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Single processor scheduling problems with various models of a due window assignment

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Abstract. In the paper we investigate four single processor scheduling problems, which deal with the process of the negotiation between a producer and a customer about delivery time of final products. This process is modeled by a due window, which is a generalization of well known classical due date and describes a time interval, in which a job should be finished. Due window assignment is a new approach, which has been investigated in the scientific literature for a few years. In this paper we consider various models of due window assignment. To solve the formulated problems we have to find such a schedule of jobs and such an assignment of due windows to each job, which minimizes a given criterion dependent on the maximum or total earliness and tardiness of jobs and due window parameters. One of the main results is the mirror image of the solutions of the considered problems and other problems presented in the scientific literature. The wide survey of the literature is also given.

Key words: scheduling, single machine, earliness/tardiness, due window assignment.

1. Introduction

The paper deals with scheduling problems which model the process of negotiation between the producer and the customer about the delivery time of the final products. The producer objective usually is to have the latest time of products delivering, while the customer tries to have them as soon as possible. The compromise of this negotiation is a time period in which the products should be completed by producer and available to be taken by customer. In scheduling problems this situation can be modeled by a due window, which describes a time interval, in which a job should be finished. This kind of scheduling problems has recently attracted considerable attention of the researchers.

The due window model examined in this paper is a generalization of the due date (see e.g. [1]) assignment model considered in scheduling problems. The extensive surveys of the results obtained for the due date assignment problems can be found in the papers of the following researchers: Cheng and Gupta [2], Chengbin, Gordon and Proth [3,4]. For example, survey [3] includes results concerning general models in which earliness and tardiness are arbitrary non-decreasing functions (considered among others by: Mosheiov and Federgruen [5,6], Cai, Lun and Chan [7]). Scheduling problems with due windows have been introduced by Anger, Lee and Martin-Vega [8]. The recent list of publications concerning scheduling problems with fixed or assignable due windows includes the papers of the following authors: Cheng [9], Lee [10], Kramer and Lee [11], Weng and Ventura [12–14], Liman and Ramaswamy [15], Mosheiov and Lann [16, 17], Koulamas [18], Azizoglu and Webster [19], Wu and Wang [20], Linn, Yen and Zhang [21], Yeung, Oguz and Cheng [22], Yoo and Martin-Vega [23], Chen and Lee [24], Biskup and Feldmann [25], and Wodecki [26]. The due window assignment in scheduling problems with a general sum-type criterion (the most interesting for us) have been considered in the following papers by: Janiak and Marek [27, 28], Kramer and Lee [29], and Liman, Panwalker and Thongmee [30, 31]. In the papers [30] and [29] the authors focused on the optimal, common for all the jobs, due window assignment in some single and parallel processors scheduling problems, respectively. In their model it is assumed that the size of the due window is given in advance. The model of due window considered in [29, 30] have been extended in [27, 28, 31]. This extension concerns the size of due window, which is as well as the location of due window a decision variable. Mosheiov and Sarig [32] and Yeung, Oguz and Cheng [33] considered problems with additional flow time penalty. Only Mosheiov [34], Janiak and Marek [35, 36] have discussed single and parallel processors scheduling problems with due window assignment and a general minmax-type criterion.

To be more precise in this paper we consider four scheduling problems, in which different models of due window assignment appear. To the best of our knowledge, the mentioned models have not been investigated in the scientific literature. For the considered problems, we should find a schedule of jobs, due windows and their locations such that their criterion values are minimized, which depend on the following weighted parts: the maximum or total earliness and tardiness of jobs and due window parameters.

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The remaining part of the paper is organized as follows. In the next section, we give a precise formulation of two considered problems. Additionally, we define some auxiliary problems, which were solved in the scientific literature. Finally, we prove the mirror image property of the optimal solutions of the considered problems and the auxiliary problems. In Section 3 the remaining problems are considered. Due to the derived properties of the solutions we constructed polynomial-time exact algorithms. Some conclusions are given in Section 4.

2. Model with due windows depended on processing times

In the section we consider two problems, in which there is given the set $J = \{1, \ldots, n\}$ of n independent and non-preemptive jobs to be scheduled on a single processor. The processor can process only one job at the moment. We assume that the processor executes the jobs without idle times. For each job its processing time p_j is given and due windows are defined as follows $\langle d'_j = p_j + q_1; d''_j = p_j + q_2 \rangle$, where q_1 and q_2 ($q_1 \le q_2$) denote common due window parameters. The only difference between the problems under consideration, denoted, respectively, by P_1 and P_2, is a criterion.

Problems P_1 and P_2 are to find a schedule π (a permutation of jobs) and such values of the parameters q_1 and q_2, which minimize the following criteria, respectively:

$$f_1(\pi, q_1, q_2) = \max\left(\alpha \max_{j\in J} E_j, \beta(q_2 - q_1), \gamma \max_{j \in J} T_j\right), \quad (1)$$

$$f_2(\pi, q_1, q_2) = \sum_{j\in J} (\alpha E_j + \beta(q_2 - q_1) + \gamma T_j), \quad (2)$$

where: $E_j = \max(d'_j - C_j, 0)$ is the earliness of the job j, $T_j = \max(0, C_j - d''_j)$ is the tardiness of the job j, C_j is the completion moment of the job j, and α, β and γ are positive weights.

Using the three-field classification [37], problems P_1 and P_2 can be described, respectively, as follows:

$$1 \left| \langle d'_j = p_j + q_1; d''_j = p_j + q_2 \rangle \right| \max\left(\alpha \max E_j, \beta\left(q_2 - q_1\right), \gamma \max T_j\right)$$

and

$$1 \left| \langle d'_j = p_j + q_1; d''_j = p_j + q_2 \rangle \right| \sum \left(\alpha E_j + \beta(q_2 - q_1) + \gamma T_j\right).$$

2.1. Auxiliary problems. At first we consider two auxiliary scheduling problems denoted by P'_1 and P'_2, which help us to find the optimal solutions of P_1 and P_2, respectively. In the three-field classification the problems P'_1 and P'_2 are given, respectively, as follows:

$$1 \left| \langle d'_j = k_1; d''_j = k_2 \rangle \right| \max(\alpha \max E_j, \beta(k_2 - k_1)_j, \gamma \max T_j),$$

$$1 \left| \langle d'_j = k_1; d''_j = k_2 \rangle \right| \sum \left(\alpha E_j + \beta\left(k_2 - k_1\right) + \gamma T_j\right),$$

where k_1 and k_2 ($k_1 \le k_2$) denote common due window parameters.

In [24] the following $O(n)$ optimal algorithm for P'_1 is presented.

Optimal algorithm A(P'_1)

Step 1: Schedule the job with the largest processing time on the first position. The remaining jobs are scheduled in an arbitrary order.

Step 2: For the schedule π obtained in Step 1 calculate:

$$k_1^*(\pi) = \frac{\beta\gamma \sum_{j=1}^{n} p_j + \alpha\left(\beta + \gamma\right) p_{\pi(1)}}{\alpha\beta + \alpha\gamma + \beta\gamma},$$

and

$$k_2^*(\pi) = \frac{\gamma\left(\alpha + \beta\right) \sum_{j=1}^{n} p_j + \alpha\beta p_{\pi(1)}}{\alpha\beta + \alpha\gamma + \beta\gamma},$$

where $p_{\pi(j)}$ denotes the processing time of the job placed on the position j in the schedule π.

Stop: The obtained schedule π and the values of parameters k_1^*, k_2^* are the optimal solution of P'_1.

In [22] the following $O(n \log n)$ optimal algorithm for P'_2 is presented.

Optimal algorithm A(P'_2)

Step 1: For $j = 1, \ldots, n$ calculate position weight $w_j = \min(\alpha(j-1), n\beta, \gamma(n-j+1))$. If $w_j = \alpha(j-1)$, then let us say j is an early position. If $w_j = n\beta$, then let us say j is a window position. If $w_j = \gamma(n-j+1)$, then let us say j is a tardy position.

Step 2: Renumber the jobs according to the non-increasing order of their processing times, i. e., $p_1 \ge p_2 \ge \ldots \ge p_n$.

Step 3: Schedule the successive jobs on positions according to the non-decreasing order of their weights, i. e., an unscheduled job with the largest processing time should be assigned to the free position with the smallest weight.

Step 4: The value of the parameter k_1^* is equal to the sum of the processing times of the jobs scheduled on the early positions and the value of the parameter k_2^* is equal to the sum of the processing times of the jobs scheduled on early and window positions.

Stop: The obtained schedule π and the values of parameters k_1^*, k_2^* are the optimal solution of P'_2.

2.2. Mirror image of optimal solutions of the considered problems. In this subsection, we will use the following notation. The upper indexes q and k will indicate the values of parameters of the problems P_1, P_2 and P'_1, P'_2, respectively.

At first we consider Theorem 1, which helps us to find the optimal solutions of P_1 and P_2, from the optimal solutions of P'_1 and P'_2, respectively.

Theorem 1. If $\alpha^q = \gamma^k$, $\beta^q = \beta^k$ and $\gamma^q = \alpha^k$, then the optimal schedule of the problem P_1 (P_2) or P'_1 (P'_2) can be obtained from an optimal schedule of the other problem P'_1 (P'_2) or P_1 (P_2)) by reversing the order of the jobs on the processor, and determining the appropriate due window parameters from the following equations: $\sum_{j=1}^{n} p_j = q_1 + k_2 = q_2 + k_1$. Moreover, the optimal criterion values for both problems are equal.

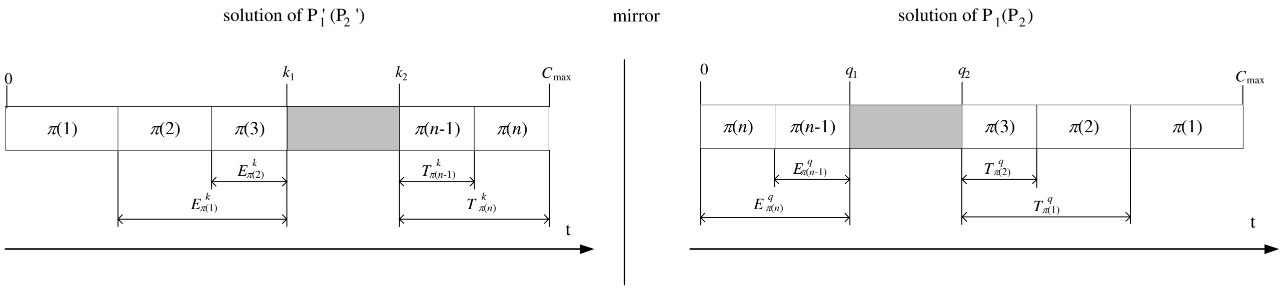

Fig. 1. Mirror image of solutions of P_1 (P_2) and P_1' (P_2')

Proof. Assume π^{rev} denotes a schedule, in which the jobs are executed in the reversed order on the processor respect to the schedule π. It is easy to see that the makespan value ($C_{\max}$) for both schedules is the same and equal to $C_{\max} = \sum_{j=1}^{n} p_j$. From the reversing execution of the jobs in the schedule π it follows that for the schedule π^{rev} we have:

$$S_j(\pi^{\text{rev}}) = C_{\max} - S_j(\pi) - p_j \quad \text{for} \quad j = 1, \ldots, n, \tag{3}$$

where S_j is the starting moment of the job j.

To prove the theorem, at first we need to show that:

$$\begin{aligned} &f_1'(\pi, k_1, k_2) \\ &= \max\left(\alpha^k \max_{j\in J} E_j^k(\pi), \beta^k(k_2 - k_1), \gamma^k \max_{j\in J} T_j^k(\pi)\right) \\ &= \max\left(\alpha^q \max_{j\in J} E_j^q(\pi^{\text{rev}}), \beta^q(q_2 - q_1)\gamma^q \max_{j\in J} T_j^q(\pi^{\text{rev}})\right) \\ &= f_1(\pi^{\text{rev}}, q_1, q_2), \end{aligned} \tag{4}$$

and

$$\begin{aligned} f_2'(\pi, k_1, k_2) &= \sum_{j\in J} \left(\alpha^k E_j^k(\pi) + \beta^k(k_2 - k_1) + \gamma^k T_j^k(\pi)\right) \\ &= \sum_{j\in J} \left(\alpha^q E_j^q(\pi^{\text{rev}}) + \beta^q(q_2 - q_1) + \gamma^q T_j^q(\pi^{\text{rev}})\right) \\ &= f_2(\pi^{\text{rev}}, q_1, q_2). \end{aligned} \tag{5}$$

According to the expressions (3) and $C_{\max} = \sum_{j=1}^{n} p_j = q_1 + k_2 = q_2 + k_1$ we can formulate the following equations:

$$\begin{aligned} &E_j^q(\pi^{\text{rev}}) = \max\left(d'^q_j - C_j(\pi^{\text{rev}}), 0\right) \\ &= \max\left(p_j + q_1 - S_j(\pi^{\text{rev}}) - p_j, 0\right) = \max\left(q_1 - S_j(\pi^{\text{rev}}), 0\right) \\ &= \max\left(C_{\max} - k_2 - C_{\max} + S_j(\pi) + p_j, 0\right) \\ &= \max\left(C_j(\pi) - k_2, 0\right) = \max\left(C_j(\pi) - d''^k_j, 0\right) = T_j^k(\pi), \end{aligned}$$

$$\begin{aligned} &T_j^q(\pi^{\text{rev}}) = \max\left(0, C_j(\pi^{\text{rev}}) - d''^q_j\right) \\ &= \max\left(0, S_j(\pi^{\text{rev}}) + p_j - p_j - q_2\right) = \max\left(0, S_j(\pi^{\text{rev}}) - q_2\right) \\ &= \max\left(0, C_{\max} - S_j(\pi) - p_j - C_{\max} + k_1\right) \\ &= \max\left(0, k_1 - C_j(\pi)\right) = \max\left(0, d'^k_j - C_j(\pi)\right) = E_j^k(\pi), \\ &q_2 - q_1 = C_{\max} - k_1 - (C_{\max} - k_2) = k_2 - k_1. \end{aligned}$$

Since $\alpha^q = \gamma^k$, $\beta^q = \beta^k$ and $\gamma^q = \alpha^k$, then the above equations imply that the Eqs. (4) and (5) are satisfied. Some example is given in Fig. 1.

Let us pass to proving, that if (π^*, k_1^*, k_2^*) is the optimal solution to P_1' (P_2'), then the corresponding solution ($\pi^{*\text{rev}}$, q_1^*, q_2^*) to P_1 (P_2) is also optimal, and vice versa.

Assume that (π^*, k_1^*, k_2^*) is the optimal solution to P_1' (P_2') and the Eq. (4) ((5)) is satisfied and the solution ($\pi^{*\text{rev}}$, q_1^*, q_2^*) is not the optimal one to P_1 (P_2). Then, there exists a solution (π', q'_1, q'_2) such that $f_1(\pi', q'_1, q'_2) < f_1(\pi^{*\text{rev}}, q_1^*, q_1^*)$ ($f_2(\pi', q'_1, q'_2) < f_2(\pi^{*\text{rev}}, q_1^*, q_1^*)$). Observe that (π', q_1', q_2') is a solution reversed with respect to some solution $(\pi'^{\text{rev}}, k_1', k_2')$ to P_1' (P_2'). Then, according to (4) ((5)), we must have $f_1'(\pi'^{\text{rev}}, k_1', k_2') = f_1(\pi', q_1', q_2') < f_1'(\pi^{*\text{rev}}, q_1^*, q_2^*) = f_1'(\pi^*, k_1^*, k_2^*)$ ($f_2'(\pi'^{\text{rev}}, k_1', k_2') = f_2(\pi', q_1', q_2') < f_2'(\pi^{*\text{rev}}, q_1^*, q_2^*) = f_2'(\pi^*, k_1^*, k_2^*)$), which contradicts the optimality of (π^*, k_1^*, k_2^*).

Similar result can be obtained if we assume that ($\pi^{*\text{rev}}$, q_1^*, q_2^*) is the optimal solution to P_1 (P_2) and the solution (π^*, k_1^*, k_2^*) is not the optimal one to P_1' (P_2').

Notice that Theorem 1 extends the result obtained in the paper [38] by Kahlbacher, which concerned similarity between CON and SLK models of the due date assignment (see surveys [8] and [9]).

It follows from Theorem 1 that to solve optimally the problem P_1 (P_2) as first we have to solve the problem $1\left|\langle d_j' = k_1; d_j'' = k_2\rangle\right| \max(\gamma \max E_j, \beta(q_2 - q_1)_j, \alpha \max T_j)$ $\left(1\left|\langle d_j' = k_1; d_j'' = k_2\rangle\right| \sum(\gamma E_j + \beta(q_2 - q_1) + \alpha T_j)\right)$ by the algorithm A(P_1') (A(P_2')). For the obtained schedule of jobs we reverse their processing order on the processor. Next, according to Theorem 1 we calculate the values of the parameters $q_1^* = C_{\max} - k_2^*$ and $q_2^* = C_{\max} - k_1^*$.

3. Mixed models of due windows

In this section we consider the remaining (two) problems. For the first one, denoted by P_3, due windows are defined as follows $\langle d_j' = k; d_j'' = p_j + q\rangle$, where k and q ($k \leq q$) denote common due window parameters. For the second problem, denoted by P_4, due windows are defined as follows $\langle d_j' = p_j + q; d_j'' = k\rangle$ ($p_{\max} + q \leq k$, where $p_{\max} = \max_{j\in J} p_j$). Notice that the presented condition guarantees the existence of the due window for each job. Both problems consist of finding schedules π on a single processor and such values of the parameters k and q, which minimize the following criterion:

$$g(\pi, k, q) = \max\left(\alpha \max_{j\in J} E_j, \beta\left|q-k\right|, \gamma \max_{j\in J} T_j\right), \quad (6)$$

where $|x|$ denotes the absolute value of x.

Using the three-field classification [10], problems P_3 and P_4 can be described, respectively, as follows: $1\left|\langle d'_j = k; d''_j = p_j + q\rangle\right| \max(\alpha E_{\max}, \beta(q - k), \gamma T_{\max})$, $1\left|\langle d'_j = p_j + q; d''_j = k\rangle\right| \max(\alpha E_{\max}, \beta(k-q), \gamma T_{\max})$.

3.1. Optimal solution of P_3. Before we start with searching the optimal solution of P_3, at first we consider the following lemma, in which for a given π we assume that: $C_{\min}(\pi) = \min_{j\in J} C_j(\pi)$ and $S_{\max}(\pi) = \max_{j\in J} S_j(\pi)$.

Lemma 1. For a given schedule π of P_3, the optimal values of the parameters $k^*(\pi)$ and $q^*(\pi)$ satisfy the following inequalities: $k^*(\pi) \geq C_{\min}(\pi)$ and $q^*(\pi) \leq S_{\max}(\pi)$, respectively.

Proof. Assume that π is a schedule in P_3, where the inequalities $k^*(\pi) \geq C_{\min}(\pi)$ and $q^*(\pi) \leq S_{\max}(\pi)$ are not satisfied. There are two cases, which should be considered, namely:

$1°$. for a given schedule π and the value of k ($k \leq q$), the optimal value of $k^*(\pi)$ is equal to $k'(\pi) = C_{\min}(\pi) - \varepsilon$,

$2°$. for a given schedule π and the value of q ($k \leq q$), the optimal value of $q^*(\pi)$ is equal to $q'(\pi) = S_{\max}(\pi) + \varepsilon$, where ε is some positive value, ($\varepsilon > 0$).

Ad $1°$. Let $g(\pi, k'(\pi), q)$ and $g(\pi, k''(\pi), q)$ denote the values of the criterion (6) obtained for the values of the parameters: $k'_1(\pi) = C_{\min}(\pi) - \varepsilon$ and $k''_1(\pi) = C_{\min}(\pi)$, respectively. We have:

$$\begin{aligned}
g(\pi, k'(\pi), q) &= \max\left(\alpha \max_{j\in J} E_j, \beta\left|q - k'(\pi)\right|, \gamma \max_{j\in J} T_j\right)\\
&= \max\left(\alpha \max_{j\in J} E_j, \beta(q - k'(\pi)), \gamma \max_{j\in J} T_j\right)\\
&= \max\Big(\alpha \max(k'(\pi) - C_{\min}(\pi), 0), \beta(q - k'(\pi)), \gamma \max_{j\in J} T_j\Big)\\
&= \max\left(0, \beta(q - C_{\min}(\pi) + \varepsilon), \gamma \max_{j\in J} T_j\right)\\
&\geq \left(0, \beta(q - C_{\min}(\pi)), \gamma \max_{j\in J} T_j\right) = g(\pi, k''(\pi), q).
\end{aligned}$$

Ad $2°$. Similar results (to $1°$) can be obtained for $g(\pi, k, q'(\pi))$ and $g(\pi, k, q''(\pi))$, where the values of the parameters $q'(\pi)$ and $q''(\pi)$ are equal to $q'(\pi) = S_{\max}(\pi) + \varepsilon$ and $q''(\pi) = S_{\max}(\pi)$, respectively.

These results contradict the assumptions that k' and q' are optimal. Thus, it follows from above, that the optimal values of parameters $k^*(\pi)$ and $q^*(\pi)$ should satisfy the following inequalities: $k^*(\pi) \geq C_{\min}(\pi)$ and $q^*(\pi) \leq S_{\max}(\pi)$.

It follows from Lemma 1, that for any feasible π the criterion value (6) is equal to:

$$\begin{aligned}
&g(\pi, k^*(\pi), q^*(\pi)) =\\
&= \max\left(\alpha \max_{j\in J} E_j, \beta\left|q^*(\pi) - k^*(\pi)\right|, \gamma \max_{j\in J} T_j\right)\\
&= \max\Big(\alpha \max_{j\in J}(\max(d'_j - C_j(\pi), 0)),\\
&\quad \beta(q^*(\pi) - k^*(\pi)), \gamma \max_{j\in J}(\max(0, C_j(\pi) - d''_j))\Big)\\
&= \max(\alpha(k^*(\pi) - C_{\min}(\pi)), \beta(q^*(\pi) - k^*(\pi)),\\
&\quad \gamma(S_{\max}(\pi) - q^*(\pi))).
\end{aligned} \quad (7)$$

In the following we prove two lemmas for the minimization of the following general function $h : \Re^3 \to \Re$ (notice that (7) is a special case of this function):

$$h(u, v, w) = \max(A_1 u, A_2 v, A_3 w), \quad (8)$$

subject to:

$$u + v + w = A, \quad (9)$$

where u, w and v are some nonnegative variables, A is given a nonnegative constant and A_1, A_2 and A_3 are given nonnegative weights.

Lemma 2. If $A_1 u = A_2 v = A_3 w$, then the value of the function (8) is minimal.

Proof. Let h' denote the value of the function (8) obtained for the following values of the variables u', v', w', for which $A_1 u' = A_2 v' = A_3 w'$.

Assume now that the value of at least one variable u, w or v, let us say u, is smaller than value u'. It follows from the constraint (9) that in this case the value of at least one from the remaining variable, let us say v, has to be greater than value v'. It means that the function (8) can be estimated by: $h(u, v, w) = \max(A_1 u, A_2 v, A_3 w) > h' = \max(A_1 u', A_2 v', A_3 w')$, which ends the proof.

Lemma 3. The optimal values of the variables u^*, w^* or v^*, which minimize the function (8) are as follows:

$$u^* = \frac{A_2 A_3 A}{A_1 A_2 + A_1 A_3 + A_2 A_3},$$

$$v^* = \frac{A_1 A_3 A}{A_1 A_2 + A_1 A_3 + A_2 A_3},$$

$$w^* = \frac{A_1 A_2 A}{A_1 A_2 + A_1 A_3 + A_2 A_3}.$$

Proof. Based on Lemma 2 and the constraint (9), to find the values of u^*, v^*, w^* we have to solve the following system of equations:

$$\begin{cases} A_1 u^* = A_2 v^* \\ A_2 v^* = A_3 w^* \\ u^* + v^* + w^* = A. \end{cases}$$

Some optimal solution properties for P_3, which concern the optimal schedule of jobs and the optimal values of $k^*(\pi)$ and $q^*(\pi)$, are given below.

Let $C_{\pi(j)}$ and $p_{\pi(j)}$ denote, respectively, the completion moment and the processing time of the job placed at the position j in the schedule π.

Property 1. For a given schedule π of jobs in P_3, the optimal values of the parameters k^* and q^* are equal to

$$k^*(\pi) = \frac{\beta\gamma \sum_{j=1}^{n-1} p_{\pi(j)} + \alpha(\beta+\gamma)\,p_{\pi(1)}}{\alpha\beta+\alpha\gamma+\beta\gamma}$$

and

$$q^*(\pi) = \frac{\gamma(\alpha+\beta)\sum_{j=1}^{n-1} p_{\pi(j)} + \alpha\beta p_{\pi(1)}}{\alpha\beta+\alpha\gamma+\beta\gamma}.$$

Proof. Based on Lemma 3 and the expression (7), we have:

$$\left(u^* = \frac{A_2A_3A}{A_1A_2+A_1A_3+A_2A_3}\right) \Rightarrow \left(k^*(\pi) - C_{\min}(\pi) = \frac{\beta\gamma}{\alpha\beta+\alpha\gamma+\beta\gamma}\left(S_{\max}(\pi) - C_{\min}(\pi)\right)\right),$$

thus

$$k^*(\pi) = \frac{\beta\gamma}{\alpha\beta+\alpha\gamma+\beta\gamma}\left(S_{\max}(\pi) - C_{\min}(\pi)\right) + C_{\min}(\pi).$$

Based on Lemma 3 and the expression (7), we have also

$$\left(v^* = \frac{A_1A_3A}{A_1A_2+A_1A_3+A_2A_3}\right) \Rightarrow \left(q^*(\pi) - k^*(\pi) = \frac{\alpha\gamma}{\alpha\beta+\alpha\gamma+\beta\gamma}\left(S_{\max}(\pi) - C_{\min}(\pi)\right)\right),$$

therefore

$$q^*(\pi) = k^*(\pi) + \frac{\alpha\gamma}{\alpha\beta+\alpha\gamma+\beta\gamma}\left(S_{\max}(\pi) - C_{\min}(\pi)\right) = \frac{\alpha\gamma+\beta\gamma}{\alpha\beta+\alpha\gamma+\beta\gamma}\left(S_{\max}(\pi) - C_{\min}(\pi)\right) + C_{\min}(\pi).$$

For the equations $C_{\min}(\pi) = p_{\pi(1)}$ and $S_{\max}(\pi) = \sum_{j=1}^{n-1} p_{\pi(j)}$ we obtain values of the parameters $k^*(\pi)$ and $q^*(\pi)$.

Property 2. There exists an optimal solution of P_3, in which two jobs with the largest processing times are executed on the first and the last positions. Moreover, it does not matter if the job with the largest processing time is performed on the first position or on the last one.

Proof. It follows from Lemmas 2 and 3 and the expression (7) that for a given π and the optimal values $k^*(\pi)$ and $q^*(\pi)$ the criterion (6) is equal to:

$$g(\pi, k^*(\pi), q^*(\pi)) = \frac{\alpha\beta\gamma}{\alpha\beta+\alpha\gamma+\beta\gamma}\left(S_{\max}(\pi) - C_{\min}(\pi)\right) = \frac{\alpha\beta\gamma}{\alpha\beta+\alpha\gamma+\beta\gamma}\left(\sum_{j=1}^{n-1} p_{\pi(j)-p_{\pi(l)}}\right) = \frac{\alpha\beta\gamma}{\alpha\beta+\alpha\gamma+\beta\gamma}\left(\sum_{j=1}^{n} p_j - p_{\pi(l)} - p_{\pi(n)}\right).$$

It follows from the above expression that the criterion value (6) is minimal, if two jobs with the largest processing times are executed on the first position and on the last one.

The optimal algorithm solving P_3 can be realised in $O(n)$ time, since the values of the parameters k^* and q^* depend only on the values of the processing times of the jobs which are executed on the first and the last positions of π^*, i.e., the jobs with the largest processing times and the remaining jobs can be scheduled in π^* in an arbitrary order.

3.2. Optimal solution of P_4. Before we start with searching the optimal solution of P_4, at first we consider the following lemmas, in which we assume that $S_{\min} = \min_{j\in J} S_j(\pi) = 0$.

Lemma 4. For a given schedule π of P_4, the optimal values of the parameters $q^*(\pi)$ and $k^*(\pi)$ satisfy the following inequalities: $q^*(\pi) \geq S_{\min}(\pi)$ and $k^*(\pi) \leq C_{\max}(\pi)$, respectively.

Proof. It is similar to the proof of Lemma 1.

It follows from Lemma 4, that for any feasible π the criterion value (6) is equal to:

$$\begin{aligned} &g(\pi, k^*(\pi), q^*(\pi)) \\ &= \max\left(\alpha \max_{j\in J} E_j, \beta\,|q^*(\pi) - k^*(\pi)|\,, \gamma \max_{j\in J} T_j\right) \\ &= \max\Big(\alpha \max_{j\in J}\left(\max\left(p_j + q^*(\pi) - C_j(\pi), 0\right)\right), \\ &\quad \beta\left(k^*(\pi) - q^*(\pi)\right), \gamma \max_{j\in J}\left(\max\left(0, C_j(\pi) - k^*(\pi)\right)\right)\Big) \\ &= \max\left(\alpha q^*(\pi), \beta\left(k^*(\pi) - q^*(\pi)\right), \gamma\left(C_{\max} - k^*(\pi)\right)\right). \end{aligned} \tag{10}$$

Some optimal solution properties for P_4, which concern the optimal schedule of jobs and the optimal values of k^* and q^* are given below.

Property 3. For a given schedule π of jobs in P_4, the optimal values of the parameters q^* and k^* are equal to:
if

$$p_{\max} \leq \frac{\alpha\gamma \sum_{j=1}^{n} p_j}{\alpha\beta+\alpha\gamma+\beta\gamma}$$

then

$$q^* = \frac{\beta\gamma \sum_{j=1}^{n} p_j}{\alpha\beta+\alpha\gamma+\beta\gamma}$$

and

$$k^* = \frac{(\alpha\gamma+\beta\gamma)\sum_{j=1}^{n} p_j}{\alpha\beta+\alpha\gamma+\beta\gamma},$$

otherwise

$$q^* = \frac{\gamma\left(\sum_{j=1}^{n} p_j - p_{\max}\right)}{\alpha+\beta}$$

and

$$k^* = \frac{\gamma \sum_{j=1}^{n} p_j + \alpha p_{\max}}{\alpha + \gamma},$$

respectively.

Proof. It follows from Lemmas 2 and 3 and the expression (10), that:

$$g\left(\pi, k^*(\pi), q^*(\pi)\right) = \alpha q^*(\pi) = \beta\left(k^*(\pi) - q^*(\pi)\right)$$
$$= \gamma\left(C_{\max} - k^*(\pi)\right) = \frac{\alpha\beta\gamma C_{\max}}{\alpha\beta + \alpha\gamma + \beta\gamma},$$

and we can calculate the expressions for the parameters $q^*(\pi)$ and $k^*(\pi)$. However, it may appear that the parameters $q^*(\pi)$ and $k^*(\pi)$ calculated in such a way do not satisfy our assumption that $p_{\max} + q \leq k$, i.e. it may turn out that $p_{\max} > k^*(\pi) - q^*(\pi)$. Taking into consideration the above situations, we have:

$$g\left(\pi, k^*(\pi), q^*(\pi)\right) = \beta(k^*(\pi) - q^*(\pi))$$
$$= \max\left(\beta p_{\max}, \frac{\alpha\beta\gamma C_{\max}}{\alpha\beta + \alpha\gamma + \beta\gamma}\right)$$
$$= \beta \max\left(p_{\max}, \frac{\alpha\gamma \sum_{j=1}^{n} p_j}{\alpha\beta + \alpha\gamma + \beta\gamma}\right). \tag{11}$$

If $p_{\max} \leq \frac{\alpha\gamma \sum_{j=1}^{n} p_j}{\alpha\beta + \alpha\gamma + \beta\gamma}$, then based on Lemma 3 and the expression (10), we have

$$\left(u^* = \frac{A_2 A_3 A}{A_1 A_2 + A_1 A_3 + A_2 A_3}\right) \Rightarrow \left(q^* = \frac{\beta\gamma C_{\max}}{\alpha\beta + \alpha\gamma + \beta\gamma}\right),$$

therefore

$$q^* = \frac{\beta\gamma \sum_{j=1}^{n} p_j}{\alpha\beta + \alpha\gamma + \beta\gamma},$$

moreover it follows from the expression (11) that

$$k^* = q^* + \frac{\alpha\gamma \sum_{j=1}^{n} p_j}{\alpha\beta + \alpha\gamma + \beta\gamma} = \frac{(\alpha\gamma + \beta\gamma) \sum_{j=1}^{n} p_j}{\alpha\beta + \alpha\gamma + \beta\gamma}.$$

If $p_{\max} > \frac{\alpha\gamma \sum_{j=1}^{n} p_j}{\alpha\beta + \alpha\gamma + \beta\gamma}$, then the minimal size of due window is equal to $p_{\max}$ and it follows from (11) that the minimal value of the criterion (6) is equal to $\beta p_{\max}$. This value of the criterion exists for many pairs of values of the parameters k and q (where $p_{\max} + q \leq k$). It follows from the expression (10), that one of these pairs is the solution of the following system of the equations:

$$\begin{cases} \alpha q = \gamma\left(C_{\max} - k\right) \\ k - q = p_{\max}. \end{cases}$$

Notice that values of the parameters k^* and q^* computed above do not depend on a schedule π of jobs.

The optimal algorithm solving P_4 can be realised in $O(n)$ time, since the values of the parameters k^* and q^* depend only on the sum of all the job-processing times and an arbitrary job order is an optimal one π^*.

It is worth to stress that although the models of due windows of P_3 and P_4 are symmetric, however the solutions of P_3 and P_4 are not symmetric, e.g. for P_4 π^* is an arbitrary one, but for P_3 two jobs with the largest processing times are scheduled on the first and last positions in π^*.

4. Conclusions

We considered four new problems of scheduling jobs on a single processor where a due window should be assigned to each job. In the considered problems we minimized the criterion, which consisted of the following parts: the maximum or total earliness and tardiness and the due window parameters. For the problems with the due window model $\langle d'_j = p_j + q_1; d''_j = p_j + q_2\rangle$ the most important result is Theorem 1 in which we established the mirror image of solutions of the considered problems and the problems known from the scientific literature. Due to this theorem we can optimally solve the considered problems. In the future we are going to generalize Theorem 1 to solve the case with parallel processors.

For the problems with the due window models $\langle d'_j = k; d''_j = p_j + q\rangle$ and $\langle d'_j = p_j + q; d''_j = k\rangle$ we proved the optimal solutions properties which concerned only minmax-type criterion. In the future we are going to widen our consideration to solve cases with the sum-type criterion.

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