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Application of uncertain variables to production planning in a class of manufacturing systems

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Abstract. A production planning problem is addressed in the paper. It consists in the determination of a production plan of products to maximize a total utility connected with their manufacture taking into account a limited amount of resources of different types which are necessary for the production. An uncertain version of the problem is considered when an amount of the resources are not precisely known. The formalism of uncertain variables is proposed in the paper to solve this problem. The solution algorithms for two versions of the uncertain production planning problem are presented. It turned out possible to replace the uncertain problems by their deterministic counterparts. A simple numerical example illustrates the solution algorithms.

Key words: production planning, decision making, intelligent manufacturing systems, uncertainty, knowledge-based systems.

1. Introduction

A production planning is an important matter of concern both in decision science (in particular in operations research) and in economics (in particular in production management). The problems investigated by decision makers concern mainly procedures for finding a production plan, while economists are mostly interested in making a good plan.

There are a variety of particular production planning problems considered and represented in literature, see e.g. [1–3]. Apart from the basic ones which deal with the determination of an amount of products taking into account the consumption of a restricted amount of resources needed, the more complex their versions, closer to real-world applications, have been formulated and solved. This research direction is connected with the development of manufacturing systems, on the one hand, as well as of tools for solving corresponding decision making problems, on the other hand. A supply chain management can be mentioned as an example. For a supply chain, being a complex structure of individual companies (or manufacturing functions) having its own objectives, finding the optimum supply chain management strategy is a very hard problem. It concerns also corresponding production planning problems where decisions deal with not only a production but also a transportation of raw materials and products as well as a management of inventories and even employment. Such an approach, which allows for a better utilization of the human and equipment resources of a company to meet some anticipated consumer demand, is called an aggregate production planning, see e.g. [2].

1.1. Production planning under uncertainty. Another factor which forces the necessity to investigate a new production planning problems is an uncertainty in data of these problems. Among many factors involved in practical decision making problems in manufacturing enterprises such as the randomness of arriving orders, the uncertainty in a competitive environment, the crucial role plays the imprecise information of available resources. Decision makers employ different approaches for modeling the uncertainty in manufacturing systems, in particular, when solving different production planning problems. The most popular and grounded is the stochastic approach where an uncertain parameter (e.g. total amount of resources) is assumed to be the realization of a random variable, e.g. [4]. For this approach, it is assumed that a certain probability distribution exists over the space of all the possible realizations (scenarios) of all the random parameters of the problem and the objective is to determine a solution that fulfils a selected probabilistic performance index. Assigning probabilities or determining probability distributions is mostly a difficult task. In many cases, it is difficult to express the future realizations of variables in terms of probabilities. This particularly applies to production planning problems where a few factors (such as products and resources) determine the uncertainty of many parameters. The most serious drawback of the stochastic approach is its inability to recognize that depending on what parameter scenario is actually realized, a whole distribution of outcomes is associated with every decision, and, so, any approach that evaluates decisions using only one parameter realization (either the expected one or the most likely one) is bound to fail.

The cost of a production or its profit are generally optimized. For the stochastic approach, typically, a decision which maximizes (minimizes) an expected performance measure (with the expectation taken over the assumed probability distribution) should be generated. The issues mentioned have motivated many researchers to develop other approaches which would cope with the uncertainty of parameters more adequately without using any probability distribution as the description of a parameter uncertainty. A fuzzy system and possibilistic approaches are often used, e.g. [5]. Realistic solutions in a production planning are dealt with in [6] and [7]

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when the amount of resources are not completely known, but they are specified in terms of uncertain intervals using fuzzy sets. A multi-product multi-period production planning problem is addressed in [8] where some cost coefficients in a performance index are assumed to be represented by fuzzy variables. The discrete version of a production planning problem with interval values of the resource quantities is investigated in [9]. The important feature of this approach consists in the fact that no other information about uncertainty is given besides of bounds of intervals. Some L-class enumeration algorithms are proposed to solve the problem.

A formalism of the uncertain variables [10] is used in the paper to model an uncertainty of a manufacturing enterprise. It is assumed that the quantity of resources are only values of the uncertain variables. Non-standard version of the objective function is used. A total utility function being a sum of the local utilities connected with a manufacture of individual products serves as a performance index to be maximized. The utility functions are often used as criteria for different decision making problems in manufacturing systems, e.g. [11–13].

Till now, the approach based on the uncertain variables was applied e.g. for stability analysis and stabilization in control systems [14] as well as for rate allocation and admission control in computer networks [15], task allocation in multiprocessor systems [16], and joint transportation and production problem [17–18]. The paper continues this series for the production planning problem. The basic information of the uncertain variables are presented in the next sub-section.

1.2. Uncertain variables . Let us consider an universal set Ω, $\omega \in \Omega$, a set of real valued vectors $\overline{B} \subseteq R^k$ as well as a function $g : \Omega \rightarrow \overline{B}$, i.e. $\overline{b} = g(\omega) \triangleq \overline{x}(\omega)$. The function g determines the value of a certain numerical feature assigned to the element ω. The existence of this function makes the main difference among an uncertain variables approach and other approaches which can describe the parameter uncertainty, e.g. probabilistic or fuzzy approaches. A crisp property $P(\overline{b})$ is introduced being the logic proposition in a two-valued logic. On the other hand, the property $P(\overline{b})$ and the function g define a crisp property $\Psi(\omega,\ P)$: "For value $\overline{b}$ assigned to ω the crisp property P is fulfilled". This sentence is a proposition in a two-valued logic which means that a logic value of $\Psi(\omega,\ P)$ is equal to 0 or 1, i.e. $w[\Psi(\omega,P)] \in \{0,1\}$. Let us introduce now a soft property $G_\omega(b)$ = "$\overline{b}(\omega) \cong b$" for $b \in B \subseteq \overline{B}$ which means that "$\overline{b}$ is approximately equal to b" or "b is an approximate value of $\overline{b}$". The properties P and G_ω generate a new soft property $\overline{\Psi}(\omega,P)$ in Ω: "an approximate value $\overline{b}(\omega)$ fulfills P", i.e.

$$\overline{\Psi}(\omega,P) = \text{“}\overline{b}(\omega)\tilde{\in}D_b\text{”}, \qquad D_b = \{\overline{b} \in \overline{B}:\ P(\overline{b})\},$$

which means that "$\overline{b}$ approximately belongs to D_b" or "an approximate value of $\overline{b}$ belongs to D_b". For given ω, it is impossible to say whether G_ω, $\overline{\Psi}$ are true (their logic values are equal to 1) or false (their logic values are equal to 0). Therefore, a multi-valued logic is proposed, i.e. $w[G_\omega(b)] \in [0,1]$ and $w[\overline{\Psi}(\omega,P)] \in [0,1]$ where w denotes a logic value of the property. There exist different interpretations of the logic value in multi-valued logic. For the uncertain variable approach, it is assumed that $w[G_\omega(b)]$ and $w[\overline{\Psi}(\omega,P)]$ denote a degree of an expert's certainty that for fixed b the property G_ω and $\overline{\Psi}$ are satisfied, respectively. The logic value of G_ω will be denoted by $h_\omega(b)$ or $v[G_\omega(b)]$ which is called a *certainty index* of the soft property G_ω. Analogously, $v[\overline{\Psi}(\omega,P)]$ is a *certainty index* (logic value) of $\overline{\Psi}$. Namely,

$$w\ [G_\omega(x)] = w\ [\overline{b} \cong b] \triangleq v\ [\overline{b} \cong b] = h_\omega(b),$$

where $h_\omega(b) \in [0,1]$ and

$$w[\overline{\Psi}(\omega,P)] = w[b(\omega)\tilde{\in}D_b] \triangleq v[b(\omega)\tilde{\in}D_b] = = \begin{cases} \max\limits_{b\in D_b} h_\omega(b), & \text{for} \quad D_b \neq \varnothing, \\ 0, & \text{for} \quad D_b = \varnothing. \end{cases}$$

The function $h_\omega(b)$ is given by an expert which can observe an element ω, collect any information concerning $\overline{b}$ and use them for a numerical evaluation of his opinion that $\overline{b} \cong b$. The expert, on the basis of the information on ω collected and his (her) experience, can, for example, give different approximate values $b_1,\ b_2,\ ...,\ b_m$ of $\overline{b}(\omega)$ and for each of them can present the degree of certainty $v[\overline{b}(\omega) \cong b_i] = h_\omega(b_i)$.

The variable $\overline{b}$ for *fixed* ω is called the uncertain variable and is defined by the set of values B, the function $h(b) \triangleq h_\omega(b) = v[\overline{b} \cong b]$, $\omega \in \Omega$, (i.e. *certainty distribution* given by the expert; for simplicity $h_\omega(b)$ is denoted as $h(b)$) and the following definitions

$$v\left(\overline{b}\tilde{\in}D_b\right) = \begin{cases} \max\limits_{b\in D_b} h(b), & \text{for} \quad D_b \neq \varnothing, \\ 0, & \text{for} \quad D_b = \varnothing, \end{cases} \tag{1}$$

$$v\left(\overline{b}\tilde{\notin}D_b\right) = 1 - v\left(\overline{b}\tilde{\in}D_b\right), \tag{2}$$

$$v\left(\overline{b}\tilde{\in}D_1 \vee \overline{b}\tilde{\in}D_2\right) = \max\{v\left(\overline{b}\tilde{\in}D_1\right),\ v\left(\overline{b}\tilde{\in}D_2\right)\}, \tag{3}$$

$$v\left(\overline{b}\tilde{\in}D_1 \wedge \overline{b}\tilde{\in}D_2\right) = = \begin{cases} \min\{v(\overline{b}\tilde{\in}D_1),\ v(\overline{b}\tilde{\in}D_2)\} & \text{for} \quad D_1 \cap D_2 \neq \varnothing, \\ 0 & \text{for} \quad D_1 \cap D_2 = \varnothing. \end{cases} \tag{4}$$

For given ω, it is not possible to state whether the crisp property "$b \in D_b$" is true or false because the function g and consequently the value of $\overline{b}$ corresponding to ω are unknown. The exact information i.e. the knowledge of the function g, which enables us to calculate $\overline{b}(\omega)$, is replaced by the certainty distribution $h_\omega(b) \triangleq h(b)$ which for given ω characterizes the different possible approximate values of $\overline{b}(\omega)$. The expert, giving the function $h(b)$ in this way, determines for the different values b his (her) degree of the certainty that $\overline{b}$ is approximately equal to b. The certainty index may be given directly by the expert or may be determined when $\overline{b}$ is a known function of an uncertain variable $\overline{e}$ described by a certainty distribution $h_e(e)$ given by the expert. Usually, the certainty distribution is characterized by: two parameters b^* and d_b which values indicate respectively the most certain value of the unknown parameter according to the expert and the range of possible values of the unknown parameter (i.e., $[b^* - d_b, b^* + d_b]$) as well as by the shape which illustrates the degree of certainty for possible values of an unknown parameter.

For the uncertain variable, one can define a mean value $M(\overline{b})$ in a similar way as an expected value for a random variable, i.e.

$$M(\overline{b}) = \int_B bh(b)db \left[\int_B h(b)db \right]^{-1},$$

$$M(\overline{b}) = \sum_{i=1}^{m} b_i h(b_i) \left[\sum_{i=1}^{m} h(b_i) \right]^{-1},$$

for continuous, discrete case, respectively. A joint certainty distribution for a pair of uncertain variables as well as corresponding marginal certainty distributions are also defined [19], [20].

It is interesting to compare uncertain variables with random and fuzzy variables. The formal part of the definition of a random variable $\widetilde{b}$, a fuzzy number $\hat{b}$ and an uncertain variable $\overline{b}$ is the same and can be expressed using a pair $\langle B,\ \overline{\mu}(b) \rangle$ where $\overline{\mu} : B \to R^1$ and $\overline{\mu}(b) \geq 0$, $b \in B$. The random variable $\widetilde{b}$ is defined by B and probability distribution $\overline{\mu}(b) = F(b)$ (probabilities $p_i(b_i)$ for a discrete case). The function $F(b)$ is an objective characteristics of Ω *as a whole* while $h(b)$ is a subjective characteristics given by an expert and describes his (her) individual opinion of the *fixed particular* ω. Moreover, for the random variables the property of additivity is required (e.g. when $B = \{b_1,\ b_2,\ ...,\ b_m\}$ then $\sum_{i=1}^{m} \overline{\mu}(b_i) = 1$). The definition of the fuzzy variable is more general then the definition of the uncertain variable. For the case of the fuzzy number, $\hat{b}(\omega)$ ($\overline{\mu}(b) = \hat{\mu}(b)$) is a membership function which is a logic value (degree of possibility) of the soft property "it is possible that the value b is assigned to ω". In the case of uncertain variables, there exists a function $\overline{b} = g(\omega)$, the value $\overline{b}$ is determined for the fixed ω but is unknown to an expert who formulates the degree of certainty that $\overline{b}(\omega) \cong b$ for different values $b \in B$. In the case of $\hat{b}(\omega)$, the function g may not exist. Instead, a property of the type "it is possible that the value b is assigned to ω" is assumed. Then, $\hat{\mu}(b)$ for fixed ω means the degree of possibility for different values $b \in B$ given by an expert.

The presentation in the paper is organized as follows. The production planning problem considered is formulated in Sec. 2 both for the deterministic and the uncertain case. The solution algorithms for the two versions of the uncertain problem are described in Sec. 3. Section 4 provides a simple numerical example. Final remarks complete the paper.

2. Problem formulation

2.1. Deterministic case. Let us consider a manufacturing system with a set of n types of products (production tasks) to be produced with the utilization of m types of resources (raw materials). It means that different resources are necessary to manufacture the products. The resources may include labor, raw material, pollution allowance, etc. The problem consists in finding a production plan $x = [x_1,\ x_2,\ ...,\ x_j,\ ...,\ x_n]^{\mathrm{T}}$ with real non-negative components which maximizes a total utility (profit) under resource constraints, e.g. [3]. The total utility is composed of local utilities, to be reached when individual products are manufactured, described by functions $f_j(x_j, a_j)$ where $a_j = [a_j^{(1)}, a_j^{(2)}, ..., a_j^{(k_j)}]^{\mathrm{T}}$ is a vector of k_j parameters characterizing function f_j. It is assumed that functions f_j are increasing, strictly concave and continuously differentiable with respect to x_j. A linear form of functions f_j is mostly considered in literature. However, non-linear functions reflect better the utility for the real-world applications. The following their form can be presented as an example

$$f_j(x_j, a_j) = \begin{cases} \left(1 - a_j^{(1)}\right)^{-1} x_j^{\left(1 - a_j^{(1)}\right)}, & 0 < a_j^{(1)} < 1, \\ \ln(x_j + 1), & a_j^{(1)} = 1, \end{cases} \tag{5}$$

i.e. $k_j = 1$. The total utility for all products is a composition of the utility functions for individual products. In general, the total utility function can be expressed in the form of the mapping $\overline{F}(f_1(x_1, a_1), f_2(x_2, a_2), .., f_n(x_n, a_n)) \triangleq F(x)$. In particular, F is the sum

$$F(x) = \sum_{j=1}^{n} f_j(x_j, a_j). \tag{6}$$

Consequently, the manufacturing system with the set of products and the set of resources can be treated as an input-output decision making plant with the production plan x as the input and the total utility F as the output (see Fig. 1).

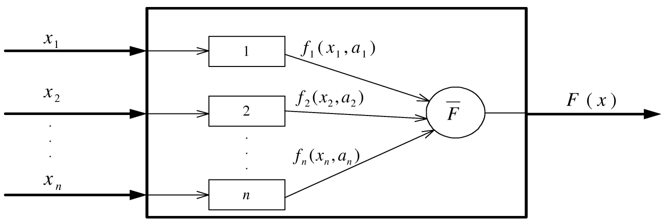

Fig. 1. The manufacturing system as an input-output decision-making plant

It is assumed that the restricted amount b_i of each type of the resource is available and the following inequalities are true

$$\sum_{j=1}^{n} c_{ij} x_j \leq b_i, \qquad i = 1,\ 2,\ ...,\ m, \tag{7}$$

where c_{ij} denotes the consumption of the resource i for the product j unit. The decision consists in the determination of $x_1,\ x_2,\ ...,\ x_n$ to enable fulfilling constraints imposed on the resources. In particular, the optimal decision making problem **P** can be formulated and then solved. Namely, for given: b_i, c_{ij}, f_j, $i = 1,\ 2,\ ...,\ m$, $j = 1,\ 2,\ ...,\ n$

$$\max_{x \in D_x} F(x), \tag{8}$$

where $D_x = \{x \colon (x_j \geq 0,\ j = 1, 2, ..., n) \wedge (7) \text{ holds}\}$. The optimal solution (the optimal production plan) x^* is obtained as the result. Let us denote additionally $F(x^*) \triangleq \max_{x \in D_x} F(x)$.

It is worth noting that **P** is the multidimensional knapsack problem for integer components of x.

2.2. Uncertain cases. The exact values of resource amounts b_i are not known in many cases, e.g. [4, 7, 8, 13, 15]. Moreover, there are often no empirical data to estimate probability distributions or their moments, and, in the consequence, the application of the probabilistic approach as the description of uncertain b_i is not justified in such a case. Therefore the subjective description of the uncertainty based on the subjective opinion of an expert, can be only used. Let us use the approach based on the formalism of uncertain variables and assume that b_i are values of independent uncertain variables $\overline{b}_i$, described in the form of certainty distributions $h_i(b_i)$ given by an expert. In fact, such an assumption means that constraints imposed on the consumption of the resources can be satisfied only in a soft way. Consequently, it is possible only to define the certainty index that constraints of the resource consumption are approximately satisfied. The formulation of an uncertain decision making problem corresponding to problem **P** is not unique. Different cases can be considered. The following two of them are addressed in the paper.

The optimization of the objective function taking into account that the resource constraints are satisfied with the certainty index not less than level $\overline{v}$ given by an user, referred to as $\mathbf{UP}_1$.

The maximization of the certainty index that "constraints of the resources are approximately satisfied" when the total utility is not less than value α given by an user, further called as $\mathbf{UP}_2$.

The data for both uncertain problems are as for **P** instead of b_i which is replaced by $h_i(b_i)$ for $i = 1, 2, ..., m$. Moreover, the certainty level $\overline{v}$ and the value of α must be additionally known for $\mathbf{UP}_1$ and $\mathbf{UP}_2$, respectively. Then, to solve $\mathbf{UP}_1$ it is necessary to perform the following maximization

$$\max_{x \in D_{x,1}} F(x)$$

where

$$D_{x,1} = \Bigg\{ \{x\colon (x_j \geq 0,\ j = 1,\ 2,\ ...,\ n) \wedge \wedge \left(v \left[\forall i = 1,\ 2,\ ...,\ m \ \sum_{j=1}^{n} c_{ij} x_j \tilde{\leq} \overline{b}_i \right] \geq \overline{v} \right) \Bigg\}.$$

The vector x' is obtained as the solution, i.e.

$$F(x') \triangleq \max_{x \in D_{x,1}} F(x).$$

Similarly, problem $\mathbf{UP}_2$ consists in

$$\max_{x \in D_{x,2}} v \left[\forall i = 1,\ 2,\ ...,\ m \ \sum_{j=1}^{n} c_{ij} x_j \tilde{\leq} \overline{b}_i \right]$$

where $D_{x,2} = \{x\colon (x_j \geq 0,\ j = 1,\ 2,\ ...,\ n) \wedge (F(x) \geq \alpha\}$. The vector $\hat{x}$ and the maximal value of the certainty index denoted as $\hat{v}$ are the results of the maximization for which, additionally, the value $F(\hat{x})$ of the total utility function can be calculated.

3. Solution algorithms for uncertain cases

In the paper, the solution algorithms for the uncertain production planning problem are considered under the assumption that for unknown parameters b_i the certainty distributions are given in the form

$$h_i(b_i) = \begin{cases} \overline{h}_i(b_i) & \text{for } b_i^* - d_{bi} \leq b_i \leq b_i^*, \\ \underline{h}_i(b_i) & \text{for } b_i^* < b_i \leq b_i^* + d_{bi}, \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

where b_i^* and d_{bi} are parameters of the certainty distribution described in Subsec. 1.2, $\overline{h}_i(b_i)$ is the increasing function, $\underline{h}_i(b_i)$ is the decreasing function, and $\overline{h}_i(b_i^* - d_{bi}) = 0$, $\overline{h}_i(b_i^*) = \underline{h}_i(b_i^*) = 1$, $\underline{h}_i(b_i^* + d_{bi}) = 0$. The examples of such functions are presented in Fig. 2. The choice by an expert the hyperbolic version of the certainty distributions means his (her) stronger opinion of the soft property "$\overline{b}_i$ is approximately equal to b_i^*" then for the other versions.

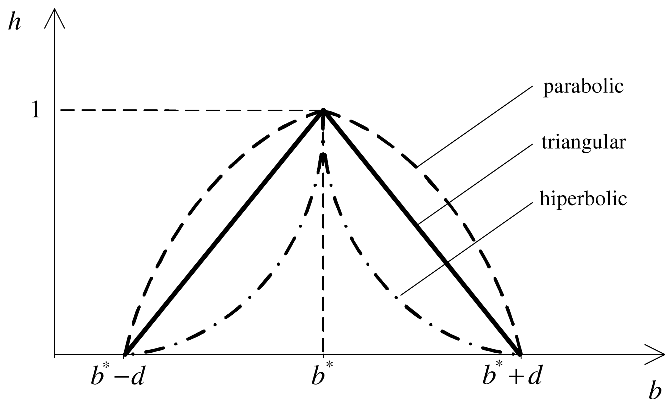

Fig. 2. The examples of the certainty distributions

3.1. Solution algorithm for $\mathbf{UP}_1$. According to the properties of the uncertain variables (1)–(4), the certainty index that "constraints of the resources are approximately satisfied" fulfills the following equations

$$v \left[\forall i = 1,\ 2,\ ...,\ m \ \sum_{j=1}^{n} c_{ij} x_j \tilde{\leq} \overline{b}_i \right] = $$

$$= v \left[\sum_{j=1}^{n} c_{1j} x_j \tilde{\leq} \overline{b}_1 \wedge \sum_{j=1}^{n} c_{2j} x_j \tilde{\leq} \overline{b}_2 \wedge ... \wedge \sum_{j=1}^{n} c_{mj} x_j \tilde{\leq} \overline{b}_m \right] =$$

$$= \min_i v \left[\sum_{j=1}^{n} c_{ij} x_j \tilde{\leq} \overline{b}_i \right].$$

Let

$$v_i \triangleq v \left[\sum_{j=1}^{n} c_{ij} x_j \tilde{\leq} \overline{b}_i \right]$$

then

$$v_i = v\left[\bar{b}_i \tilde{\in} \left[\sum_{j=1}^{n} c_{ij}x_j, \infty\right)\right] = \max_{b_i \tilde{\in} \left[\sum_{j=1}^{n} c_{ij}x_j, \infty\right)} h_i(b_i).$$

Consequently, the certainty index v_i can be given in the form of

$$v_i = \begin{cases} 1 & \text{for } \sum_{j=1}^{n} c_{ij}x_j \le b_i,^* \\ \underline{h}_i\left(\sum_{j=1}^{n} c_{ij}x_j\right) & \text{for } b_i^* < \sum_{j=1}^{n} c_{ij}x_j \le b_i^* + d_{bi}, \\ 0 & \text{otherwise.} \end{cases}$$

Taking into account the transformation of

$$\max_{b_i \tilde{\in} \left[\sum_{j=1}^{n} c_{ij}x_j, \infty\right)} h_i(b_i) = \overline{v}$$

to

$$\sum_{j=1}^{n} c_{ij}x_j = \underline{h}_i^{-1}(\overline{v})$$

as well as the fact that function $\underline{h}_i$ is monotonic, one can obtain the set $D_{x,1}$ which, in fact, is dependent on $\overline{v}$, i.e.

$$D_{x,1}(\overline{v}) = \Bigg\{x\colon (x_j \ge 0,\ j = 1,\ 2,\ \ldots,\ n) \wedge \wedge \left(\forall i = 1,\ 2,\ \ldots,\ m \sum_{j=1}^{n} c_{ij}x_j \le \underline{h}_i^{-1}(\overline{v})\right)\Bigg\}. \tag{10}$$

For example, when $h_i(b_i)$ are triangular certainty distributions the set (10) has the form of

$$D_{x,1} = \Bigg\{x\colon (x_j \ge 0,\ j = 1,\ 2,\ \ldots,\ n) \wedge \wedge \left(\forall i = 1,\ 2,\ \ldots,\ m \sum_{j=1}^{n} c_{ij}x_j \le \underline{b}_i^* + (1-\overline{v})d_{bi}\right)\Bigg\}.$$

It can be noticed that there are no uncertain parameters in (10), the parameters characterizing an expert's knowledge are present only. So, the solution algorithms as for the deterministic case **P** can be applied now, e.g. [21].

3.2. Solution algorithm for UP$_2$. Actually, the solutions of **UP**$_1$, i.e. x' and $F(x')$ are dependent on $\overline{v}$ which can be expressed as $x'(\overline{v})$ and $F(x'(\overline{v})) \stackrel{\Delta}{=} F'(\overline{v})$. The following property is true.

Property

For every $v_a, v_b \in (0,1)$ and $v_a > v_b$ the inequality $F'(v_a) \le F'(v_b)$ holds.

Let us assume indirectly that a pair $v_a, v_b \in (0,1)$, $v_a > v_b$ exists that

$$F'(v_a) > F'(v_b). \tag{11}$$

It is easy to see that

$$\forall_i\ \underline{h}_i^{-1}(v_a) < \underline{h}_i^{-1}(v_b)$$

because $\underline{h}_i$ and $\underline{h}_i^{-1}$ are decreasing functions. Therefore, inclusion $D_{x,1}(v_a) \subseteq D_{x,1}(v_b)$ for the corresponding sets of feasible solutions along with the following implication are true: If $x'(v_a) = \arg\max_{x \in D_{x,1}(v_a)} F(x)$ and consequently $x'(v_a) \in D_{x,1}(v_a)$ then $x'(v_a) \in D_{x,1}(v_b)$. Let $x'(v_b) = \arg\max_{x \in D_{x,1}(v_b)} F(x)$ which means $\neg\left[\exists_{x \in D_{x,1}(v_b)} F(x'(v_b)) < F(x)\right]$, in particular

$$\neg\left[F(x'(v_b)) < F(x'(v_a))\right]. \tag{12}$$

On the other hand, $F(x'(v_a)) = F'(v_a) > F'(v_b) = F(x'(v_b))$ according to (11) which contradicts (12).

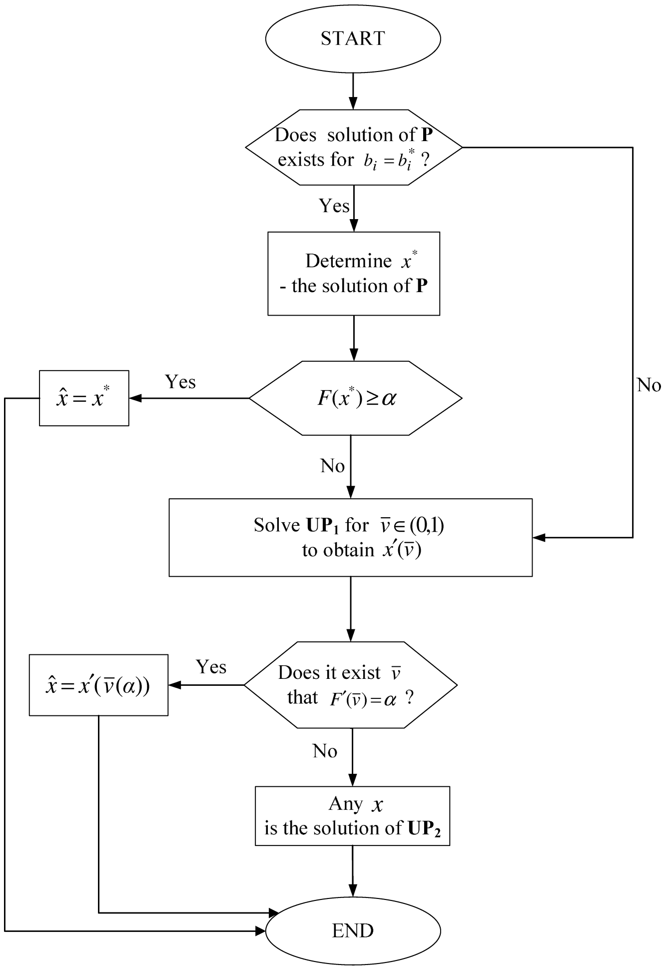

Fig. 3. The solution algorithm for **UP**$_2$

The solution algorithm for **UP**$_2$, which block scheme is presented in Fig. 3, can be introduced in three following steps:

1. If there exists any production plan for which the certainty index is equal to 1 it is the optimal solution because the maximal value of the certainty index cannot be greater than 1. To check this, one should determine if set

$\widetilde{D}_x = \{x : x \in D_{x,2} \wedge \forall i = 1,\ 2,\ ...,\ m\ v_i = 1\}$ is empty or not. It can be easily done by finding x^*, i.e. the optimal solution of the deterministic production planning problem **P**, assuming $b_i = b_i^*$. If $F(x^*) \geq \alpha$ then $\hat{x} = x^*$ is also an optimal solution of $\mathbf{UP}_2$.

2. If $\widetilde{D}_x$ is empty the maximal value of the certainty index is less than 1. According to Property, $F'(\overline{v})$ is a non-increasing function of $\overline{v}$, so the solution of $\mathbf{UP}_2$ can be obtained as $\hat{x} = x'(\overline{v}(\alpha))$ where $\overline{v}(\alpha)$ is the solution of $F'(\overline{v}) = \alpha$ if it exists.
3. If $\widetilde{D}_x$ is empty and $F'(\overline{v}) = \alpha$ has no solutions then for every production plan x the certainty index

$$v\left[\forall i = 1,\ 2,\ ...,\ m\ \sum_{j=1}^{n} c_{ij}x_j \widetilde{\leq} \overline{b}_i\right] = 0,$$

which means that every production plan x is the solution of $\mathbf{UP}_2$.

4. Numerical example

A simple numerical example presented in this section can refer to a production of four types of a juice (e.g. orange, apple, grape and multifruit) manufactured from three types of fruits (orange, apple, grape). It is needed to provide 1.2 units of fruits to obtain one unit of juice. The multifruit juice is made from oranges, apples and grapes mixed with equal proportions. Consequently, the following numerical data result from the assumptions: $n = 4$, $m = 3$, $c_{ij} = 1.2$ for $i, j = 1, 2, 3$, $i = j$, $c_{ij} = 0$ for $i, j = 1, 2, 3$, $i \neq j$, $c_{i4} = 0.4$ for $i = 1, 2, 3$, and the constraints (7) for the consumption of resources can be expressed by the following inequalities

$$1.2x_i + 0.4x_4 \leq b_i, \quad \text{for} \quad i = 1, 2, 3, \qquad (13)$$

where b_i denotes the available amount of each type of resources being the values of the uncertain variables. However, the precise values of b_i are not known in advance and only expert's knowledge characterized by triangular certainty distributions with the parameters $b_1^* = 1$, $d_{b1} = 1$, $b_2^* = 2$, $d_{b2} = 1$, $b_3^* = 3$, $d_{b3} = 1$ is given. The local (consumers') utilities concerning the juice production are described by the logarithmic function for each type of juice. So, the total utility function is in the form of $F(x) = \sum_{j=1}^{n} \ln(x_j + 1)$. To solve such a problem, a numerical algorithm for solving the set of nonlinear equations has to be applied. In this case, Newton's method was used. The results determined for $\mathbf{UP}_1$ and for different user's requirements $\overline{v}$ are given in Table 1.

Table 1
The results of $\mathbf{UP}_1$ for different $\overline{v}$

$\overline{v}$	x_1'	x_2'	x_3'	x_4'	$F(x')$
0.7	0.65	1.48	2.32	1.29	3.44
0.8	0.59	1.42	2.26	1.23	3.33
0.9	0.53	1.36	2.19	1.16	3.21

One can notice that the greater is certainty level $\overline{v}$ the less is the total utility which may be obtained, and, so, the amount of resources needed for the production are closer to the values b_i^*. The decreasing of certainty level $\overline{v}$ implies the increasing of the total utility, and, in the consequence, the amount of resources needed for the production is closer to the values $b_i^* + d_{bi}$. The values of the left-hand side in (13) for $i = 1, 2, 3$ and for $\overline{v} = 0.7,\ 0.8,\ 0.9$ are given in Table 2 (notice that $b_1^* = 1$, $b_2^* = 2$ and $b_3^* = 3$). Another speaking, the greater is the certainty level $\overline{v}$ that the constraints for resources are approximately satisfied the closer is a real consumption of resources to the values b_i^* which the expert is the most certain.

Table 2
The values of the left-hand side in (13) for different $\overline{v}$

$\overline{v}$	$i = 1$ $(1.2x_1' + 0.4x_4')$	$i = 2$ $(1.2x_2' + 0.4x_4')$	$i = 3$ $(1.2x_3' + 0.4x_4')$
0.7	1.296	2.292	3.300
0.8	1.200	2.196	3.204
0.9	1.100	2.096	3.092

Analogously, the results for $\mathbf{UP}_2$ and for different user's requirements α are presented in Table 3.

Table 3
The results for UP_2 and different α

α	$F(\hat{x})$	$\hat{x}_1$	$\hat{x}_2$	$\hat{x}_3$	$\hat{x}_4$	$\hat{v}$
2.00	2.839728	0.83	1.66	2.50	0.00	1.00
3.50	3.503816	0.69	1.52	2.35	1.33	0.64
3.75	3.752855	0.84	1.67	2.50	1.48	0.40
4.00	4.004201	0.99	1.83	2.66	1.66	0.14

The values of $\hat{v}$ obtained inform a decision maker how realistic is the production plan according to the expert's knowledge. It can be noticed that the greater is the requested value of the total utility the less is the certainty index that "constraints of the resources are approximately satisfied", i.e. the less realistic is the production plan. In the second column of Table 3, the values of the total utility calculated for $\hat{x}$ are presented. It is easy to notice that if the minimal acceptable value of the utility α is low like in the first row of Table 3 one can find such an allocation for which the total utility is greater than user's expectation and the certainty index concerning constraints is maximal, i.e., equal to 1. For greater values of α the total utility $F(\hat{x})$ calculated is almost equal to α. The small differences between α and $F(\hat{x})$, which can be noticed in two first columns of Table 3, are effects of the rounding up and the calculation errors caused by the numerical algorithms applied to solve problem $\mathbf{UP}_2$ which could not be solved analytically.

5. Final remarks

The solution algorithms for the production planning problem with uncertain amount of the resources are presented. The formalism of the uncertain variables has been used. The information of the uncertain problem's data (the amount of resources) is given in the form of the certainty distributions.

In further investigations the more general case will be considered which deals with taking into account a minimal positive size of the production $x_{j,\min} > 0$. Then, the set of feasible solutions can be stated as

$$D_x = \{x\colon (x_j \geq x_{j,\min} > 0,\ j = 1,\ 2,\ ...,\ n) \wedge (7)\ \text{holds}\}.$$

It is obvious that too small size of products might lead to a loss not to a profit as it could be seen for the numerical example given in Sec. 4 when the local utility functions of the form $f_j(x_j, a_j) = \ln\ x_j$ would be used instead of (5).

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