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Non-Parametric Test for the Existence of the Common Deterministic Cycle: The Case of the Selected European Countries

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Abstract

The aim of the article is to construct an asymptotically consistent test, based on a subsampling approach, to verify hypothesis about existence of the individual or common deterministic cycle in coordinates of multivariate macroeconomic time series. By the deterministic cycle we mean the periodic or almost periodic fluctuations in the mean function in cyclical fluctuations. To construct test we formulate a multivariate non-parametric model containing the business cycle component in the unconditional mean function. The construction relies on the Fourier representation of the unconditional expectation of the multivariate Almost Periodically Correlated time series and is related to fixed deterministic cycle presented in the literature. The analysis of the existence of common deterministic business cycles for selected European countries is presented based on monthly industrial production indexes. Our main findings from the empirical part is that the deterministic cycle can be strongly supported by the data and therefore should not be automatically neglected during analysis without justification.

Keywords: testing deterministic cycles, subsampling, spectral analysis, almost periodic mean function, Almost Periodically Correlated time series

JEL Classification: C14, C46, E32

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1 Introduction

Economic integration, which has substantially accelerated during the last 40 years, has made national growth more correlated across the globe. Globalization processes, influencing the degree of cross-border integration in markets, initiated decades of a more synchronized growth across countries. Additionally, there is no doubt that international trade has strongly intensified recently. The observed correlation of changes in world economic activity prompted new studies on the field of empirical macroeconomics, focused on the construction of appropriate measures of synchronization for the business cycles rather than on examining well established properties of the cycle in most economies; see Stock and Watson (2005); Doyle and Faust (2005); Kose et al. (2003); Yetman (2011); Crucini et al. (2011); Otto et al. (2001); Ayhan–Kose et al. (2008); Imbs (2004); Artis et al. (2011).

The analyses of the comovements of business cycle fluctuations are particularly focused on European countries. The continued progress toward economic integration in Europe and the monetary union in 1999 have prompted new studies concerning the nature of business cycle synchronization of the member countries. The thorough empirical verification of the hypothesis that the euro area is becoming an optimal currency area requires detailed insight into the nature of changes in economic activity across countries. Initially, Bayoumi and Eichengreen (1997) conducted empirical analysis on whether the implications of the theory of optimal currency areas are empirically supported in the case of European countries. Bayoumi and Eichengreen propose a procedure for cross country data, indicating the existence of three groups of countries with respect to their participation in the European monetary union. The empirical analysis of correlations of the business cycles of small European countries with the Euro area business cycle was presented by Gouveia and Correia (2008). Some changes of correlations were observed, and the level of synchronization has declined since 1997. Changes in the synchronization of economic activity factors were also tested by Bergman (2004), Canova et al. (2009), Christodoulakis et al. (1995), Duecker and Wesche (1999) and many others.

Additionally, changes in the nature of world business cycles are of extreme importance from a policy perspective. In particular, as suggested by Obstfeld and Rogoff (2002), with stronger business cycle transmission, policy measures taken by one country could have a larger impact on economic activity in other countries, implying that the degree of synchronization of business cycle fluctuations has important implications for international policy coordination. Understanding the evolution of the world business cycle requires detailed insight into the correlation of changes in economic activity among different regional economies. Because the globalization processes may result in faster transmission of the economic downturns from one economy to another, analysis of the synchronization of the business cycles of the largest economies is important. Ayhan–Kose et al. (2008) studied the importance of rising trade and financial linkages in business cycle correlation in G-7 countries. An extended analysis of the evolution of the comovement properties of the main macroeconomic aggregates has been conducted by many authors. The literature overview clearly indicates the diversity of conclusions about the temporal evolution of business cycle synchronization. In particular, a decline in the output correlation over the last three decades is reported by Helbling and Bayoumi (2002), Heathcote and Perri (2004), Stock and Watson (2005) and others. In contrast, strengthening business cycle linkages have also been reported; see Kose et al. (2003), Bordo and Helbling (2003). Doyle and Faust (2005) conclude that there have been no significant changes in the correlation between the growth rates of output in the case of G-7 countries.

One of the most popular method applied in the problem of analysing the business cycle properties relies on the model of unobserved components. In this framework the cyclical component is assumed to be stationary ARMA process with complex conjugate roots in AR polynomial; see Harvey and Trimbur (2003), Trimbur (2006), Koopman and Shephard (2015), Pelagatti (2016). The extension to multivariate case was considered in Azevedo et al. (2006) and also in Harvey et al. (2007), but only trivariate example was considered. In Koopman and Azevedo (2008) the multivariate model with stationary multiple cyclical process with common frequency at each coordinate was examined. The phase shifts incorporated in this model is flexible and allow for increasing or diminishing.

Despite the importance of detailed and precise analysis of synchronization of the business cycle, analyses have been conducted to date mainly with the use of simple methods involving basic descriptive statistics of the extracted business cycle component. Analysis of sample correlations and cross-correlations seems to be the standard approach. Factor-structural VAR models were applied by Stock and Watson (2005). In addition, factor models were utilized from the Bayesian viewpoint by Ayhan–Kose et al. (2008). A measure of business cycle synchronization constructed on the basis of cross wavelets and multidimensional mapping was proposed by Aguiar-Conraria and Soares (2011). In the empirical literature, the approach involving spectral analysis has also played an important role; see Ftiti (2010); Metz (2009); Orlov (2006, 2009); Pakko (2004); McAdam and Mestre (2008); Uebele and Ritschl (2009). The theoretical counterpart can be found in Croux et al. (2001); Hamilton (1994); Priestley (1981) and others. The spectral representation of the unconditional moment of the processes describing the time evolution of macroeconomic time series is commonly used despite its disadvantages. However, formal statistical inferences about the nature of the spectra of the modeled, possibly nonstationary, processes are still missing. Hence, in the presence of such a deficiency, the development of statistical tests verifying the significance of business cycle comovements is necessary.

Unfortunately the above predominant approaches imposes stationarity assumption of cyclical components with zero-mean expectation function. Hence the issue of formal statistical testing concerning the existence of deterministic cycle or common deterministic cycle has not been explored in detail. In the most applications the existence of only zero-mean stochastic cycle is assumed. It is known that the deterministic cycle is not flexible enough to describe dynamics of business fluctuations.

In this paper we agree with this statement. But the problem is if deterministic cycle and stochastic cycle can be modeled jointly? Or, if deterministic cycle can exist as complementary dynamics jointly with stochastic cycle. To put a light to this question, we propose in this paper the non-parametric procedure for testing the existence of the deterministic cycle. Note, that our procedure is not to distinguish between the existence of deterministic cycle versus stochastic cycle. Hence, our procedure test the existence of just deterministic cycles versus the constant mean function under general assumptions concerning second order structure. Note that the joint movement of stochastic and deterministic cycle may be assumed only in some time interval.

In this paper the theory of the multivariate Almost Periodically Correlated (APC) time series is applied in building a formal statistical test of the individual or common deterministic cycle. Asymptotic results concerning the mean function in the case of multivariate APC time series are discussed. In this framework formal statistical inference about the frequencies associated with mean function is obtained. Based on these results, we propose the test statistics for the problem of testing the existence of individual or common deterministic cycles in the coordinates of the multivariate APC time series. It should be emphasized that we do not propose any new definition of business cycle or common business cycle. We would like to shed the light only on the problem of the existence of deterministic cycle, rather than model business fluctuations. Our approach differs from common practice of business cycle analysis where synchronization and convergence is analyzed using parametric approach based on the multivariate stationary model. Starting from the Fourier representation of the unconditional expectation of the process, we build test statistics to make formal inferences about the existence of similar frequencies in every time series modeled jointly by a nonstationary multivariate process. The distribution of the natural test statistics is approximated by the subsampling distribution according to the method described in Politis et al. (1999). The subsampling approach is considered because the asymptotic distribution of test statistics is too complicated to be approximated using standard techniques. We discuss the asymptotic consistency of both test statistics and subsampling approximations of quantiles, utilized in the procedure as critical values. The problem of formal testing of common frequencies in Fourier representations has not been considered in the literature. In application to the univariate case, the statistical inference concerning unknown frequency in Fourier representation was considered in Walker (1971), He (1997), Li and Song (2002), Lavielle and Levy–Leduc (2005) and recently by Lenart (2013).

In section 2, we present basic notation and definitions. Section 3 presents an application of the general concept of almost periodic mean function and continuous spectral analysis in modeling business cycle fluctuations. In section 4, the theoretical results are presented. Based on almost periodic function, we formulate a multivariate nonparametric model of changes in economic activity (section 5). The formal test for individual or common deterministic business cycle is defined in section 6, and the usefulness of the proposed testing procedure is discussed in section 7. In addition, we discuss the robustness of the proposed procedure to structural changes (i.e., the previous global financial crisis) and we present the results of estimations given proposed time interval windows. We also compare the results with the concepts based on continuous spectral characteristics (cross-spectral analysis), commonly applied in the literature.

2 Basic notation and definitions for APC time series

We develop the spectral theory of Almost Periodically Correlated (APC) or Cyclostationary time series. We start from the definition of the almost periodic (ap) function.

Definition 1. *(see Corduneanu (1989)) A real-valued function $f(t) : \mathbb{Z} \longrightarrow \mathbb{R}$ of an integer variable is called almost periodic if for any $\epsilon > 0$ there exists an integer $L_\epsilon > 0$ such that among any L_ϵ consecutive integers there is an integer p_ϵ with the property* $\sup_{t\in\mathbb{Z}} |f(t+p_\epsilon) - f(t)| < \epsilon$.

Formally, we say that a multivariate, $\mathbb{R}^r$-valued time series, denoted by $\{\boldsymbol{X}_t = (X_{1,t}, X_{2,t}, \ldots, X_{r,t}) : t \in \mathbb{Z}\}$, is Almost Periodically Correlated (APC) if for any $k, l \in \{1, 2, \ldots, r\}$ the mean function $\mu_k(t) = E(X_{k,t})$ and the cross-covariance functions $B(t,\tau)_{k\,l} = \mathrm{cov}(X_{k,t}, X_{l,t+\tau})$ are almost periodic functions at time t for any $\tau \in \mathbb{Z}$. In the APC case, for any $k \in \{1, 2, \ldots, r\}$, the unconditional expectation has the Fourier representation of the form

$$\mu_k(t) \sim \sum_{\psi\in\Psi_k} m_k(\psi) e^{i\psi t}, \tag{1}$$

with the Fourier coefficients $m_k(\psi) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mu_k(j) e^{-i\psi j}$; see Hurd (1989, 1991); Dehay and Hurd (1994). The most important result, making Fourier analysis on the basis of the spectral atoms of the process possible, is that the sets $\Psi_k = \{\psi \in [0, 2\pi) : m_k(\psi) \neq 0\}$ for $k = 1, 2, \ldots, k$ are countable; see Corduneanu (1989). In addition to the unique representation of the unconditional expectation, one may also consider for each $k, l \in \{1, 2, \ldots, r\}$ a representation of the autocovariance function $B(t,\tau)_{k\,l} = \mathrm{cov}(X_{k,t}, X_{l,t+\tau})$, $\tau \in \mathbb{Z}$. In this case, the Fourier expansion takes the form

$$B(t,\tau)_{k\,l} \sim \sum_{\lambda\in\Lambda_{k\,l}} a(\lambda,\tau)_{k\,l} e^{i\lambda t}, \tag{2}$$

where $a(\lambda,\tau)_{k\,l} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} B(j,\tau)_{k\,l} e^{-i\lambda j}$.

The sets $\Lambda_{k\,l} = \bigcup_{\tau\in\mathbb{Z}} \{\lambda \in [0, 2\pi) : a(\lambda,\tau)_{k\,l} \neq 0\}$ for $k, l \in \{1, 2, \ldots, r\}$ are also countable. In general, the unique characterisation of both the unconditional mean (1) and the covariance matrix (2) of the multivariate APC times series exists; this paper presents detailed insight into the properties of the first unconditional moment. From the definition of the APC time series, the almost periodic time pattern for the unconditional mean and covariance is assured for the whole domain of the process, namely for each $t \in \mathbb{Z}$. We generalise the notion of the APC time series by considering a class of processes with an almost periodic structure of moments restricted only to a subset $A \subseteq Z$. For a proper definition of such a process, the definition of the almost periodic function on a subset A is necessary.

Definition 2. *A real-valued function $f(t) : A \longrightarrow \mathbb{R}$ of an integer variable $t \in A \subseteq \mathbb{Z}$ is called almost periodic on the set A (ap_A) if there exists a real-valued almost periodic function $h(t) : \mathbb{Z} \longrightarrow \mathbb{R}$ of an integer variable such that $f(t) = h(t)$, for any $t \in A$.*

We say that a multivariate, $\mathbb{R}^r$-valued time series, denoted by $\{\boldsymbol{X}_t = (X_{1,t}, X_{2,t}, \ldots, X_{r,t}) : t \in \mathbb{Z}\}$, is Almost Periodically Correlated of order one on the set $A = A_1 \times A_2 \times \ldots \times A_r$ ($\text{APC}(1)_A$ in short) if for any $k, l \in \{1, 2, \ldots, r\}$ the mean function $\mu_k(t) = E(X_{k,t})$ is an ap_{A_k} function and the cross-correlation function $B(t, \tau)_{k\,l} = \text{cov}(X_{k,t}, X_{l,t+\tau})$ are ap functions of an integer variable for every $\tau \in \mathbb{Z}$.
For further analysis, let us consider the $\mathbb{R}^r$-valued APC case on the set $A = A_1 \times A_2 \times \ldots \times A_r$, where the sets $\mathbb{Z} \setminus A_k$, $k = 1, 2, \ldots, r$ are finite. We establish this condition in the following assumption.

Assumption 1. *An $\mathbb{R}^r$-valued time series is APC on the set $A = A_1 \times A_2 \times \ldots \times A_r$, where the complementary of the sets A_k, $k = 1, 2, \ldots, r$ are finite.*

According to assumption 1, the unconditional mean of the analysed APC time series has a unique representation of the form (1), but this unique representation is valid only for $t \in A$. Now we adopt Assumption 1.1 from Lenart (2013) to the multivariate setting considered here to justify asymptotic results.

Assumption 2. *Let $\{\boldsymbol{X}_t : t \in \mathbb{Z}\}$ be an $\mathbb{R}^r$-valued APC on the set $A = A_1 \times A_2 \times \ldots \times A_r$ time series. For any $x \in [0, 2\pi)$ and $k \in \{1, 2, \ldots, r\}$ there exists a constant $B_k(x)$ (which may depend only on x), such that $\sum_{\psi \in \Psi_k \setminus \{x\}} \left| m_k(\psi) \text{cosec}\left(\frac{\psi - x}{2}\right) \right| < B_k(x) < \infty.$*

Because we need asymptotic independence, we assume that the considered $\mathbb{R}^r$-valued time series is so-called α-mixing. We recall this definition from Doukhan (1994). Formally, the $\mathbb{R}^r$-valued time series $\{\boldsymbol{X}_t : t \in \mathbb{Z}\}$ is called α-mixing, with corresponding α-mixing sequence $\alpha(\cdot)$, if $\alpha(s) \to 0$ for $s \to \infty$, where $\alpha(s) = \sup_{t \in \mathbb{Z}} \sup_{\substack{A \in \mathcal{F}_X(-\infty, t) \\ B \in \mathcal{F}_X(t+s, \infty)}} |P(A \cap B) - P(A)P(B)|$ and $\mathcal{F}_{\boldsymbol{X}}(t_1, t_2)$ stands for the σ-algebra generated by $\{\boldsymbol{X}_t : t_1 \leq t \leq t_2\}$.

Assumption 3. *The $\mathbb{R}^r$-valued time series $\{\boldsymbol{X}_t : t \in \mathbb{Z}\}$ is α-mixing with corresponding mixing sequence denoted by $\alpha(\cdot)$.*

The next assumption means that the so-called population spectrum matrix exists for the multivariate time series $\{\boldsymbol{X}_t : t \in \mathbb{Z}\}$.

Assumption 4. *Suppose we have an $\mathbb{R}^r$-valued time series $\{\boldsymbol{X}_t = (X_{1,t}, X_{2,t}, \ldots, X_{r,t}) : t \in \mathbb{Z}\}$. For each univariate time series $\{X_{k,t} : t \in \mathbb{Z}\}$, where $k = 1, 2, \ldots, r$, the assumptions (A3)-(A5) from Lenart (2011) hold.*

Under Assumption 4, for any $k, l \in \{1, 2, \ldots, r\}$ and $\lambda \in \Lambda_{k\,l}$ the spectral density equals $g_\lambda(\nu)_{k\,l} = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} a(\lambda, \tau)_{k\,l} e^{-i\nu\tau}$, where $a(\lambda, \tau)_{k\,l} = \int_0^{2\pi} e^{i\nu\tau} g_\lambda(\nu)_{k\,l} \mathrm{d}\nu$. We denote by $\mathbf{P}(\nu, \omega) = \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} \mathbf{a}(\nu - \omega, \tau) e^{-i\nu\tau}$ the generalisation of the spectral density matrix to the square $[0, 2\pi)^2$, where $\mathbf{a}(\nu - \omega, \tau) = [a(\nu - \omega, \tau)_{k\,l}]_{r\times r}$ and $\mathbf{P}(\nu, \omega) = [P(\nu, \omega)_{k\,l}]_{r\times r}$ are complex matrices.

3 Spectral analysis and business cycle comovement

We use the spectral analysis of the APC time series in making formal inferences about the business cycle features as well as inferences about synchronisation. Because the class of harmonizable time series plays a key role in spectral theory, we recall some details concerning this idea. The property, initially proposed in the 1940s, was examined in detail by Hurd and Miamee (2007) in the case of univariate time series. The following definition generalizes the concept of a harmonizable process to the non-zero mean multivariate framework (see Hurd and Miamee (2007), p. 99-102, 141).

Definition 3. *We say that $\mathbb{R}^r$-valued time series $\{\boldsymbol{X}_t = (X_{1,t}, X_{2,t}, \ldots, X_{r,t}) : t \in \mathbb{Z}\}$, with each element $\boldsymbol{X}_t$ of finite covariance matrix, is weakly harmonizable if for any $k \in \{1, 2, \ldots, r\}$ there exists a bounded vector measure $\xi = [\xi_1, \xi_2, \ldots, \xi_k]$ on Borel subsets of $[0, 2\pi) \times [0, 2\pi) \times \ldots \times [0, 2\pi)$ such that*

$$X_{k,t} = \int_0^{2\pi} e^{it\psi} \xi_k(d\psi) \tag{3}$$

for any $k = 1, 2, \ldots, r$. Additionally let us assume that for any $k, l \in \{1, 2, \ldots, r\}$ there exists a scalar valued measure $F_{k\,l}$ of bounded variation on Borel subsets of $[0, 2\pi)^2$ such that

$$\mathrm{cov}(X_{k,s}, X_{l,t}) = \int_0^{2\pi} \int_0^{2\pi} e^{i(s\nu - t\omega)} F_{k\,l}(\mathrm{d}\nu, \mathrm{d}\omega). \tag{4}$$

If the measure $F_{k\,l}$ exists for any $k, l \in \{1, 2, \ldots, r\}$ then we say that time series is strongly harmonizable with spectral measure $\mathbf{F} = [F_{k\,l}]_{r\times r}$.

Now if the process $\{X_t : t \in \mathbb{Z}\}$ is APC and harmonizable with the spectral measure $\mathbf{F} = [F_{k\,l}]_{r\times r}$ and if $\mathrm{card}(\bigcup_{i=1}^{r}\Psi_i) < \infty$, then $E(X_{k,t}X_{l,t+\tau}) = \int_0^{2\pi}\int_0^{2\pi} e^{it(\xi_1-\xi_2)-i\tau\xi_2}\tilde{F}_{k\,l}(d\xi_1, d\xi_2)$, where

$$\tilde{F}_{k\,l} = F_{k\,l} + \sum_{\psi_1\in\Psi_k}\sum_{\psi_2\in\Psi_l} m_k(\psi_1)\overline{m_l(\psi_2)}\delta(\xi_1-\psi_1,\xi_2-\psi_2),\ k,l\in\{1,2,\ldots,r\} \quad (5)$$

and $\delta(x,y) = 1 \Leftrightarrow x = y = 0$; see Hurd and Miamee (2007) (p. 174-175) in the univariate Periodically Correlated case. According to equation (5), the measure can be easily decomposed into two separable elements. The first element in the sum corresponds to the continuous spectrum, while the second is related to the spectral atoms. Hence, $F_{k\,l}$ can be interpreted as a joint measure (in the spectral domain) of the dependence between coordinates of the multivariate time series. This idea can be found for example in Priestley (1981) and Hamilton (1994), but so far theoretical considerations have been conducted only in the zero mean strictly stationary case.
Note that the existence of the spectral density matrix is connected with the property that the proces is harmonizable (see: Priestley (1981), Hamilton (1994) for stationary case). For any $(\nu,\omega)\in(0,2\pi]^2$ and $\lambda\in\Lambda_{k\,l}$ we have $g_\lambda(\nu)_{k\,l}\,\mathrm{d}\nu = \mathrm{cov}\{\xi_k(\mathrm{d}\nu), \xi_l(\mathrm{d}\nu-\lambda)\} = F(\mathrm{d}\nu,\mathrm{d}\nu-\lambda)_{k\,l}$, where $F([a,b),[c,d))_{k\,l} = \mathrm{cov}\{\xi_k([a,b)),\overline{\xi_l([c,d))}\}$, and $a(\lambda,\tau)_{k\,l} = \int_0^{2\pi}\int_0^{2\pi} \mathrm{I}\{(\nu-\omega-\lambda) \text{ modulo } 2\pi = 0\}e^{i\tau\nu}F(\mathrm{d}\nu,\mathrm{d}\omega)_{k\,l}$. Consequently, for any $\lambda\in\Lambda_{k\,l}$, $g_\lambda(\nu)_{k\,l}\,\mathrm{d}\nu$ represents the covariance between $\xi_k(\mathrm{d}\nu)$ and $\xi_l(\mathrm{d}\nu-\lambda)$.
The APC time series is a generalization of the stationary sequence. One may also consider another class of processes, called asymptotically stationary time series; see Hurd and Miamee (2007) page 172. The main property of the harmonizable sequence is that it is also asymptotically stationary.
On the basis of the aforementioned property, several measures of comovement in cases of $\mathbb{R}^r$-valued time series have been proposed, provided that the assumption $\Lambda = \bigcup_{k,l=1,2,\ldots,r}\Lambda_{k\,l} = \{0\}$ holds. In analyses of comovements of economic processes, the most frequently used are cross-spectrum $g_0(\cdot)_{k\,l}$, modulus of the value of the coherency function

$$\rho(\omega)_{k\,l} = \frac{|g_0(\omega)_{k\,l}|}{\sqrt{g_0(\omega)_{k\,k}g_0(\omega)_{l\,l}}},$$

dynamic correlation

$$\tilde{\rho}(\omega)_{k\,l} = \frac{Re[g_0(\omega)_{k\,l}]}{\sqrt{g_0(\omega)_{k\,k}g_0(\omega)_{l\,l}}},$$

dynamic correlation on a frequency band (denoted by Λ_+)

$$\tilde{\rho}(\Lambda_+)_{k\,l} = \frac{\int_{\Lambda_+} Re[g_0(\omega)_{k\,l}]\mathrm{d}\omega}{\sqrt{\int_{\Lambda_+} g_0(\omega)_{k\,k}\mathrm{d}\omega\int_{\Lambda_+} g_0(\omega)_{l\,l}\mathrm{d}\omega}},$$

cohesion

$$\mathrm{coh}_{\mathbf{X}}(\omega) = \frac{\sum_{i \neq j} w_i w_j \rho_{i\,j}(\omega)}{\sum_{i \neq j} w_i w_j}$$

and cohesion within the frequency band Λ_+ :

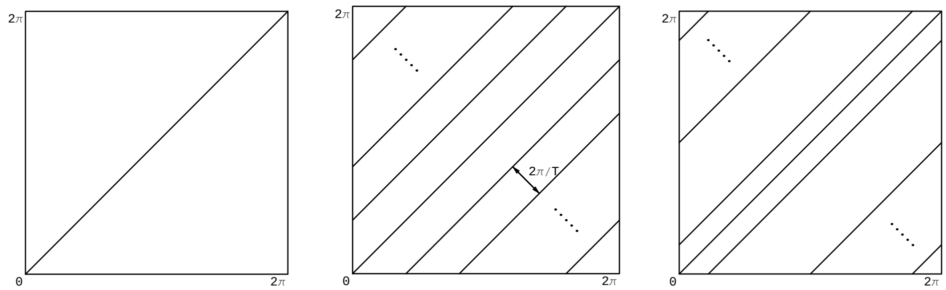
$$\mathrm{coh}_{\mathbf{X}}(\Lambda_+) = \frac{\sum_{i \neq j} w_i w_j \rho_{i\,j}(\Lambda_+)}{\sum_{i \neq j} w_i w_j},$$

where $w_1, w_2, \ldots, w_r$ are positive weights. The details concerning the interpretation of the measures as well as empirical applications are in Priestley (1981), Hamilton (1994) and Croux et al. (2001).
In parametric approach based on modelling cyclical pattern by stationary process (see Harvey and Trimbur (2003), Harvey et al. (2007), Koopman and Azevedo (2008), Azevedo et al. (2006), Koopman and Shephard (2015), Pelagatti (2016)) the zero mean assumption with $\Lambda = \bigcup_{k,l=1,2,\ldots,r} \Lambda_{k\,l} = \{0\}$ plays a central role. In this approach the univariate or multivariate cyclical pattern is modelled in parametric way by stationary process with concentration of the spectral density (or spectral density matrix) around the frequency corresponding to cyclical fluctuations.
Figures 1 and 2 illustrate the differences between supports of the measure $F_{k\,l}$ and $\tilde{F}_{k\,l}$, respectively. We draw the support for the stationary case (on the left) and for two nonstationary cases, represented by Periodically Correlated processes with the period T (a plot in the middle) and APC time series (on the left). In the current econometric literature the extraction of the business cycle component is commonly carried out under the scheme of the stationary property of this component. This is related to the case of the domain of the measures $F_{k\,l}$ and $\tilde{F}_{k\,l}$ limited to the diagonal of the square $(0, 2\pi]^2$, given the additional assumption that the process has a zero or constant mean function. Our approach is based on the more relaxed assumption, making the domain of the aforementioned measure more extended, as in case (c) in Figures 1 and 2. Consequently, the stationary property can be a testable restriction and if it is rejected, the formal theory is still valid. In our approach the existence of the business cycle comovement can be formally tested on the basis of the properties of the localization of the spectral atoms, i.e., the sets Ψ_k, corresponding Fourier coefficients $m_k(\cdot)$ and interactions. This clearly distinguishes our approach from the predominant analyses, relying on the continuous spectrum.
In this paper the spectral atoms are related to non-zero mean assumption of the cyclical pattern. Let us recall the fixed deterministic cycle model:

$$y_t = \alpha \cos(\lambda_c t) + \beta \sin(\lambda_c t) + \epsilon_t, \tag{6}$$

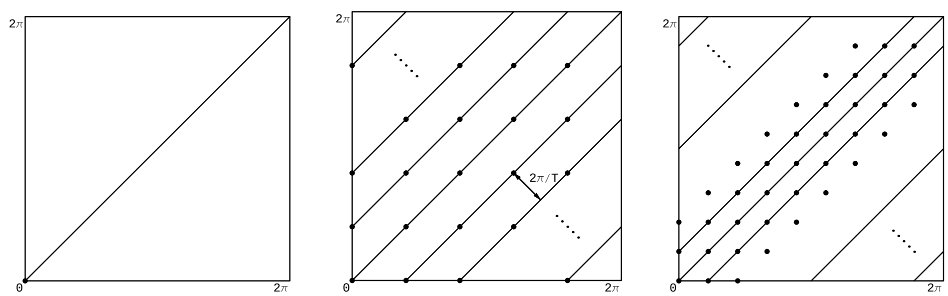
where λ_c is a frequency related to cycle with length $2\pi/\lambda$ and α, β are parameters and ϵ_t is a white noise; see Harvey (2004). This model produce one spectral atom

Figure 1: Example of the support of measure $F_{k\,l}$ on square $(0, 2\pi]^2$



(a) Stationary case. (b) PC case with period T. (c) APC case.

Figure 2: Example of the support of measure $\tilde{F}_{k\,l}$ on square $(0, 2\pi]^2$



(a) Stationary case. (b) PC case with period T. (c) APC case.

associated with frequency λ_c and constant value of spectral density function. In this case the cyclical fluctuations are modelled only by a single spectral atom. Therefore it is traditional paradigm that the above model is not enough flexible to model business cycle phenomenon. In this paper the cyclical pattern is assumed to be much more flexible than in (6). We assume the possible existence of stochastic cycle and the deterministic cycle jointly. In testing procedure introduced in the next sections our statistical inference concerns existence of the individual or common spectral atoms associated with the first order properties of the cyclical pattern. These spectral atoms corresponds easily to the frequencies in the deterministic cycle.

4 Asymptotic results

Let us take any frequency $\psi \in [0, 2\pi)$. Following Lenart (2013), we generalize the estimator of the parameter $m_k(\psi)$ based on a sample

$\{X_{k,c_n+1}, X_{k,c_n+2}, \ldots, X_{k,c_n+d_n}\}$, to the following form:

$$\hat{m}_{k,n}^{c,d}(\psi) = \hat{m}_{k,n}^{c_n,d_n}(\psi) = \frac{1}{d_n}\sum_{j=c_n+1}^{c_n+d_n} \mathbf{I}\{j \in A_k\} X_{k,j} e^{-i\psi j}, \tag{7}$$

for any sequence of positive integers $\{d_n\}_{n\in\mathbb{N}}$ tending to infinity with n, and for any sequence $\{c_n\}_{n\in\mathbb{N}}$ of integers, where $\mathbf{I}\{B\}$ is an indicator function of the event B. If $c_n \equiv 0$ and $d_n = n$, then we denote $\hat{m}_{k,n}^{c,d}(\psi) = \hat{m}_{k,n}(\psi)$. The following theorem establishes the conditions under which the normalized vector of estimators $[\hat{m}_{1,n}^{c,d}(\psi)\ \hat{m}_{2,n}^{c,d}(\psi) \ldots \hat{m}_{r,n}^{c,d}(\psi)]^T$ is asymptotically normally distributed. Denote $\hat{\mathbf{m}}_n^{c,d}(\psi) =$ $= [\mathrm{Re}(\hat{m}_{1,n}^{c,d}(\psi))\,\mathrm{Im}(\hat{m}_{1,n}^{c,d}(\psi))\,\mathrm{Re}(\hat{m}_{2,n}^{c,d}(\psi))\,\mathrm{Im}(\hat{m}_{2,n}^{c,d}(\psi))\ldots\mathrm{Re}(\hat{m}_{r,n}^{c,d}(\psi))\,\mathrm{Im}(\hat{m}_{r,n}^{c,d}(\psi))]^T$ and $\mathbf{m}(\psi) = [\mathrm{Re}(m_1(\psi))\ \mathrm{Im}(m_1(\psi))\ \mathrm{Re}(m_2(\psi))\ \mathrm{Im}(m_2(\psi))\ \ldots$ $\mathrm{Re}(m_r(\psi))\,\mathrm{Im}(m_r(\psi))]^T$.

Theorem 1. *Suppose that* $\{\boldsymbol{X}_t = (X_{1,t}, X_{2,t}, \ldots, X_{r,t}) : t \in \mathbb{Z}\}$ *is an* $\mathbb{R}^r$*-valued time series and let assumptions 1-4 hold. In addition, assume that there exist constants* $\delta > 0$, $\Delta < \infty$ *and* $K < \infty$, *such that* $\sup_{s=1,2,\ldots,r}\sup_{t\in\mathbb{Z}} \|X_{s,t}\|_{2+\delta} \le \Delta$ *and* $\sum_{k=0}^{\infty} \alpha(k)^{\frac{\delta}{2+\delta}} \le K$, *where* $\|X_{s,t}\|_{2+\delta} := \left(E|X_{s,t}|^{2+\delta}\right)^{1/(2+\delta)}$. *Then, for any* $\psi \in [0, 2\pi)$, *the following convergence holds:*

$$\sqrt{d_n}\left(\hat{\mathbf{m}}_n^{c,d}(\psi) - \mathbf{m}(\psi)\right) \xrightarrow{d} \mathcal{N}_{2r}(0, \Omega(\psi)),$$

where $\Omega(\psi) = [\Omega_{i,j}]_{2r\times 2r}$ *is a covariance matrix, such that for any* $s_1, s_2 \in \{1, 2, \ldots, r\}$ *we have* $\Omega_{2s_1-1,2s_2-1} = \pi(\mathrm{Re}[P(\psi,\psi)_{s_1\,s_2}] + \mathrm{Re}[P(\psi, 2\pi+$ $-\psi)_{s_1\,s_2}])$, $\Omega_{2s_1,2s_2} = \pi(\mathrm{Re}[P(\psi,\psi)_{s_1\,s_2}] - \mathrm{Re}[P(\psi, 2\pi - \psi)_{s_1\,s_2}])$, $\Omega_{2s_1-1,2s_2} =$ $= -\pi(\mathrm{Im}[P(\psi,\psi)_{s_1\,s_2}] - \mathrm{Im}[P(\psi, 2\pi - \psi)_{s_1\,s_2}])$.

Now we formulate two theorems that are crucial for proving the asymptotic consistency of the testing problems concerning the existence of common frequencies in multivariate time series.

Theorem 2. *Let all the assumptions of Theorem 1 hold. Then, we have convergence:*

$$\sqrt{d_n}\left(\sqrt[r]{\prod_{k=1}^{r} |\hat{m}_{k,n}^{c,d}(\psi)|} - \sqrt[r]{\prod_{k=1}^{r} |m_k(\psi)|}\right) \xrightarrow{d} J^{A,\psi};$$

$$J^{A,\psi} = \begin{cases} \mathcal{L}(\tilde{Z}_A), & \text{for } \prod_{k=1}^{r} |m_k(\psi)| = 0 \\ \mathcal{N}_1(0, \mathrm{C}_0\Omega(\psi)\mathrm{C}_0^{\mathrm{T}}), & \text{for } \prod_{k=1}^{r} |m_k(\psi)| \neq 0, \end{cases}$$

where

$$C_0 = \frac{1}{r}\left(\sqrt[r]{\prod_{k=1}^{r}|m_k(\psi)|}\right)^{1-2r}\begin{bmatrix} \mathrm{Re}(m_1(\psi))\cdot\prod_{k\neq 1}|m_k(\psi)|^2 \\ \mathrm{Im}(m_1(\psi))\cdot\prod_{k\neq 1}|m_k(\psi)|^2 \\ \mathrm{Re}(m_2(\psi))\cdot\prod_{k\neq 2}|m_k(\psi)|^2 \\ \mathrm{Im}(m_2(\psi))\cdot\prod_{k\neq 2}|m_k(\psi)|^2 \\ \vdots \\ \mathrm{Re}(m_r(\psi))\cdot\prod_{k\neq r}|m_k(\psi)|^2 \\ \mathrm{Im}(m_r(\psi))\cdot\prod_{k\neq r}|m_k(\psi)|^2 \end{bmatrix},$$

$\tilde{Z}_A = \sqrt[r]{\prod_{j=1}^{r}\sqrt{B_{2j-1}^2 + B_{2j}^2}}$, *and the random vector* $[B_1\, B_2\, \ldots\, B_{2r}]^T$ *follows a* $2r$*-dimensional normal distribution with zero mean and covariance matrix* $\Omega(\psi)$*. If we assume additionally that* $g_0(\xi)_{k\,k} > 0$ *for any* $\xi \in [0, 2\pi)$, $k \in \{1, 2, \ldots, r\}$ *and* $det(\Omega(\psi)) > 0$ *for* $\prod_{k=1}^{r}|m_k(\psi)| \neq 0$ *and* $\psi \in (0, \pi)$*, then the law* $J^{A,\psi}$ *is continuous.*

Theorem 3. *Let all the assumptions of Theorem 1 hold. Then, the following convergence holds:*

$$\sqrt{d_n}\left(\sqrt{\sum_{k=1}^{r}|\hat{m}_{k,n}^{c,d}(\psi)|^2} - \sqrt{\sum_{k=1}^{r}|m_k(\psi)|^2}\right) \xrightarrow{d} J^{B,\psi};$$

$$J^{B,\psi} = \begin{cases} \mathcal{L}(\tilde{Z}_B), & \text{for } \sum_{k=1}^{r}|m_k(\psi)| = 0 \\ \mathcal{N}_1(0, \mathrm{A}_0\Omega(\psi)\mathrm{A}_0^{\mathrm{T}}), & \text{for } \sum_{k=1}^{r}|m_k(\psi)| \neq 0, \end{cases}$$

where $A_0 = \frac{1}{\sqrt{\sum_{k=1}^{r}|m_k(\psi)|^2}}\mathbf{m}(\psi)$, $\tilde{Z}_B = \sqrt{\sum_{j=1}^{2r}B_j^2}$, *and the random vector* $[B_1\, B_2\, \ldots\, B_{2r}]^T$ *follows a* $2r$*-dimensional normal distribution with zero mean and covariance matrix* $\Omega(\psi)$*. If we assume additionally that* $g_0(\xi)_{k\,k} > 0$ *for any* $\xi \in [0, 2\pi)$, $k \in \{1, 2, \ldots, r\}$ *and* $det(\Omega(\psi)) > 0$ *for* $\sum_{k=1}^{r}|m_k(\psi)| \neq 0$ *and* $\psi \in (0, \pi)$*, then the law* $J^{B,\psi}$ *is continuous.*

5 Non-parametric model with deterministic cycle

In this section, we propose a model for testing existence of individual or common deterministic cycle on the basis of the multivariate APC time series. We do not base this concept on filtering methods, but rather, according to the properties of the APC class, we formulate a model equation connected with the characteristics of the first unconditional moment of the process. Let $\{\mathbf{P}_t = (P_{1,t}, P_{2,t}, \ldots, P_{r,t}) \in \mathbb{R}^r : t \in \mathbb{Z}\}$ be a multivariate, $\mathbb{R}^r$-valued time series observed T times during the year. We assume that an unconditional expectation of the time series exists, and can be written in the following form:

$$E(P_{k,t}) = f_k(t) + \mu_k(t) \tag{8}$$

where $f_k(\cdot)$ is a polynomial of order d_k and $\mu_k(t)$ is an ap_{A_k} function with the representation as in (1) of the form $\mu_k(t) = \sum_{\psi \in \Psi_{P_k}} m_{P_k}(\psi) e^{i\psi t}$, true for any $t \in A_k$, $k \in \{1, 2, \ldots, r\}$. Just as in the univariate case considered by Lenart and Pipień (2013a) and Lenart and Pipień (2013b), we consider the decomposition of the set $\Psi_{P_k} \subset [0, 2\pi)$ to $\Psi_{P_k} = \Psi_{P_k,1} \cup \Psi_{P_k,2} \cup \Psi_{P_k,3}$, for $k = 1, 2, \ldots, r$, where sets $\Psi_{P_k,i}$ are mutually disjoint (for $i = 1, 2, 3$) and $\Psi_{P_k,1} \subset (0, 2\pi/1.5T)$, $\Psi_{P_k,2} \subseteq \{2j\pi/T : j = 0, 1, 2, \ldots, T-1\}$. Each element $\psi \in (0, 2\pi)$ corresponds to the length of the cycle equal to $2\pi/\psi$, which means that $\Psi_{P_k,1}$ contains frequencies corresponding to the length of the cycle greater then 1.5 year, while the set $\Psi_{P_k,2}$ contains frequencies interpreted as being seasonal. The representation (8) is true for a large variety of models describing the time evolution of macroeconomic time series. In particular, representation (8) holds for any multivariate strict stationary process with a constant mean function.

Our approach based on representation (8) is different from others presented in the literature since our non-parametric model contains tested complementary component called deterministic cycle. To relate assumption (8) with the most popular approach based on stochastic cycle, let us recall a popular approach based on latent component models presented in Harvey and Jaeger (1993) and Harvey et al. (2007). The aforementioned construct assumes that:

$$\begin{cases} \mathbf{P}_t = \boldsymbol{\mu}_t + \boldsymbol{\psi}_{n,t} + \boldsymbol{\epsilon}_t \\ \boldsymbol{\mu}_t = \boldsymbol{\mu}_{t-1} + \boldsymbol{\beta}_{t-1} \\ \boldsymbol{\beta}_t = \boldsymbol{\beta}_{t-1} + \boldsymbol{\zeta}_t \end{cases} \tag{9}$$

where $\boldsymbol{\epsilon}_t \sim NID(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}})$ and $\boldsymbol{\zeta}_t \sim NID(\mathbf{0}, \boldsymbol{\Sigma}_{\boldsymbol{\zeta}})$. The stochastic trend component $\boldsymbol{\mu}_t$ is a multivariate integrated random walk. The vector cycle $\boldsymbol{\psi}_{n,t}$ is a generalization of cycle $\boldsymbol{\psi}_{1,t}$ presented in Harvey and Jaeger (1993). This generalisation called nth-order cycle produce smoother extracted cycle than $\boldsymbol{\psi}_{1,t}$. The basic characteristic of cyclical component $\boldsymbol{\psi}_{n,t}$ is a concentration of the spectral density function around frequency λ_c which is unknown parameter in this model being subject to estimation. The higher n (and fixed rest parameters) the more the values of the spectral density is concentrated around frequency λ_c (see illustrative example in Trimbur (2006)). Given

this approach the cyclical component has a stochastic counterpart. If we assume in model (9) that we have seed value of $\boldsymbol{\mu}_0$ then elementary calculations gives that (8) holds with $f_k(t)$ as polynomial of order two and $\mu_k(t) \equiv 0$. In our approach, presented in this paper, we test the existence of such non-zero common frequencies ψ that $\mu_k(t) \not\equiv 0$ exhibit almost periodicity with corresponding common length of the cycle $2\pi/\psi$. From this point of view we consider testing problem of common frequencies in mean function $\mu_k(t)$, for which it is assumed in the literature that equals constant value.
In Section 6, the asymptotically consistent test based on subsampling approach is constructed for testing common frequencies in the sets $\Psi_{P_k,1}$, $k = 1, 2, \ldots, r$.

6 Testing the existence of the common deterministic cycles

In this section we formulate two testing problems concerning the existence of common cycles in the multivariate $\mathrm{APC}(1)_A$ time series. Let us take any frequency $\psi \in \left(0, \dfrac{2\pi}{1.5T}\right)$, and let us consider the following hypothesis testing problem concerning the existence of the common frequency in the sets Ψ_{P_j}:

$$\overbrace{H_0^A : \psi \notin \bigcap_{j=1}^{r} \Psi_{P_j}.}^{\text{deterministic cycle with length } 2\pi/\psi \text{ is not common for each process } P_j}$$

$$\overbrace{H_1^A : \psi \in \bigcap_{j=1}^{r} \Psi_{P_j} - \text{common frequency } \psi}^{\text{we observe common deterministic cycle with length } 2\pi/\psi \text{ in each process } P_j}.$$

Note that $H_0^A \Leftrightarrow \prod_{k=1}^{r} |m_k(\psi)| = 0$ and $H_1^A \Leftrightarrow \prod_{k=1}^{r} |m_k(\psi)| \neq 0$.
Given H_1^A, a cycle with a particular length determined by the frequency $\psi \in (0, 2\pi/(1.5T))$ is common for each of the coordinates of the considered multivariate processes.
In the second testing problem, we state the hypothesis that a cycle with the considered length is not present in at least one coordinate of the considered process:

$$\overbrace{H_0^B : \psi \notin \bigcup_{j=1}^{r} \Psi_{P_j},}^{\text{deterministic cycle with considered length is not observed in any process } P_j}$$

against the negation:

$$\overbrace{H_1^B : \psi \in \bigcup_{j=1}^{r} \Psi_{P_j}}^{\text{we observe a deterministic cycle with considered length at least one process } P_j}.$$

Note that $H_0^B \Leftrightarrow \sum_{k=1}^{r} |m_k(\psi)| = 0$ and $H_1^B \Leftrightarrow \sum_{k=1}^{r} |m_k(\psi)| \neq 0$.
The testing problems stated above are difficult to solve because for the process $\{\mathbf{P}_t = (P_{1,t}, P_{2,t}, \ldots, P_{r,t}) \in \mathbb{R}^r : t \in \mathbb{Z}\}$ we impose very general assumptions about the unconditional mean, as in (8). Therefore, we applied the same linear transformations as shown in Lenart and Pipień (2013a), i.e., centered moving average 2×12MA and d_k-times differencing. Under these transformations we consider the process $\{\tilde{\mathbf{P}}_t = (\tilde{P}_{1,t}, \tilde{P}_{2,t}, \ldots, \tilde{P}_{r,t}) \in \mathbb{R}^r : t \in \mathbb{Z}\}$, where $\tilde{P}_{k,t} = L_{d_k}(B)$ $L_{2\times12}(B)P_{k,t}$, $L_{2\times12}(B) = (B^{-6} + 2B^{-5} + \ldots + 2B^{-1} + 2 + 2B + \ldots + 2B^5 + B^6)/24$, $L_{d_k}(B) = (1-B)^{d_k}$ and $B^i X_{k,t} = X_{k,t-i}$, $k = 1, 2, \ldots, r$, $i \in \mathbb{N}$.
The most important result making inferences about frequencies describing the unconditional mean of the observed process is the invariance of the sets $\Psi_{P_{k,1}}$ in the sense that

$$E(\tilde{P}_{k,t}) = c_k + \sum_{\psi \in \Psi_{P_k,1} \cup \Psi_{P_k,3}} m_{\tilde{P}_k}(\psi) e^{i\psi t},$$

where $m_{\tilde{P}_k}(\psi) = L_1(e^{-i\psi}) L_{2\times12}(e^{-i\psi}) m_{P_k}(\psi)$ and c_k is some constant (see Lenart and Pipień (2013a)).
To construct appropriate test statistics and prove the consistency of proposed tests, we need the following assumption concerning the process $\{\tilde{\mathbf{P}}_t \in \mathbb{R}^r : t \in \mathbb{Z}\}$.

Assumption 4. *For the process $\{\tilde{\boldsymbol{P}}_t = (\tilde{P}_{1,t}, \tilde{P}_{2,t}, \ldots, \tilde{P}_{r,t}) \in \mathbb{R}^r : t \in \mathbb{Z}\}$ assumptions (1)-(4) hold.*

In testing problem A, we propose the natural test statistics of the form:

$$T_n^A(\psi) = \sqrt{n} \sqrt[r]{\prod_{k=1}^{r} |\hat{r}_{k,n}(\psi)|},$$

and for testing problem B, we propose the following test statistics:

$$T_n^B(\psi) = \sqrt{n} \sqrt{\sum_{k=1}^{r} |\hat{r}_{k,n}(\psi)|^2},$$

where $\hat{r}_{k,n}(\psi) = \frac{1}{n} \sum_{j=1}^{n} (\tilde{P}_{k,j} - \overline{\tilde{P}}_{k,n}) \mathbf{1}\{j \in A_k\} e^{-ij\psi}$ and $\overline{\tilde{P}}_{k,n}$ is a sample mean based on these values from the set $\{\tilde{P}_{k,1}, \tilde{P}_{k,2}, \ldots, \tilde{P}_{k,n}\}$ for which $t \in A_k$. We subtract the

mean because we are not interested in frequency equal to zero.
Let $c_{n,b}^{\psi\,A}(1-\alpha)=\inf\{x:\tilde{L}_{n,b}^{\psi\,A}(x)\geq 1-\alpha\}$ be a critical value obtained on the basis of the subsampling distribution

$$\tilde{L}_{n,b}^{\psi\,A}(x)=\frac{1}{n-b+1}\sum_{t=1}^{n-b+1}\mathbf{1}\left\{\sqrt{b}\left(\sqrt[r]{\prod_{k=1}^{r}|\hat{r}_{k,n}^{t-1,b}(\psi)|}-\sqrt[r]{\prod_{k=1}^{r}|\hat{r}_k(\psi)|}\right)\leq x\right\}, \quad (10)$$

where $\hat{r}_{k,n}^{t-1,b}(\psi)$ is an estimator $\hat{m}_{k,n}^{t-1,b}(\psi)$ based on a subsample starting at t and with length b from sample $\{\tilde{P}_{k,1}-\overline{\tilde{P}}_{k,n},\tilde{P}_{k,2}-\overline{\tilde{P}}_{k,n},\ldots,\tilde{P}_{k,n}-\overline{\tilde{P}}_{k,n}\}$. Then, the following theorem concerning consistency of the confidence intervals holds.

Theorem 4. *Take any frequency $\psi\in(0,2\pi)$. Given the regularity conditions from Theorem 1, the subsampling confidence intervals for $\sqrt[r]{\prod_{k=1}^{r}|m_k(\psi)|}$ are asymptotically consistent, which means that: $P\left(\sqrt{n}\left(\sqrt[r]{\prod_{k=1}^{r}|\hat{r}_{k,n}(\psi)|}-\sqrt[r]{\prod_{k=1}^{r}|m_k(\psi)|}\right)\leq c_{n,b}^{\psi\,A}(1-\alpha)\right)\longrightarrow 1-\alpha$, where $b=b(n)\to\infty$, and $b/n\to 0$.*

Analogously, if $c_{n,b}^{\psi\,B}(1-\alpha)=\inf\{x:\tilde{L}_{n,b}^{\psi\,B}(x)\geq 1-\alpha\}$ where

$$\tilde{L}_{n,b}^{\psi\,B}(x)=\frac{1}{n-b+1}\sum_{t=1}^{n-b+1}\mathbf{1}\left\{\sqrt{b}\left(\sqrt{\sum_{k=1}^{r}|\hat{r}_{k,n}^{t-1,b}(\psi)|^2}-\sqrt{\sum_{k=1}^{r}|\hat{r}_k(\psi)|^2}\right)\leq x\right\}, \quad (11)$$

then the following theorem is true.

Theorem 5. *Take any frequency $\psi\in(0,2\pi)$. Given the regularity conditions from Theorem 1, the subsampling confidence intervals for $\sqrt{\sum_{k=1}^{r}|m_k(\psi)|^2}$ are asymptotically consistent. This means that: $P\left(\sqrt{n}\left(\sqrt{\sum_{k=1}^{r}|\hat{r}_{k,n}(\psi)|^2}-\sqrt{\sum_{k=1}^{r}|m_k(\psi)|^2}\right)\leq c_{n,b}^{\psi\,B}(1-\alpha)\right)\longrightarrow 1-\alpha$, where $b=b(n)\to\infty$, and $b/n\to 0$.*

The testing procedures based on critical values $c_{n,b}^{\psi\,A}(1-\alpha)$ and $c_{n,b}^{\psi\,B}(1-\alpha)$ are asymptotically consistent. If we replace the critical value by its subsampling approximations $g_{n,b}^{\psi\,A}(1-\alpha)$ and $g_{n,b}^{\psi\,B}(1-\alpha)$, calculated on the basis of subsampling distributions:

$$\tilde{G}_{n,b}^{\psi\, A}(x) = \frac{1}{n-b+1} \sum_{t=1}^{n-b+1} \mathbf{1} \left\{ \sqrt{b} \left(\sqrt[r]{\prod_{k=1}^{r} |\hat{r}_{k,n}^{t-1,b}(\psi)|} \right) \leq x \right\}, \tag{12}$$

and:

$$\tilde{G}_{n,b}^{\psi\, B}(x) = \frac{1}{n-b+1} \sum_{t=1}^{n-b+1} \mathbf{1} \left\{ \sqrt{b} \left(\sqrt{\sum_{k=1}^{r} |\hat{r}_{k,n}^{t-1,b}(\psi)|^2} \right) \leq x \right\}, \tag{13}$$

respectively, then the asymptotic consistency still holds. This result was presented in the case of the univariate APC time series ($r = 1$) by Lenart and Pipień (2013b) according to the general setting of consistency of the subsampling procedure presented in Politis et al. (1999). The approximated critical values $g_{n,b}^{\psi\, A}(1-\alpha)$ and $g_{n,b}^{\psi\, B}(1-\alpha)$) can be utilised in the empirical analysis because the probability of H_0 rejection (under H_0 in finite sample size) is lower in this case. This follows from (10) and (12) and from (11) and (13) for tests A and B, respectively.

7 Analysis of the existence of common deterministic cycles for selected European countries

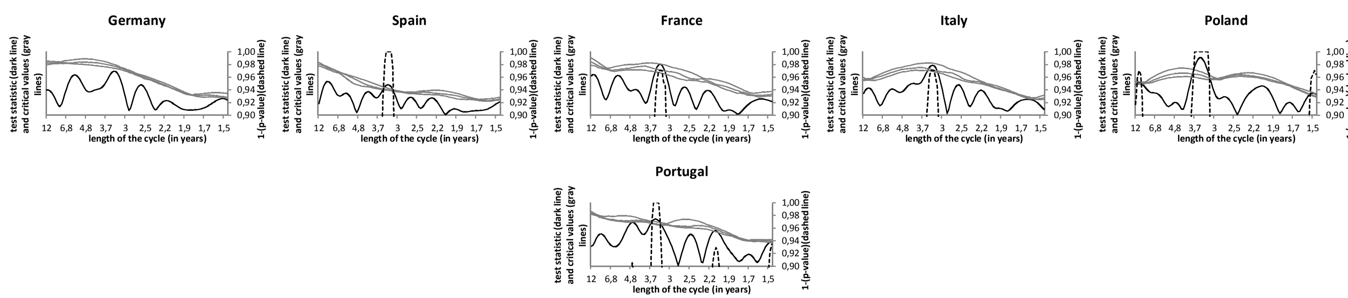
We consider the monthly data of the industrial production index in Germany, Spain, France, Italy, Poland and Portugal. In empirical analyses of the business cycle the choice of industrial production series of monthly frequency seems standard. However literature overview presented in the introductory part of the paper indicates that analyses of the quarterly data (for example GDP and other growth indicators) are at least as popular as those conducted on the basis of monthly series. In order to keep appropriate significance of statistical inference as well as to maintain the power of testing procedures at satisfactory level, we used relatively long dataset, covering the period from January 1995 to June 2011. The total number of observations in the initial dataset is the same for all countries.

The proposed subsampling procedure has some limitations and requires appropriate length of observed series. This precludes useful applications for quarterly data in case of selected EU countries mentioned above. Thus the statistical properties of developed testing scheme is the most important factor leading us to analyses utilizing monthly frequency. For developed economies, like USA, Great Britain, Japan and others, appropriately long quarterly series of growth indicators are available and the methodology presented in this paper may be applied as well.

To define sets A_k, $k = 1, 2, ..., r$ in Assumption 1 we imposed the following criteria:

$$t \in A_k \Leftrightarrow |\tilde{P}_{k,t}| < 0.01. \tag{14}$$

Figure 3: Test statistics $T_n^A(\psi)$ with critical values for nominal levels $\alpha = 0.02$, 0.05 and 0.08. The case of the r-dimensional APC time series with $r = 1$



This criteria is arbitrary and is related only to the period of the last crisis 2007-2008, where the existence of the deterministic cycle is rejected by the data in the most analyzed cases below. But the perturbations during last crisis period are not a reason to reject the existence of the deterministic cycle during whole analyzed period. The specific dynamics of business fluctuations during last crisis period confirm that stochastic cycle assumption should be predominant in this period. Hence, to cope with problematic for deterministic cycle crisis period we define the sets A_k, $k = 1, 2, \ldots, r$ via assumption (14). This assumption excludes from the analysis the observations included in last crisis period. The number of excluded observations is no longer than one year. Assumption (14) is enough to obtain significance of the deterministic cycle. Consequently, the set of observations excluded from the dataset may be different across countries. Based on (14), we excluded observations contained in the time interval from July 2008 to April 2009 in the case of Germany (10 observations), May 2008 to April 2009 in the case of Spain (12 observations), May 2008 to April 2009 in the case of France (12 observations), July 2008 to April 2009 in the case of Italy (11 observations), August 2008 in the case of Poland (single observation) and July 2008 to August 2008 in the case of Portugal (2 observations). Figures 4, 5, 6, 7 and 8 present the results of subsampling inference about the existence of common deterministic cycle (common frequency) in the cyclical components of the observed series. Formally, the r-dimensional APC time series for $r = 6$ is required in our case, but we split the set of analysed countries into subsets. Additionally we consider two ($r = 2$, Figure 4), three ($r = 3$, Figure 5), four ($r = 4$, Figure 6), five ($r = 5$, Figure 7) and six ($r = 6$, Figure 8) countries, separately. We plotted the values of the test statistics $T_n^A(\psi)$, together with subsampling approximations of the critical values for nominal levels $\alpha = 0.02$, 0.05 and 0.08. We take $b = b(n) = [2.5\sqrt{n}]$ arbitrally. In addition, we present in Figure 3 the results of subsampling inferences about the business cycle characteristics in the analysed countries, conducted on the Fourier coefficinets in the univariate case that was considered in Lenart et al. (2008) and Lenart and Pipień (2013a).

The results clearly indicate the existence of the common deterministic cycle in a predominant analysed case. The case when test statistics exceed the critical values for the frequency ψ corresponding to the aforementioned length of the cycle seems invariant with respect to r. In case of bivariate processes when considering Poland and Portugal, the data support exceptionally a common dterministic cycle of length less than 18 months. In addition, given $r = 2$, the data do not strongly support a common frequency in the pair of Germany and France and the pair of Germany and Portugal. All remaining pairs of the analysed series support the existence of a cycle of length approximately three and a half years much more strongly.

In the case with $r = 3$ (Figure 5), the trio Germany, Italy and Portugal and the trio Germany, France and Italy do not strongly support the existence of a common deterministic cycle. All remaining cases seem to have a common length of deterministic cycle of approximately three and a half years. The results of testing

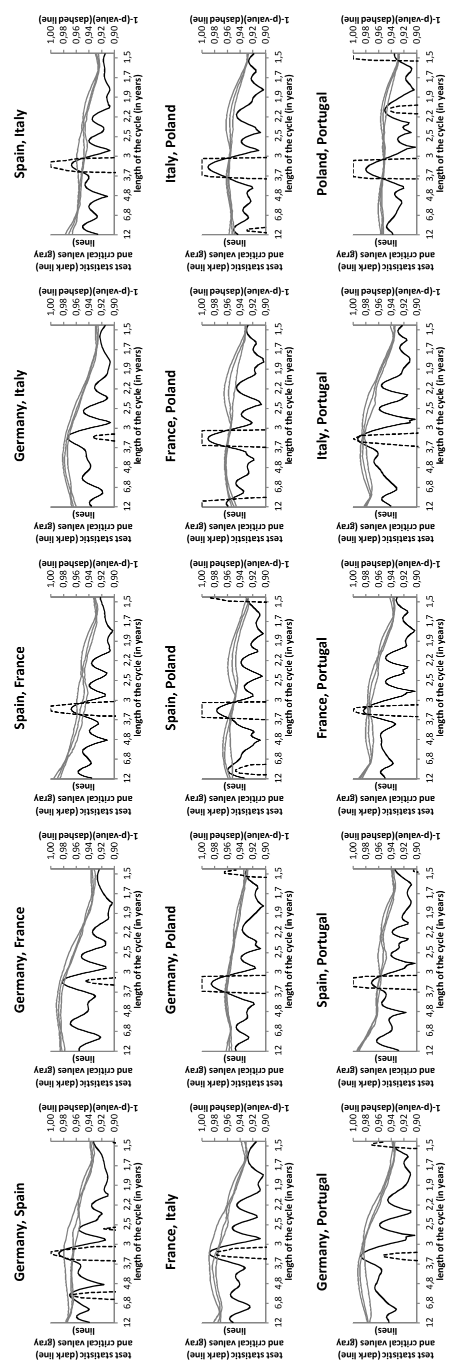
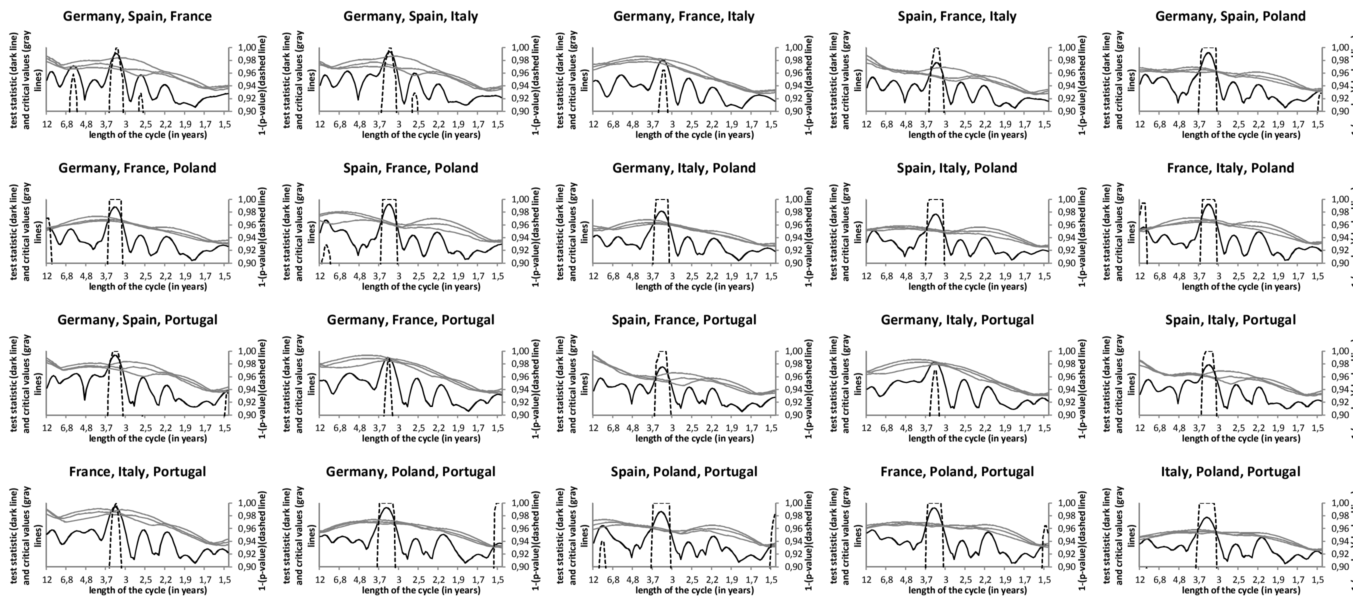


Figure 4: Test statistics $T_n^A(\psi)$ with critical values for nominal levels $\alpha = 0.02$, 0.05 and 0.08. The case of the r-dimensional APC time series with $r = 2$

Figure 5: Test statistics $T_n^A(\psi)$ with critical values for nominal levels $\alpha = 0.02$, 0.05 and 0.08. The case of the r-dimensional APC time series with $r = 3$



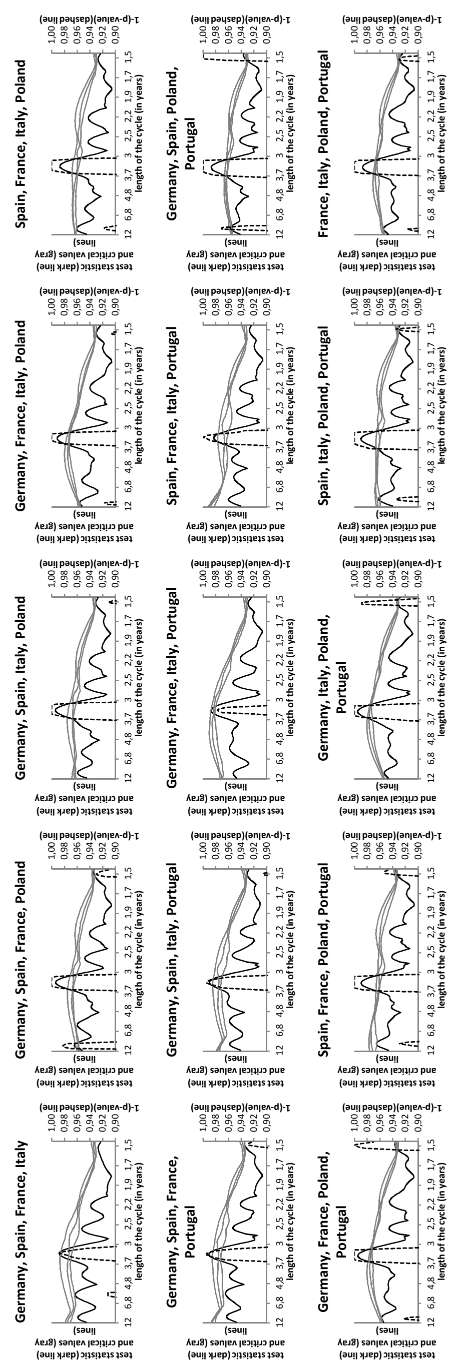


Figure 6: Test statistics $T_n^A(\psi)$ with critical values for nominal levels $\alpha = 0.02$, 0.05 and 0.08. The case of the r-dimensional APC time series with $r = 4$

Figure 7: Test statistics $T_n^A(\psi)$ with critical values for nominal levels $\alpha = 0.02$, 0.05 and 0.08. The case of the r-dimensional APC time series with $r = 5$

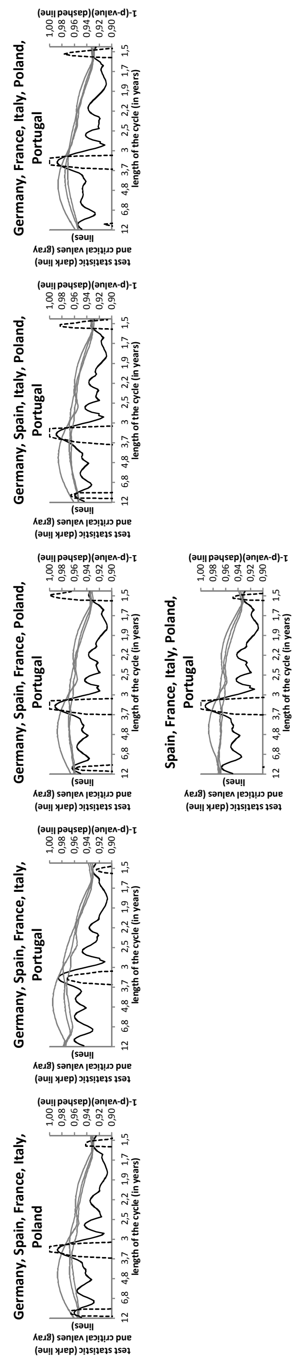
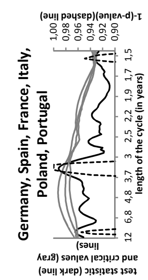


Figure 8: Test statistics $T_n^A(\psi)$ with critical values for nominal levels $\alpha = 0.02$, 0.05 and 0.08. The case of the r-dimensional APC time series with $r = 6$



the existence of the common cycle in the case of four-dimensional APC time series are presented in Figure 6. In all sets, we observed that the data support cycle with a length of three and a half years. The only exception is the case of the series of industrial production for Germany, France, Italy and Portugal modeled jointly. In the cases of $r = 5$ (Figure 7) and finally $r = 6$ (Figure 8), the inference changes slightly, extending the set of statistically significant frequencies. The existence of the rather short common deterministic cycle is supported, in addition to the predominant three and a half year cycle. Except in the case with Germany, Spain, France, Italy and Portugal modeled jointly, the series support strongly the length of this short cycle equally approximately 18 months in length.

Figures 9, 10, 11, 12 and 13 present the results of formal inferences about the existence of statistically significant frequencies in at least one cyclical component of the observed multivariate series. Just as with the test statistics $T_n^A(\psi)$, we split the set of analysed countries into subsets and consider separately groups of two ($r = 2$, Figure 9), three ($r = 3$, Figure 10), four ($r = 4$, Figure 11), five ($r = 5$, Figure 12) and six countries ($r = 6$, Figure 13). We plotted the values of the test statistics $T_n^B(\psi)$, together with subsampling approximations of the critical values for nominal levels $\alpha = 0.02$, 0.05 and 0.08. We take $b = b(n) = [2.5\sqrt{n}]$. We additionally calculated the test statistics $T_n^B(\psi)$ in the case with $r = 1$, which is equivalent to the case with $r = 1$ for the test statistics $T_n^A(\psi)$, discussed previously.

The analysed series indicate the existence of statistically significant frequencies of the same length as in the case of testing problem A. The result is invariant with respect to the dimension of the process, $r = 2$, 3 and 4. In the case of $r = 4$ the data do not strongly support the existence of a statistically significant frequency, when Germany, France, Italy and Portugal are modeled jointly.

Analysing the test statistics $T_n^A(\psi)$ for the APC time series with $r = 2$ (Figure 4), the existence of the common deterministic cycle was not supported for the Germany-France, Germany-Italy and Germany-Portugal cases. Some detailed insight into the nature of the business cycle comovements is presented in Figure 9 for the bivariate APC model. According to the values of $T_n^B(\psi)$, the modeled series of industrial production of the aforementioned pairs of countries do not support the existence of any significant frequency. This effect seems invariant with respect to changes in $r \in \{2, 3, 4, 5, 6\}$. On the one hand, if the value of the test statistics $T_n^A(\psi)$ indicated the existence of the common deterministic cycle of a particular length, the test statistics $T_n^B(\psi)$ obviously supported the existence of an appropriate frequency in at least one univariate series modeled jointly. On the other hand, in case of rejection of the hypothesis that there exists a frequency common to all series and statistically significant, the test statistics $T_n^B(\psi)$ explains the reason for that, clearly indicating the nonexistence of a significant frequency in the univariate spectrum.

To compare our results, with the framework imposed in the existing literature, suppose the modeled series are observed scraps of paths of the stationary stochastic

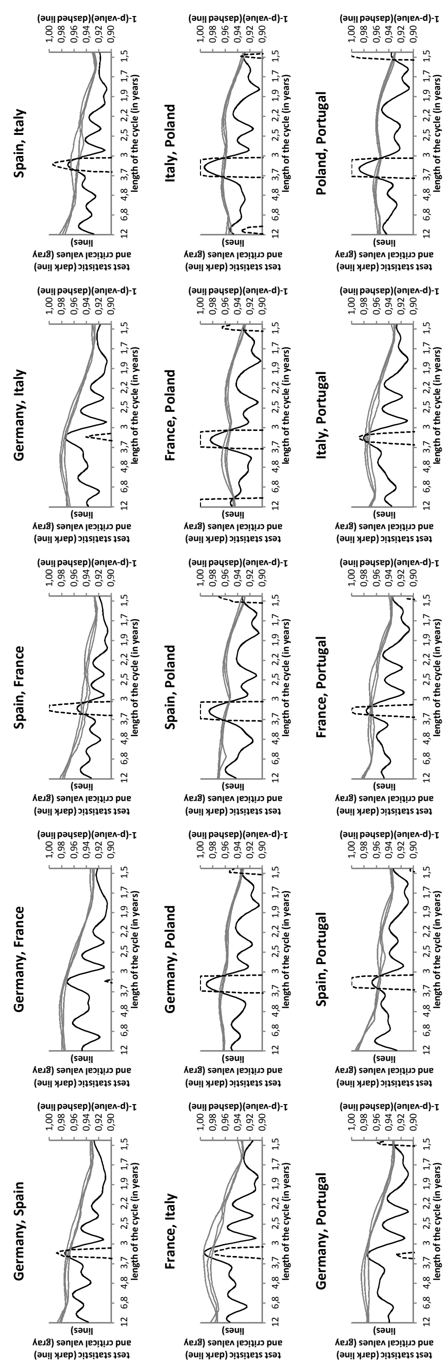


Figure 9: Test statistics $T_n^B(\psi)$ with critical values for nominal levels $\alpha = 0.02$, 0.05 and 0.08. The case of the r-dimensional APC time series with $r = 2$

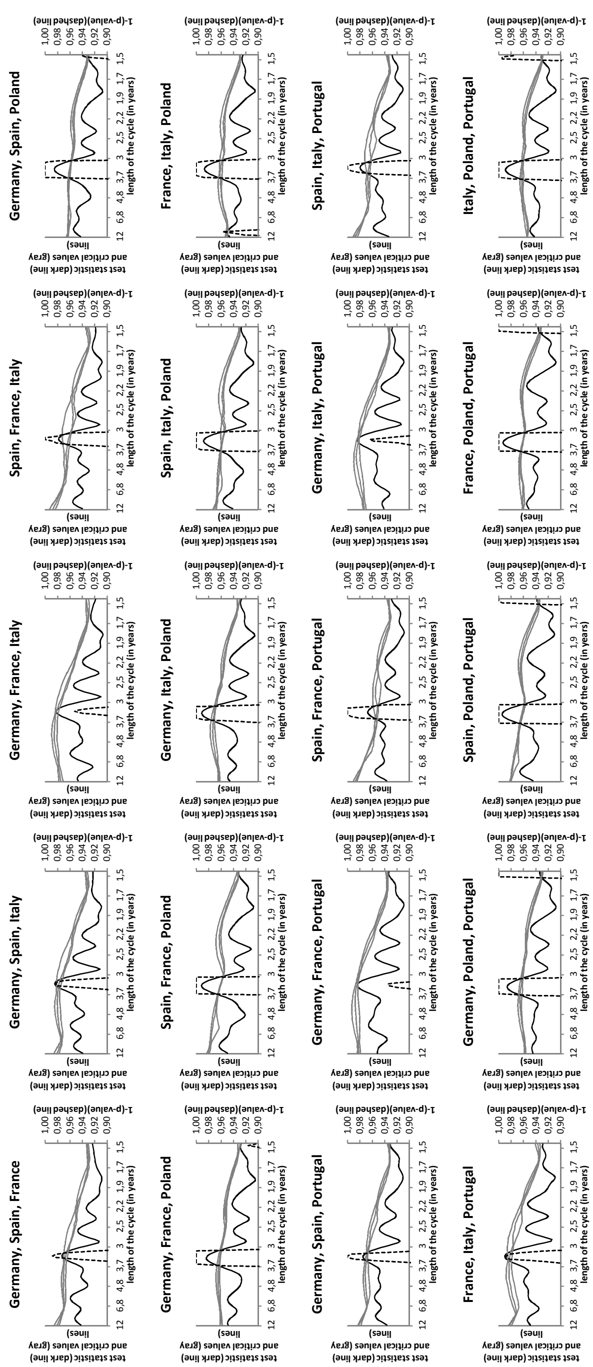


Figure 10: Test statistics $T_n^B(\psi)$ with critical values for nominal levels $\alpha = 0.02$, 0.05 and 0.08. The case of the r-dimensional APC time series with $r = 3$

Figure 11: Test statistics $T_n^B(\psi)$ with critical values for nominal levels $\alpha = 0.02$, 0.05 and 0.08. The case of the r-dimensional APC time series with $r = 4$

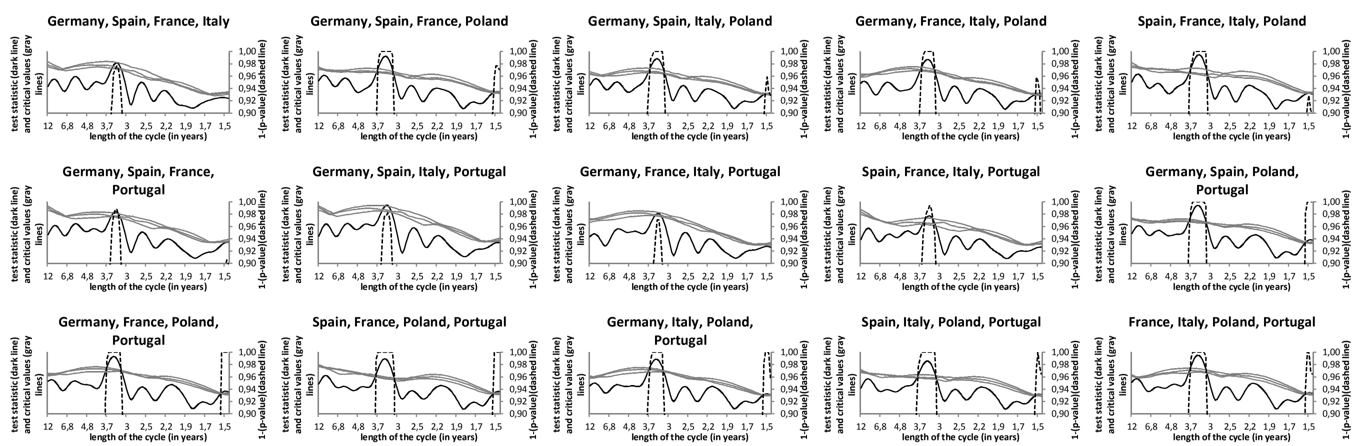


Figure 12: Test statistics $T_n^B(\psi)$ with critical values for nominal levels $\alpha = 0.02$, 0.05 and 0.08. The case of the r-dimensional APC time series with $r = 5$

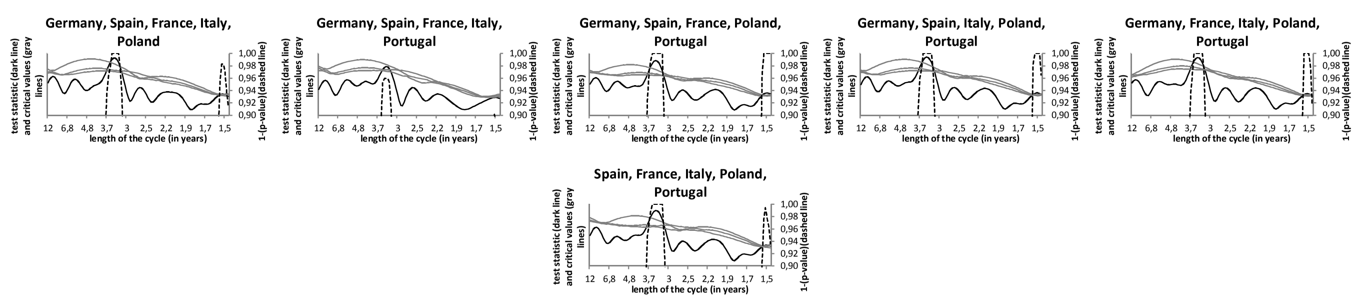


Figure 13: Test statistics $T_n^B(\psi)$ with critical values for nominal levels $\alpha = 0.02$, 0.05 and 0.08. The case of the r-dimensional APC time series with $r = 6$

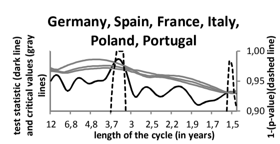


Figure 14: Cross-spectrum estimates and coherence for bivariate time series ($r = 2$) calculated on the basis of the whole dataset, i.e. $A_i = \mathbb{Z}$, $i = 1, 2, \ldots, r$

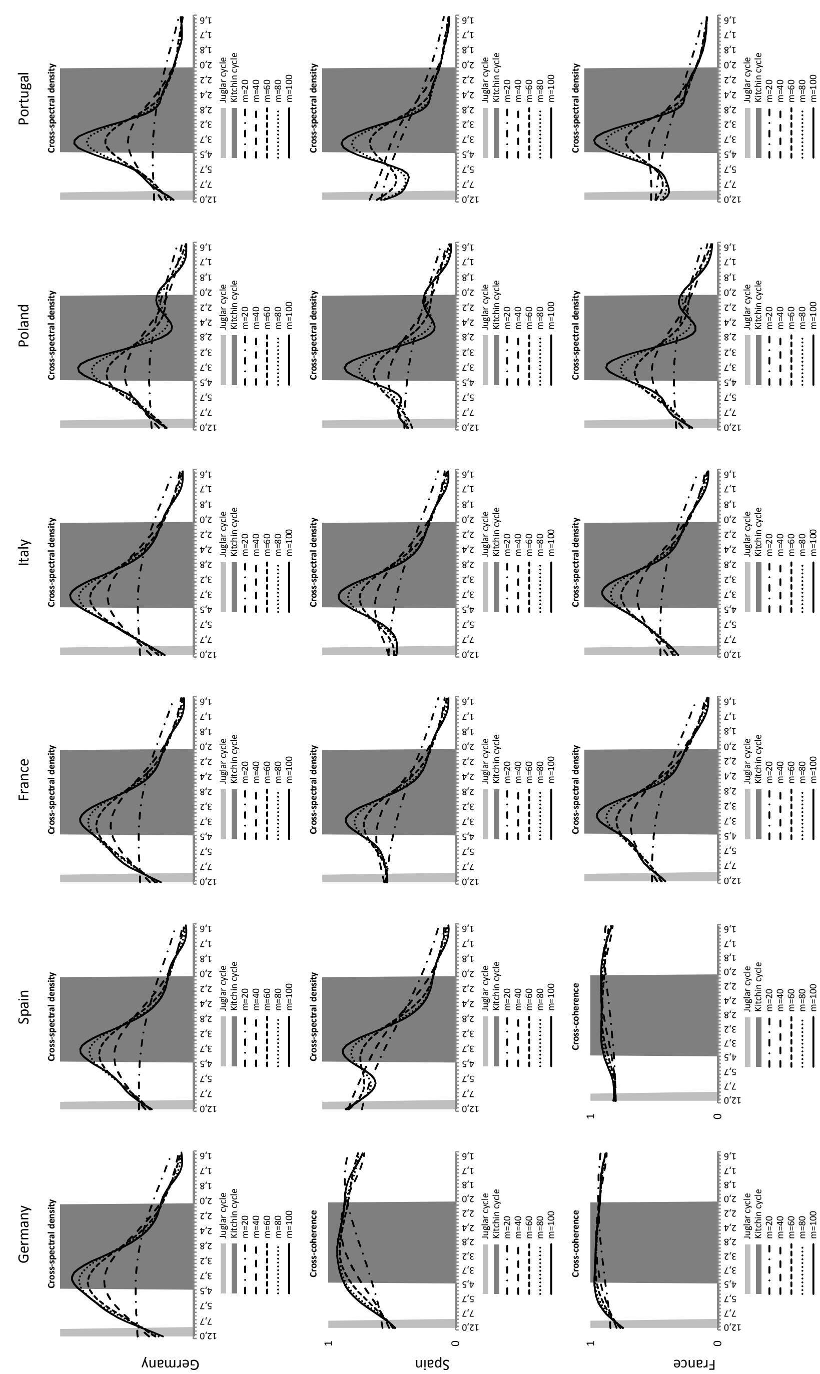
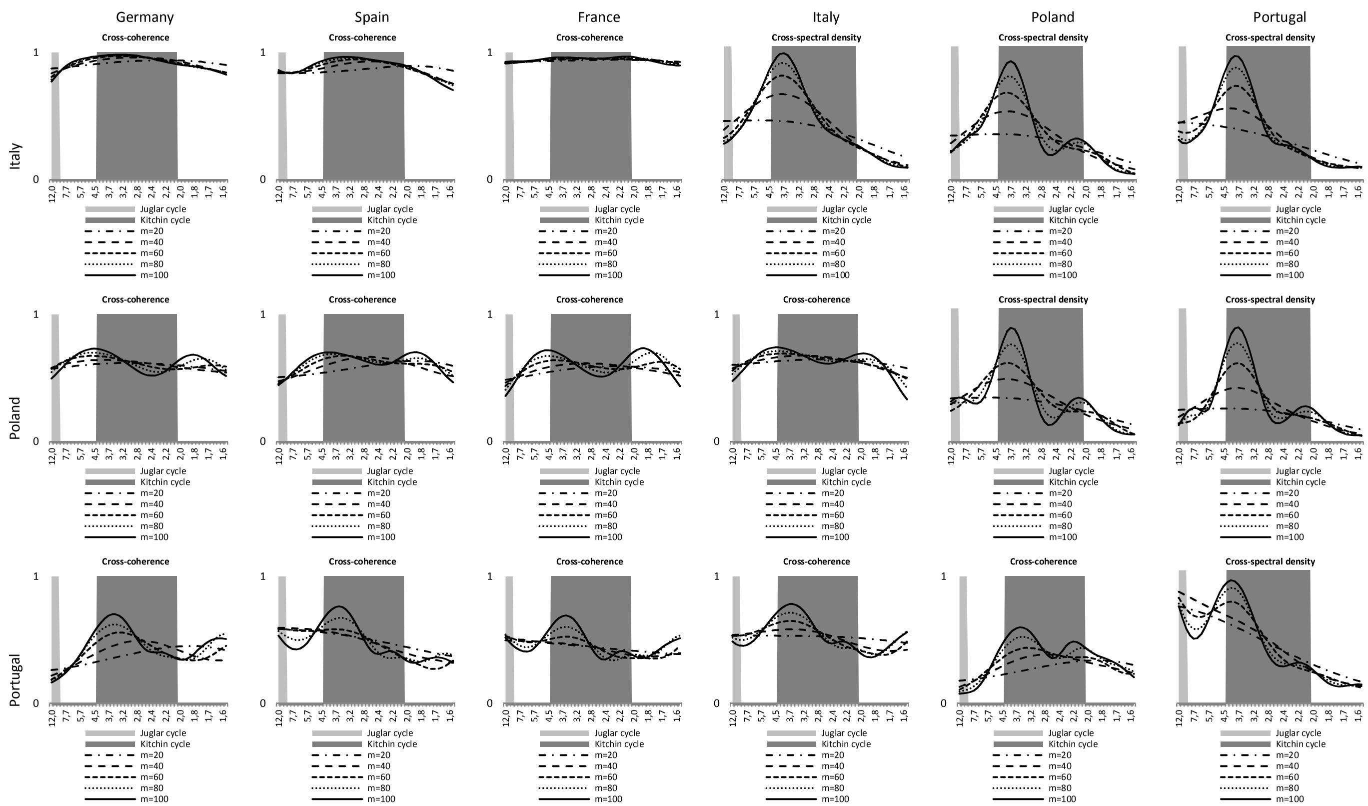


Figure 15: Cross-spectrum estimates and coherence for bivariate time series ($r = 2$) calculated on the basis of the whole dataset, i.e. $A_i = \mathbb{Z}$, $i = 1, 2, \ldots, r$



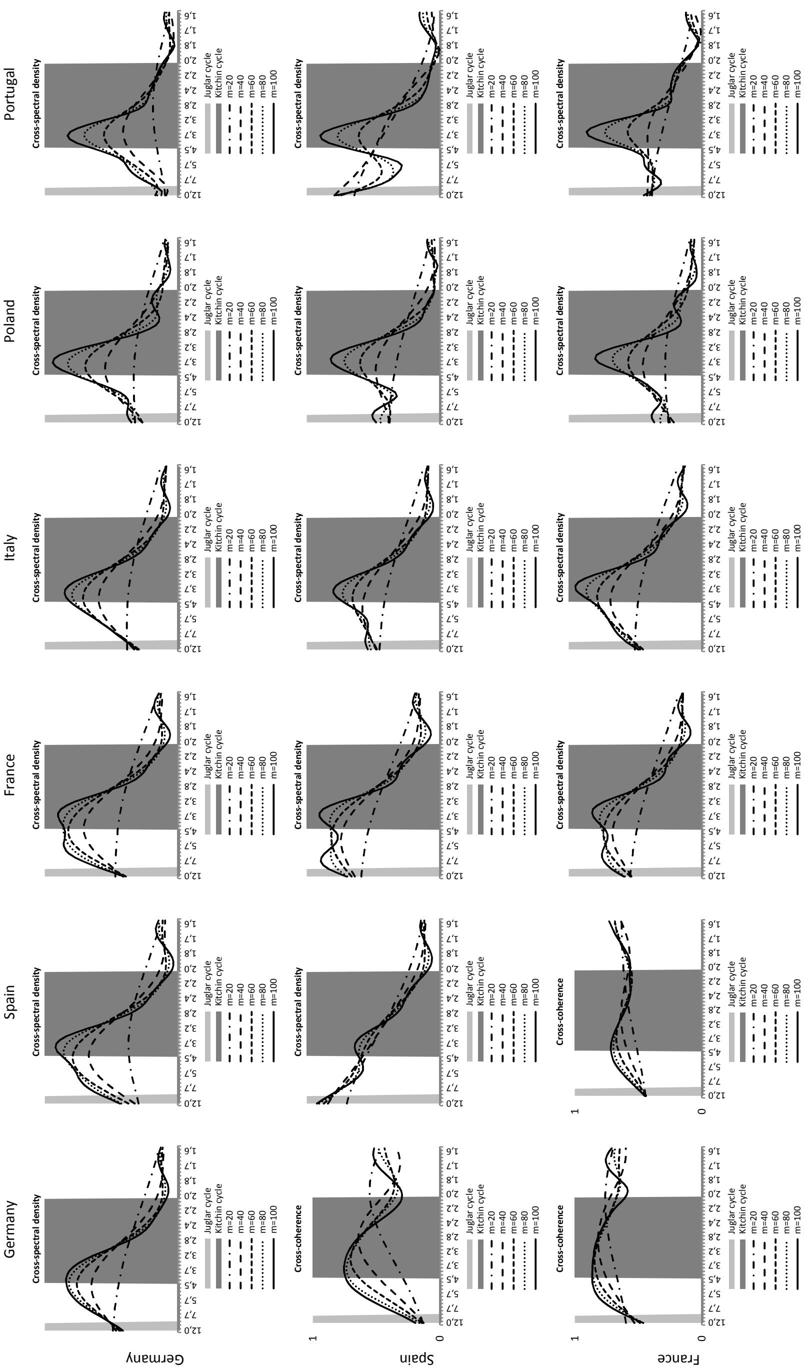
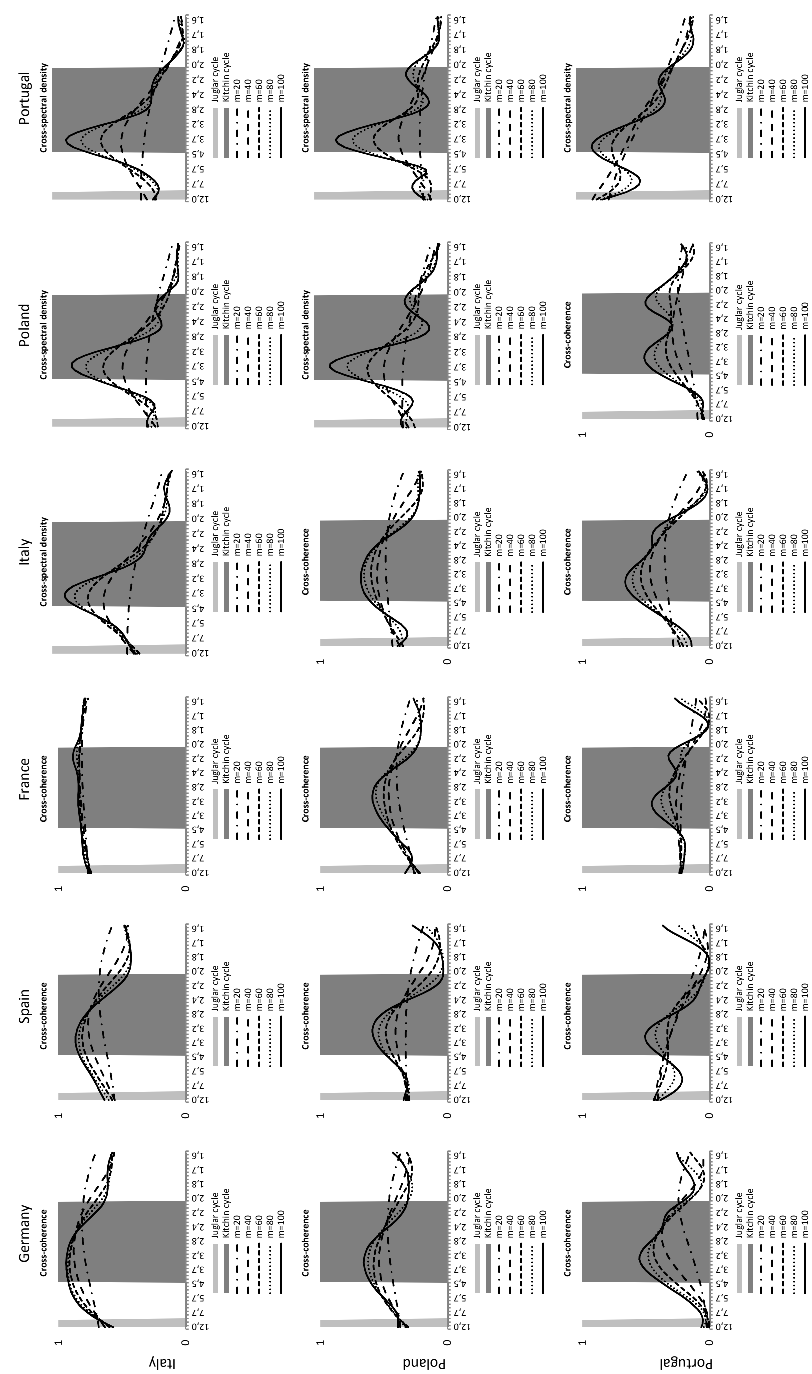


Figure 16: Cross-spectrum estimates and coherence for bivariate time series ($r = 2$) calculated on the basis of the set A with A_i chosen according to restriction (14)

Figure 17: Cross-spectrum estimates and coherence for bivariate time series ($r = 2$) calculated on the basis of the set A with A_i chosen according to restriction (14)



process with a zero mean. We calculated the estimates of the dynamic correlation on a frequency band coefficients for pairs of the series representing all countries. Initially, we impose the zero mean and stationary assumption for the process $\tilde{P}_{k,t}$ ($k = 1, 2, \ldots, r$) on the entire integer line. Alternatively, just as with the definition of the APC time series on the set $A = A_1 \times \ldots A_r$, we restrict the stationarity assumption only to the sets A_k defined in (14). The formal definition of the construct can be found in the Appendix; see Definition 8.1. In Table 1, we report the point estimates of the dynamic correlation coefficients calculated on the basis of the set of frequencies that corresponds to the length of the cycle from 1.5 years to 12 years.

Table 1: The estimates of the dynamic correlation coefficients calculated on the set of frequencies that corresponds to a length of the cycle from 1.5 to 12 years. Entries above the diagonal represent coefficients in the sets A_i and A_j. Entries below the diagonal represent coefficients obtained under the assumptions $A_i = \mathbb{Z}$ and $A_j = \mathbb{Z}$ for $i, j \in \{1, 2, \ldots, r\}$

	Germany	Spain	France	Italy	Poland	Portugal
Germany		0.616	0.876	0.886	0.654	0.467
Spain	*0.869*		0.737	0.798	0.569	0.441
France	*0.966*	*0.917*		0.901	0.586	0.438
Italy	*0.965*	*0.928*	*0.970*		0.689	0.598
Poland	*0.742*	*0.790*	*0.744*	*0.791*		0,459
Portugal	*0.632*	*0.693*	*0.650*	*0.724*	*0.555*	

Entries above the diagonal represent coefficients on the sets A_k and A_l. Entries below the diagonal represent coefficients obtained on the whole series. The dynamic correlation coefficient estimates are clearly higher if calculated on the basis of the whole series. If we restrict the stationarity to set A, estimates of the dynamic correlations inform about the much weaker effect of the business cycles comovement. The abrupt and pervasive contraction of the output in analysed economies may lead to spurious conclusions about the strength of the business cycle comovements. Figures 14-15 and 16-17 present cross-spectral densities and cross-coherence functions, calculated for the bivariate time series ($r = 2$). Above the diagonal, we report plots of cross-spectral densities, whereas cross-coherence functions are placed below the diagonal. On the diagonal, we present spectral densities for the univariate series of industrial production in the cases of Germany, Spain, France, Italy, Poland and Portugal, respectively. Figures 14-15 presents the results obtained on the basis of the whole series, i.e., $A_i = \mathbb{Z}$ and in Figures 16-17, we put the results obtained in the case of the set A with A_i chosen according to restriction (14). The shape of the cross-spectral densities is invariant with respect to the choice of set A. The corresponding plots are similar in Figures 14-15 and 16-17, with the exceptions of the Germany-France, Germany-Spain and Spain-France pairs. Only in those three cases do the qualitative inferences about the common dominant frequency change when the set of

observations from the crisis is excluded from the dataset. We observe a more irregular shape of the spectral densities given a set A chosen according to the restriction (14). In particular, in the case of Spain-France pair, the cross-spectral density is bimodal, indicating the existence of common dominant frequencies of approximately 4 and 8 years in length. Different conclusions can be drawn when the whole dataset is subject to analysis. The cross-spectral density is more regular and unimodal with a dominant frequency of approximately 4 years in length. In the case of all remaining pairs of countries, the cross spectral densities are almost the same, indicating the existence of a single dominant common frequency of approximately 4 years in length.
In contrast, the cross coherence functions, calculated on the basis of the whole time series, are much different in general than in the case of the set A chosen according to restriction (14). In the case of all analyzed pairs of countries, the removal of observations from the crisis changes the shape of cross-coherence, making inference about relationships between the dynamics of analyzed series very difficult.

8 Concluding remarks

The aim of this article was to construct an asymptotically consistent test to identify the existence of individual or common deterministic cycle in multivariate stochastic processes describing changes in economic activity. Following Lenart and Pipień (2013a), a multivariate nonparametric model containing the business cycle component in the unconditional mean was considered. In our concept, we use the Fourier representation of the unconditional expectation of the Almost Periodically Correlated (APC) stochastic process. We discussed the asymptotic normality of the normalized Fourier transform in the case of the multivariate APC time series. Consequently, we built the test statistics for the problem of testing the existence of the common frequency (identify with deterministic cycle) in the coordinates of the multivariate APC time series and proved the subsampling consistency.
Our main findings from the empirical part is that deterministic cycle can be supported by the data strongly. In particular, in the cases of Germany, Spain, France, Italy, Poland and Portugal, the changes in economic activity seem to exhibit comovement described by common deterministic cycle. The testing procedure indicated the existence of only a single frequency common to all countries. This frequency corresponds to a length of the common business cycle of approximately three and a half years. This may have serious consequence, especially while forecast of the cyclical pattern is derived.
Hence, we recommend to pay more attention to joint modeling of the business fluctuations by deterministic and stochastic cycle assumption jointly. This is the future interest of the research for the authors.

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Appendix

Definition 4. *We say that the second order $\mathbb{R}^r$-valued time series* $\{X_t = (X_{1,t}, X_{2,t}, \ldots, X_{r,t}) : t \in \mathbb{Z}\}$ *is stationary on the set* $A = A_1 \times A_2 \times \ldots \times A_r$ *if the mean function* $E(X_{k,t})$ *is a constant function on the set* A_k *for any* $k \in \{1, 2, \ldots, r\}$ *and the cross-covariance function* $cov(X_{k,t_1}, X_{l,t_2})$ *depends only on* $(t_1 - t_2)$ *on the set* $(t_1, t_2) \in A_k \times A_l$ *for any* $k, l \in \{1, 2, \ldots, r\}$.

Proof of the Theorem 1. This proof is analogous to the proof of Theorem 2.1 in Lenart (2013), but it should be changed slightly according to the multivariate case of the APC time series being analysed. Consequently, we generalise the same auxiliary Lemmas.

Lemma 1. *Let* $\{\mathbf{X}_t : t \in \mathbb{Z}\}$ *be* $\mathbb{R}^r$*-valued APC time series on the set* A *and suppose Assumptions 1-4 hold. Then, for any* $(\nu, \omega) \in (0, 2\pi]^2$ *and* $k, l \in \{1, 2, \ldots, r\}$ *the following convergence* $E\big[\hat{R}_n^{c,d}(\nu, \omega)_{k\,l}\big] \to P(\nu, \omega)_{k\,l}$ *is true, where*

$$\hat{R}_n^{c,d}(\nu, \omega)_{k\,l} = \frac{1}{2\pi d_n} \sum_{s=c_n+1}^{c_n+d_n} \sum_{t=c_n+1}^{c_n+d_n} \mathbf{1}_{\{s \in A_k \wedge t \in A_l\}} (X_{k,s} - \mu_k(s))(X_{l,t} - \mu_l(t)) e^{-i(\nu s - \omega t)}.$$

Proof of Lemma 1. Because sets $A_k^{'} = \mathbb{Z} \setminus A_k$ and $A_l^{'} = \mathbb{Z} \setminus A_l$ are finite, consequently the proof is analogous to the proof of Lemma A.5 contained in Lenart (2011). The difference is that here we consider non-zero mean multivariate APC on the set time series. Therefore, we show only the main steps. Take any $(\nu, \omega) \in (0, 2\pi]^2$ and notice that because sets A_i' are finite for any $i = 1, 2, \ldots, r$ we have $(j = t,\ \tau = s - t)$

$$\begin{aligned}
&E\big[\hat{R}_n^{c,d}(\nu, \omega)_{k\,l}\big] = \\
&= \frac{1}{2\pi d_n} \sum_{s=c_n+1}^{c_n+d_n} \sum_{t=c_n+1}^{c_n+d_n} \mathbf{1}_{\{s \in A_k \wedge t \in A_l\}} E((X_{k,s} - \mu_k(s))(X_{l,t} - \mu_l(t))) e^{-i(\nu - \omega)t} e^{-i\nu(s-t)} \\
&= \frac{1}{2\pi d_n} \sum_{j=c_n+1}^{c_n+d_n} \sum_{\tau=c_n+1-j}^{c_n+d_n-j} \mathbf{1}_{\{j+\tau \in A_k \wedge j \in A_l\}} B(j, \tau)_{k\,l} e^{-i(\nu-\omega)j} e^{-i\nu\tau} \\
&= \frac{1}{2\pi d_n} \sum_{j=c_n+1}^{c_n+d_n} \sum_{\tau=c_n+1-j}^{c_n+d_n-j} B(j, \tau)_{k\,l} e^{-i(\nu-\omega)j} e^{-i\nu\tau} + o(1).
\end{aligned}$$

The next steps are analogical (therefore we omit them). The only difference is that we write $a(\cdot, \cdot)_{k\,l}$ in place of $a(\cdot, \cdot)$ □

Lemma 2. *Let Assumptions 2-4 hold. Then, for any* $(\nu,\omega) \in (0,2\pi]^2$ *we have*

$$\lim_{n\to\infty} d_n \mathrm{cov}(\mathrm{Re}[\hat{m}^{c,d}_{k,n}(\nu)], \mathrm{Re}[\hat{m}^{c,d}_{l,n}(\omega)]) = \pi(\mathrm{Re}[P(\nu,\omega)_{k\,l}] + \mathrm{Re}[P(\nu,2\pi-\omega)_{k\,l}]),$$
$$\lim_{n\to\infty} d_n \mathrm{cov}(\mathrm{Im}[\hat{m}^{c,d}_{k,n}(\nu)], \mathrm{Im}[\hat{m}^{c,d}_{l,n}(\omega)]) = \pi(\mathrm{Re}[P(\nu,\omega)_{k\,l}] - \mathrm{Re}[P(\nu,2\pi-\omega)_{k\,l}]),$$
$$\lim_{n\to\infty} d_n \mathrm{cov}(\mathrm{Re}[\hat{m}^{c,d}_{k,n}(\nu)], \mathrm{Im}[\hat{m}^{c,d}_{l,n}(\omega)]) = -\pi(\mathrm{Im}[P(\nu,\omega)_{k\,l}] - \mathrm{Im}[P(\nu,2\pi-\omega)_{k\,l}]).$$

Proof of Lemma 2. Using the same steps as in the proof of Lemma A.3 in Lenart (2013) we obtain

$$d_n \mathrm{cov}(\mathrm{Re}[\hat{m}^{c,d}_{k,n}(\nu)], \mathrm{Re}[\hat{m}^{c,d}_{l,n}(\omega)]) = \pi E\left(\mathrm{Re}[\hat{R}^{c,d}_n(\nu,\omega)_{k\,l}] + \mathrm{Re}[\hat{R}^{c,d}_n(\nu,2\pi-\omega)_{k\,l}]\right)$$
$$d_n \mathrm{cov}(\mathrm{Im}[\hat{m}^{c,d}_{k,n}(\nu)], \mathrm{Im}[\hat{m}^{c,d}_{l,n}(\omega)]) = \pi E\left(\mathrm{Re}[\hat{R}^{c,d}_n(\nu,\omega)_{k\,l}] - \mathrm{Re}[\hat{R}^{c,d}_n(\nu,2\pi-\omega)_{k\,l}]\right)$$
$$d_n \mathrm{cov}(\mathrm{Re}[\hat{m}^{c,d}_{k,n}(\nu)], \mathrm{Im}[\hat{m}^{c,d}_{l,n}(\omega)]) = -\pi E\left(\mathrm{Im}[\hat{R}^{c,d}_n(\nu,\omega)_{k\,l}] - \mathrm{Im}[\hat{R}^{c,d}_n(\nu,2\pi-\omega)_{k\,l}]\right).$$

Hence, by Lemma 1, we complete the proof. □

Using now the same decomposition as in the proof of Theorem 2.1 in Lenart (2013), it can be shown that the elements of the vector k_2 tend to zero, and random vector $k_1(n)$ tends in distribution to $\mathcal{N}_{2r}(0,\Omega((\psi)))$ (by the same arguments). □

Proof of the Theorem 2. The proof is analogous to the proof of Theorem 2.2 in Lenart (2013). Therefore, we show only the main steps. We consider two cases.

Case 1. Assume that $\prod_{k=1}^{r} |m_k(\psi)| = 0$. Then, by Theorem 1 and the continuous mapping theorem we have $\sqrt{d_n}\left(\sqrt[r]{\prod_{k=1}^{r} |\hat{m}_{k,n}(\psi)|} - \sqrt[r]{\prod_{k=1}^{r} |m_k(\psi)|}\right) \xrightarrow{d} \mathcal{L}(\tilde{Z}_A)$, where $\tilde{Z}_A = \sqrt[r]{\prod_{j=1}^{r} \sqrt{B^2_{2j-1} + B^2_{2j}}}$, and the random vector $[B_1\, B_2\, \ldots\, B_{2p}]^T$ follows a $2r$-dimensional normal distribution with zero mean and covariance matrix $\Omega(\psi)$. Because $g_0(\xi)_{k\,k} > 0$ the law of $\sqrt{B^2_{2j-1} + B^2_{2j}}$ is continuous for any $j = 1,2,\ldots,r$. Therefore the law of $\tilde{Z}_A$ is continuous.

Case 2. Let us assume that $\prod_{k=1}^{r} |m_k(\psi)| \neq 0$. By Theorem 1 and the multivariate delta method applied for the function $f(x_1,y_1,x_2,y_2,\ldots,x_r,y_r) = \sqrt[r]{\prod_{k=1}^{r} \sqrt{x_k^2+y_k^2}}$ (that is differentiable at the point

$(\mathrm{Re}(m_1(\psi)), \mathrm{Im}(m_1(\psi)), \mathrm{Re}(m_2(\psi)), \mathrm{Im}(m_2(\psi)), \ldots, \mathrm{Re}(m_r(\psi)), \mathrm{Im}(m_r(\psi))))$ we obtain convergence $\sqrt{d_n}\left(\sqrt[r]{\prod_{k=1}^{r}|\hat{m}_{k,n}(\psi)|} - \sqrt[r]{\prod_{k=1}^{r}|m_k(\psi)|}\right) \xrightarrow{d} \mathcal{N}_1(0, C_0\Omega(\psi)C_0^T)$.

In the case with $\det(\Omega(\psi)) > 0$ for $\prod_{k=1}^{r}|m_k(\psi)| \neq 0$ and $\psi \in (0,\pi)$, we obtain $C_0\Omega(\psi)C_0^{\mathrm{T}} > 0$. In the particular case when $\psi \in \{0,\pi\}$ easy calculations show that $C_0\Omega(\psi)C_0^{\mathrm{T}} > 0$. Because $|m(\psi)| = |m(2\pi-\psi)|$, for any $\psi \in [0,\pi]$ we conclude that the law $J^{A,\psi}$ is continuous for any $\psi \in [0, 2\pi)$. This completes the proof. □

Proof of the Theorem 3. The proof is analogous to the proof of Theorem 2. The the case of $\sum_{k=1}^{r}|m_k(\psi)| = 0$ is analogical. In case $\sum_{k=1}^{r}|m_k(\psi)| \neq 0$ we use function $f(x_1, y_1, x_2, y_2, \ldots, x_r, y_r) = \sqrt{\sum_{k=1}^{r} x_k^2 + y_k^2}$ (that is differentiable at the point $(\mathrm{Re}(m_1(\psi)), \mathrm{Im}(m_1(\psi)), \mathrm{Re}(m_2(\psi)), \mathrm{Im}(m_2(\psi)), \ldots, \mathrm{Re}(m_r(\psi)), \mathrm{Im}(m_r(\psi))))$ and the multivariate delta method. This completes the proof. □

Proof of the Theorem 4. The proof is analogous to the proof of Theorem 8.2 in Lenart and Pipień (2013a) and Theorem 2.3 in Lenart (2013). Note that the spacial case $r = 1$ is equivalent to Theorem 8.2 in Lenart and Pipień (2013a). Therefore, we omit the details of the proof. □

Proof of the Theorem 5. The proof is analogous to the proof of Theorem 8.2 in Lenart and Pipień (2013a) and Theorem 2.3 in Lenart (2013). Note that the spacial case $r = 1$ is equivalent to Theorem 8.2 in Lenart and Pipień (2013a). Therefore, we omit the details of the proof. □