha)\frac{R_1+R_3}{L_1} & -\frac{R_3}{L_1} \\ -\frac{R_3}{L_2} & 1+(1-\alpha)\frac{R_2+R_3}{L_2}\end{bmatrix}^{-1}\times \tag{31a}$$
$$\times\alpha\begin{bmatrix} -\frac{R_1+R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2}\end{bmatrix},$$

$$\hat{B}=[I_n-(1-\alpha)A]^{-1}(1-\alpha)B=$$
$$=\begin{bmatrix} 1+(1-\alpha)\frac{R_1+R_3}{L_1} & -\frac{R_3}{L_1} \\ -\frac{R_3}{L_2} & 1+(1-\alpha)\frac{R_2+R_3}{L_2}\end{bmatrix}^{-1}\times \tag{31b}$$
$$\times(1-\alpha)\begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2}\end{bmatrix}.$$

From (31) it follows that the matrix $\hat{A}$ is asymptotically stable Metzler matrix and the matrix $\hat{B}$ has positive entries for all positive values of the resistances R_1, R_2, R_3 and inductances L_1, L_2. Because of complicated calculations of (31a) and (31b) we will show an example for R_1, R_2, $R_3 = 1$, L_1, L_2, $= 1$ and $\alpha = 0.5$. In this case we obtain

$$\hat{A}=\begin{bmatrix} -0.5 & 0 \\ 0 & -0.5\end{bmatrix},\ \hat{B}=\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3}\end{bmatrix}. \tag{31c}$$

The standard fractional circuit is reachable since the matrix (31b) is invertible. It is easy to check that the matrix (4) for the standard fractional circuit is also invertible.

The positive fractional circuit is reachable only if $R_3 = 0$. In this case the matrix $\hat{A}$ is diagonal of the form

$$\hat{A}=\begin{bmatrix} \frac{-\alpha R_1}{L_1+(1-\alpha)R_1} & 0 \\ 0 & \frac{-\alpha R_2}{L_2+(1-\alpha)R_2}\end{bmatrix}. \tag{32a}$$

The matrix $\hat{B}$ is also diagonal with positive diagonal entries

$$\hat{B}=\begin{bmatrix} \frac{1-\alpha}{L_1+(1-\alpha)R_1} & 0 \\ 0 & \frac{1-\alpha}{L_2+(1-\alpha)R_2}\end{bmatrix}. \tag{32b}$$

for all positive values of R_1, R_2 and L_1, L_2.

Using (4) and (32), we obtain

$$R_f=\int_0^{t_f} e^{\hat{A}t}\hat{B}\hat{B}^T e^{\hat{A}^T t}dt=\int_0^{t_f} e^{\hat{A}t}\hat{B}(e^{\hat{A}t}\hat{B})^T dt=\int_0^{t_f}\begin{bmatrix} r_f^1 & 0 \\ 0 & r_f^2\end{bmatrix}dt,$$
$$r_f^1=\left[\frac{(1-\alpha)\exp\left(\frac{-\alpha R_1}{L_1+(1-\alpha)R_1}t\right)}{L_1+(1-\alpha)R_1}\right]^2, \tag{33}$$
$$r_f^2=\left[\frac{(1-\alpha)\exp\left(\frac{-\alpha R_2}{L_2+(1-\alpha)R_2}t\right)}{L_2+(1-\alpha)R_2}\right]^2.$$

The matrix (33) is monomial and the positive fractional circuit for $R_3 = 0$ is reachable.

Now, we shall consider the minimum energy control problem for the fractional positive reachable electrical circuit shown in Fig. 1 for R_1, R_2, $= 1$, $R_3 = 0$, L_1, L_2, $= 1$ and $\alpha = 0.5$. To compute the input $u(t)$ satisfying the condition

$$\hat{u}(t)=\begin{bmatrix} e_1 \\ e_2\end{bmatrix}\le\begin{bmatrix} 5 \\ 5\end{bmatrix} \text{ for } t\in[0,5] \tag{34}$$

that steers the state of the electrical circuit from zero state to final state $x_f = [1\ \ 1]^T$ (T denotes the transpose) and minimizes the performance index (15) with

$$Q=\begin{bmatrix} 2 & 0 \\ 0 & 2\end{bmatrix}. \tag{35}$$

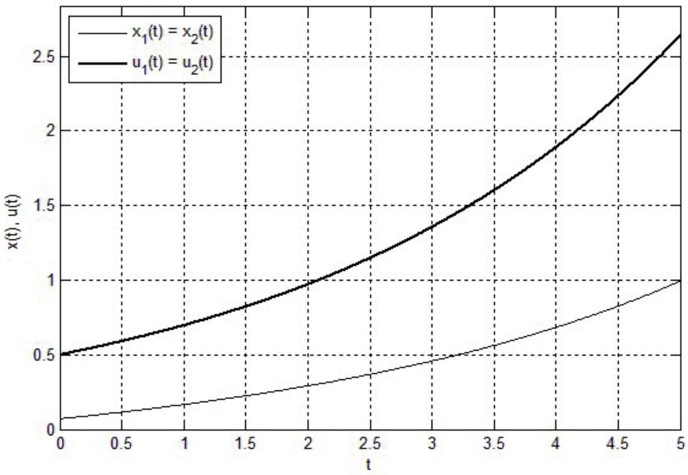

Fig. 2. Input signal and state vector for $t_f = 5$[s]

Procedure 1 will be used. Using (16, 17) and (33–35), we obtain

$$W(t_f) = \int_0^5 e^{\hat{A}(5-\tau)}\hat{B}Q^{-1}\hat{B}^T e^{\hat{A}^T(5-\tau)}d\tau = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \times \int_0^5 \begin{bmatrix} \left[\dfrac{0.5\exp\left(\dfrac{-0.5(5-\tau)}{1.5}\right)}{1.5}\right]^2 & 0 \\ 0 & \left[\dfrac{0.5\exp\left(\dfrac{-0.5(5-\tau)}{1.5}\right)}{1.5}\right]^2 \end{bmatrix} d\tau = \tag{36}$$

$$= \begin{bmatrix} 0.0804 & 0 \\ 0 & 0.0804 \end{bmatrix}$$

and

$$\hat{u}(t) = Q^{-1}\hat{B}^T e^{\hat{A}^T(5-t)}W^{-1}(t_f)x_f = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} \exp\left(\dfrac{-0.5(5-t)}{1.5}\right) & 0 \\ 0 & \exp\left(\dfrac{-0.5(5-t)}{1.5}\right) \end{bmatrix} \times$$
$$\times \begin{bmatrix} 12.4378 & 0 \\ 0 & 12.4378 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4985e^{\frac{1}{3}t} \\ 0.4985e^{\frac{1}{3}t} \end{bmatrix}. \tag{37}$$

Note that the electrical circuit is stable. Therefore, $\hat{u}(t)$ reaches its maximal value for $t = t_f$. From (23) we have the minimal value of the performance index

$$I(\hat{u}) = x_f^T W^{-1}(t_f)x_f, \tag{38}$$

where $W(t_f)$ is given by (36).

5. Concluding remarks

The Caputo-Fabrizio definition of the fractional derivative has been applied to minimum energy control of the fractional positive continuous-time linear systems with bounded inputs. Conditions for the reachability of the standard and positive fractional linear systems have been established (Theorems 2 and 3). The conditions for the existence of solution of the minimum energy control problem for the positive fractional systems with bounded inputs have been derived (Theorem 5). A procedure for computation of the optimal inputs has been proposed (Procedure 1) and illustrated by example of the positive fractional electrical circuit. The considerations can be easily extended to positive linear systems and electrical circuits with different fractional orders. An open problem is an extension of this results to the fractional nonlinear systems.

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