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Determination of optimal controllers. Comparison of two methods for electric network chain

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Abstract. In the paper the comparison of two methods for calculation optimal gains is considered. One method using a Kalman procedure and one using a Riccati equation are compared. It is proved that a Kalman procedure is much better.

Key words: optimal controllers, Kalman equation, Riccati equation.

1. Introduction

Let us consider the following optimal control problem:

The state equation has the form [8]

$$\dot{x}(t) = Ax(t) + Bu(t),\ \ t \in [0, \infty],\ \ x(0) = x_0. \tag{1}$$

The quality functional is

$$J(u) = \frac{1}{2}\int_0^{\infty}\left[x^T(t)Qx(t) + u^T(t)Ru(t)\right]dt. \tag{2}$$

The matrices A, B, Q and R are constants with appropriates dimensions. The state vector is n-dimensional and the control vector is r dimensional. The horizon of control it is time $\tau = \infty$. Without loss of generality we assume that matrices Q and R are symmetric and matrix Q is nonnegative determined and R is positive determined.

Theorem 1. [1] The necessary and sufficient condition of the existence of optimal control is the existence of the integral (2) for the admissible control. For that we assume the asymptotic stability of the state matrix A.

2. The methods of determination the optimal controllers

2.1. Riccati equation.

Theorem 2. [1–4, 10] Optimal control is described by the equation

$$u(t) = -R^{-1}B^TKx(t),\ \ t \geq 0 \tag{3}$$

where matrix K is constant and may be calculated from the Riccati equation

$$KBR^{-1}B^TK - A^TK - KA - Q = 0. \tag{4}$$

We look for the solution

$$K = K^T \geq 0. \tag{5}$$

The system (1) has the unique solution and the closed optimal control system is stable and determined by the equation

$$\dot{x}(t) = \left(A - BR^{-1}B^TK\right)x(t). \tag{6}$$

From the symmetry of the matrix K follows that the equation (4) has $\frac{1}{2}n(n+1)$ scalar equations of the second degree.

2.2. Kalman equation.

Theorem 3. [1, 7] The transfer function of the system described by the equation (1) is equal

$$G(s) = (sI - A)^{-1}B. \tag{7}$$

Denoting by

$$M(s) = \det(sI - A) \tag{8}$$

which is the denominator of the transfer function of the open system (7), we can write for the denominator of the transfer function of the optimal closed system $M_c(s)$

$$\det[M_c(s)M_c(-s)] = (-1)^n M(s)M(-s)\det[I + R^{-1}G^T(-s)QG(s)]. \tag{9}$$

This is Kalman equation [9].

In the case of one dimensional system, when the control and output are scalars and

$$R = Q = 1 \tag{10}$$

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the Kalman equation simplifies to

$$\det[M_c(s)M_c(-s)] = (-1)^n M(s)M(-s)\det[I + + G^T(-s)G(s)]. \tag{11}$$

The transfer function of the open loop system is

$$G(s) = \frac{L(s)}{M(s)}. \tag{12}$$

Finally

$$\det[M_c(s)M_c(-s)] = (-1)^n[M(s)M(-s)] + + (-1)^{-r}[L(s)L(-s)]. \tag{13}$$

Theorem 4. [1] Let us numerator of the transfer function be a polynomial of the degree r

$$L(s) = \Pi_{i=1}^{r}(s - z_i) \tag{14}$$

where z_i are zeros of $L(s)$ and the denominator be of the polynomial degree n

$$M(s) = \Pi_{i=1}^{n}(s - p_i) \tag{15}$$

where p_i are zeros of $M(s)$, it means poles of the transfer function $G(s)$.

Let the characteristic polynomial of the optimal closed system be

$$M_c(s) = \Pi_{i=1}^{n}(s - \delta_i). \tag{16}$$

The Kalman equation gives the relation between the poles of the optimal control system and the poles and zeros of the open loop system in the form

$$\Pi_{i=1}^{n}(s^2 - \delta_i^2) = \Pi_{i=1}^{n}(s^2 - p_i^2) + + (-1)^{n-r}\Pi_{i=1}^{r}(s^2 - z_i^2) = 0. \tag{17}$$

Now we compare effectiveness of these two methods on an example of the electric network chain.

3. Optimal controllers of electric network chain

3.1. Description of the system. The chain composed of n-equal elements R, L, C, G type is considered. In the Fig. 1 the chain is shown.

From the Kirchhoff laws applied to the elementary system we obtain:

$$u_{m-1} - i_m R - \frac{di_m}{dt} - u_m = 0 \tag{18}$$

$$i_m - i_{m+1} - Gu_m - C\frac{du_m}{dt} = 0, \quad m = 1, 2, \ldots, n \tag{19}$$

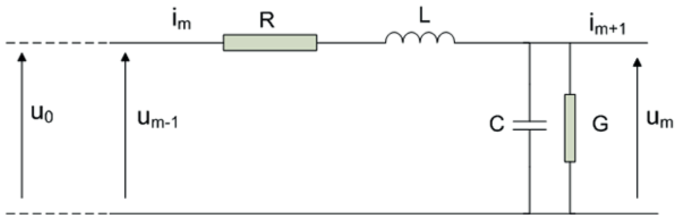

Fig. 1. Element R, L, C, G

or in the matrix form:

$$\begin{bmatrix} L\frac{di_m}{dt} \\ C\frac{du_m}{dt} \end{bmatrix} = \begin{bmatrix} -R & -1 \\ 1 & -G \end{bmatrix}\begin{bmatrix} i_m \\ u_m \end{bmatrix} + \begin{bmatrix} u_{m-1} \\ i_{m+1} \end{bmatrix}. \tag{20}$$

Denoting the state vector of the whole system by x, and the input voltage by u, where

$$x = [i_1, u_1, \ldots, i_n, u_n]^T \tag{21}$$

we obtain

$$\frac{dx}{dt} = Ax + Bu \tag{22}$$

where

$$u = u_0 \tag{23}$$

and A is $(2n \times 2n)$ matrix:

$$A = \begin{bmatrix} -\frac{1}{T_L} & -C\omega_0^2 & 0 & 0 & 0 & \cdots & 0 \\ L\omega_0^2 & -\frac{1}{T_c} & -L\omega_0^2 & 0 & 0 & \cdots & 0 \\ 0 & C\omega_0^2 & -\frac{1}{T_L} & -C\omega_0^2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & -\frac{1}{T_L} & -C\omega_0^2 \\ 0 & 0 & \cdots & \cdots & \cdots & L\omega_0^2 & -\frac{1}{T_c} \end{bmatrix} \tag{24}$$

and

$$B = \frac{1}{L}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{25}$$

$$\left.\begin{aligned} T_L &= \frac{L}{R} \\ T_C &= \frac{C}{G} \\ \omega_0^2 &= \frac{1}{LC} \end{aligned}\right\}. \tag{26}$$

3.2. Eigenvalues. We consider the case of different eigenvalues. The characteristic polynomial of the matrix A is denoted by $P_{2n}(\lambda)$:

$$P_{2n}(\lambda)=\begin{vmatrix} \lambda+\frac{1}{T_L} & C\omega_0^2 & 0 & 0 & 0 & \cdots & 0 \\ -L\omega_0^2 & \lambda+\frac{1}{T_c} & L\omega_0^2 & 0 & 0 & \cdots & 0 \\ 0 & -C\omega_0^2 & \lambda+\frac{1}{T_L} & C\omega_0^2 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \lambda+\frac{1}{T_L} & C\omega_0^2 \\ 0 & 0 & \cdots & \cdots & \cdots & -L\omega_0^2 & \lambda+\frac{1}{T_c} \end{vmatrix} \quad (27)$$

The principal minor whose rows and columns have indices $k+1, k+2, \ldots, k+l$; $0 \le k \le 2n-1$, $1 \le l \le 2n-k$ is denoted by $P_l(\lambda)$ if k is even and by $Q_l(\lambda)$ if k is odd. Developing determinant $P_l(\lambda)$ along its first row we obtain following recurrence formula:

$$P_l(\lambda)=\left(\lambda+\frac{1}{T_L}\right)Q_{l-1}(\lambda)+\omega_0^2P_{l-2}(\lambda),\ 3\le l\le 2n. \quad (28)$$

Analogically we have

$$Q_{l-1}(\lambda)=\left(\lambda+\frac{1}{T_C}\right)P_{l-2}(\lambda)+\omega_0^2Q_{l-3}(\lambda),\quad 4\le l\le 2n \quad (29)$$

and

$$P_{l-2}(\lambda)=\left(\lambda+\frac{1}{T_L}\right)Q_{l-3}(\lambda)+\omega_0^2P_{l-4}(\lambda),\quad 5\le l\le 2n \quad (30)$$

From (28), (29) and (30) we have the equation

$$P_l(\lambda)-\left[\left(\lambda+\frac{1}{T_L}\right)\left(\lambda+\frac{1}{T_C}\right)+2\omega_0^2\right]P_{l-2}(\lambda)+ +\omega_0^4P_{l-4}(\lambda)=0,\ \ 5\le l\le 2n. \quad (31)$$

From (31) we can calculate $P_l(\lambda)$, separately for *l-even* knowing P_2, P_4 and for *l-odd* knowing P_1, P_3.

The initial values from (27) are:

$$\left.\begin{aligned} P_1(\lambda)&=\lambda+\frac{1}{T_L}\\ P_2(\lambda)&=\left(\lambda+\frac{1}{T_L}\right)\left(\lambda+\frac{1}{T_C}\right)+\omega_0^2\\ P_3(\lambda)&=\left(\lambda+\frac{1}{T_L}\right)\left[\left(\lambda+\frac{1}{T_L}\right)\left(\lambda+\frac{1}{T_C}\right)+2\omega_0^2\right]\\ P_4(\lambda)&=\left(\lambda+\frac{1}{T_L}\right)^2\left(\lambda+\frac{1}{T_C}\right)^2+\left(\lambda+\frac{1}{T_L}\right)\left(\lambda+\frac{1}{T_C}\right)3\omega_0^2+\omega_0^4 \end{aligned}\right\}. \quad (32)$$

For the sequences $P_1, P_3, \ldots,$ and $P_2, P_4, \ldots,$ we can treat (31) as a difference equation of the second order with the characteristic equation:

$$r^2-\left[\left(\lambda+\frac{1}{T_L}\right)\left(\lambda+\frac{1}{T_C}\right)+2\omega_0^2\right]r+\omega_0^4=0. \quad (33)$$

The roots are

$$r=\frac{1}{2}\left[\left(\lambda+\frac{1}{T_L}\right)\left(\lambda+\frac{1}{T_C}\right)+2\omega_0^2\right]\pm \pm\sqrt{\frac{1}{4}\left[\left(\lambda+\frac{1}{T_L}\right)\left(\lambda+\frac{1}{T_C}\right)+2\omega_0^2\right]^2-\omega_0^4}\,. \quad (34)$$

We put

$$\frac{1}{2}\left[\left(\lambda+\frac{1}{T_L}\right)\left(\lambda+\frac{1}{T_C}\right)+2\omega_0^2\right]=\omega_0^2\cos\varphi \quad (35)$$

where φ may be a complex number. We have

$$r=\omega_0^2(\cos\varphi+i\sin\varphi). \quad (36)$$

The formulas (32) are now

$$\left.\begin{aligned} P_1(\lambda)&=-\lambda+\frac{1}{T_L}\\ P_2(\lambda)&=(2\cos\varphi-1)\omega_0^2\\ P_3(\lambda)&=2\left(\lambda+\frac{1}{T_L}\right)\omega_0^2\cos\varphi\\ P_4(\lambda)&=\left[4(\cos\varphi)^2-2\cos\varphi-1\right]\omega_0^4 \end{aligned}\right\}. \quad (37)$$

The solutions of (31) are

$$P_{2m}=\left[c_1(\cos m\varphi+i\sin m\varphi)+ +c_2(\cos m\varphi-i\sin m\varphi)\right]\omega_0^{2m} \quad (38)$$

$$P_{2m-1}=\left[c_3(\cos m\varphi+i\sin m\varphi)+ +c_4(\cos m\varphi-i\sin m\varphi)\right]\omega_0^{2m} \quad (39)$$

The constants $c_1, \ldots, c_4$ are determined by (37):

$$\left.\begin{aligned} c_1&=\frac{1}{2}+\frac{1-\cos\varphi}{2-\sin\varphi}i\\ c_2&=\frac{1}{2}-\frac{1-\cos\varphi}{2-\sin\varphi}i\\ c_3&=-c_4=-\frac{\left(\lambda+\frac{1}{T_L}\right)}{2\omega_0^2\sin\varphi}i \end{aligned}\right\}. \quad (40)$$

We can assume $\varphi\neq 0$ because the polynomial under the square root in (34) is not identically equal zero.

Substitution of (40) into (38) and (39) gives:

$$P_{2m}=\frac{\omega_0^{2m}}{\sin\varphi}\left[\sin(m+1)\varphi-\sin m\varphi\right],\ m=1,2,\ldots,n \quad (41)$$

$$P_{2m-1} = \frac{\left(\lambda + \frac{1}{T_L}\right)\omega_0^{2m-2}}{\sin\varphi}\sin m\varphi, \quad m = 1, 2, \ldots, n. \tag{42}$$

We have from (28):

$$P_{2m}(\lambda) = \left(\lambda + \frac{1}{T_L}\right)Q_{2m-1}(\lambda) + \omega_0^2 P_{2m-2}(\lambda) \tag{43}$$

and

$$P_{2m-1}(\lambda) = \left(\lambda + \frac{1}{T_L}\right)Q_{2m}(\lambda) + \omega_0^2 P_{2m-1}(\lambda). \tag{44}$$

According to (41) ÷ (44) we have

$$Q_{2m-1}(\lambda) = \frac{\omega_0^{2m-2}}{\sin\varphi}\left(\lambda + \frac{1}{T_L}\right)\sin m\varphi, \quad m = 1, 2, \ldots, n. \tag{45}$$

$$Q_{2m}(\lambda) = \frac{\omega_0^{2m}}{\sin\varphi}\left[\sin(m+1)\varphi - \sin m\varphi\right]. \tag{46}$$

The eigenvalues λ_i, $i = 1, 2, \ldots, 2n$ of the whole system we obtain from (41) putting $m = n$

$$\sin(n+1)\varphi - \sin n\varphi = 0 \tag{47}$$

and taking into account (35). From (41) and (42) we have:

$$P_{2n}(\lambda) = \frac{\omega_0^{2n}\cos(2n+1)\frac{\varphi}{2}}{\cos\frac{\varphi}{2}} \tag{48}$$

and

$$P_{2n-1}(\lambda) = \frac{\left(\lambda + \frac{1}{T_L}\right)\omega_0^{2n-2}\sin n\varphi}{\sin\varphi}. \tag{49}$$

Finally from (48) we have the eigenvalues

$$\varphi = \frac{(2j-1)\pi}{2n+1}, \quad j = 1, \ldots, n. \tag{50}$$

3.3. The transfer function. By putting $m+1$ in place m in the equation (18) after using Laplace transform we obtain:

$$(u_{m+1} - u_m) + i_{m+1}(R + sL) = 0, \quad m = 1, 2, \ldots, n. \tag{51}$$

After elimination currents $i_m - i_{m+1}$ from equation (19) and using equations (18), (21) and $i_{n+1} = u_n/R_0$ we obtain the following relation between voltages:

$$-u_{m-1} + A(s)u_m - u_{m+1} = 0, \quad m = 0, 1, \ldots, n-1 \tag{52}$$

$$-u_{n-1} + \left[A(s) - 1 + \frac{R(1+sT_L)}{R_0}\right]u_n = 0, \quad m = n \tag{53}$$

where

$$A(s) = \frac{1}{\omega_0^2}\left[\left(s + \frac{1}{T_L}\right)\left(s + \frac{1}{T_C}\right) + 2\omega_0^2\right]. \tag{54}$$

For the unloaded chain the resistance $R_0 = \infty$ and current $i_{n+1} = 0$.

The recurrent relation (52) can be written in the matrix form:

$$\begin{bmatrix} A(s) & -1 & 0 & \cdots & 0 & 0 \\ -1 & A(s) & -1 & \cdots & 0 & 0 \\ 0 & -1 & A(s) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A(s) & -1 \\ 0 & 0 & 0 & \cdots & -1 & B(s) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \cdots \\ u_{n-1} \\ u_n \end{bmatrix} = \begin{bmatrix} u_0 \\ 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{bmatrix} \tag{55}$$

where

$$B(s) = A(s) - 1. \tag{56}$$

Let $T_n(s)$ denote the $n \times n$ matrix presented by (55).

The determinant

$$M_n(s) = \det T_n(s) \tag{57}$$

is the polynomial of $2n$-th degree.

The eigenvalues accordingly are for $P_{2n}(\lambda) = 0$

$$\cos(2n+1)\frac{\varphi}{2} = 0 \tag{58}$$

$$\varphi = \frac{(2j-1)\pi}{2n+1}, \quad j = 1, 2, \ldots, n. \tag{59}$$

Then, using (35)

$$\left(\lambda + \frac{1}{T_L}\right)\left(\lambda + \frac{1}{T_C}\right) + 4\omega_0^2\sin^2\frac{(2j-1)\pi}{4n+2} = 0, \quad j = 1, 2, \ldots, n. \tag{60}$$

Finally

$$\left(\lambda + \frac{1}{T_L}\right)\left(\lambda + \frac{1}{T_C}\right) + 2\omega_0^2\left(1 - \cos\frac{(2j-1)\pi}{2n+1}\right) = 0, \quad j = 1, 2, \ldots, n-1. \tag{61}$$

Applying Cramers method to the matrix equation (55), we find that

$$u_m(s) = \frac{M_{n-m}(s)}{M_n(s)}u_0(s) \quad \text{for} \quad m = 1, 2, \ldots, n \tag{62}$$

(which may be proved by inspection), where

$$M_0(s) \stackrel{\text{def}}{=} 1,\; M_1(s) = B(s),\; M_2(s) = A(s)B(s) - 1. \quad (63)$$

The transfer function is equal to

$$H(s) = \frac{1}{M_n(s)} \quad (64)$$

The poles of the transfer function it is the roots of the equation

$$M_n(s) = 0 \quad (65)$$

are the eigenvalues determined by equations (61) and we have no zeros of the transfer function (64).

The Kalman equation (17) in this case takes a form

$$\Pi_{j=1}^{n}\left(s^2 - \delta_j^2\right) = \Pi_{j=1}^{n}\left(s^2 - \lambda_j^2\right) + (-1)^n = 0 \quad (66)$$

where the roots λ_j from equation (60) are

$$\lambda_{j1,2} = -\frac{1}{2}\left(\frac{1}{T_L} + \frac{1}{T_C}\right) \pm$$
$$\pm\sqrt{\frac{1}{4}\left(\frac{1}{T_L} + \frac{1}{T_C}\right)^2 - \left[\frac{1}{T_L T_C} + 4\omega_0^2 \sin^2\frac{(2j-1)\pi}{4n+2}\right]}, \quad (67)$$
$$j = 1, 2, \ldots, n$$

$$\left(s^2 - \lambda_j^2\right) = \left(s^2 - \lambda_{j1}^2\right)\left(s^2 - \lambda_{j2}^2\right), \quad j = 1, 2, \ldots, n.$$

Finally

$$\left(s^2 - \lambda_j^2\right) = s^4 + \left[\frac{\left(8\omega_0^2 \sin^2\frac{(2j-1)\pi}{4n+2}\right)T_L^2 T_C^2 - T_L^2 - T_C^2}{T_L^2 T_C^2}\right]s^2 +$$
$$+ \frac{8\omega_0^2 \sin^2\frac{(2j-1)\pi}{4n+2}}{T_L T_C} + 16\omega_0^4 \sin^4\frac{(2j-1)\pi}{4n+2} + \frac{1}{T_L^2 T_C^2} + 1, \quad (68)$$
$$j = 1, 2, \ldots, n$$

and the poles of the transfer function of the closed system can be calculated from the Kalman equation (69)

$$\Pi_{j=1}^{n}\left(s^2 - \delta_j^2\right) = \Pi_{j=1}^{n}\left\{s^2 - \lambda_j^2\right\} + (-1)^n = 0, \quad j = 1, 2, \ldots, n. \quad (69)$$

From the other side, after using Laplace transform to the optimal equation (6), we have

$$M_c(s) = \left\|[sI - A] + [BR^{-1}B^T K]\right\| \quad (70)$$

$$M_c(s) = \left\|[sI - A + BB^T K]\right\| \quad (71)$$

and

$$M_c(s)M_c(-s) = \left\|[sI - A + BB^T K]\right\| \left\|[-sI - A + BB^T K]\right\|. \quad (72)$$

Finally

$$M_c(s)M_c(-s) =$$
$$\begin{vmatrix} s+\frac{1}{T_L}+K_{11} & C\omega_0^2+K_{12} & K_{13} & K_{14} & \cdots & K_{1n} \\ -L\omega_0^2 & s+\frac{1}{T_C} & L\omega_0^2 & 0 & \cdots & 0 \\ 0 & -C\omega_0^2 & s+\frac{1}{T_L} & C\omega_0^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & -C\omega_0^2 & s+\frac{1}{T_L} & C\omega_0^2 \\ 0 & 0 & 0 & 0 & -L\omega_0^2 & s+\frac{1}{T_C} \end{vmatrix} \circ$$
$$\begin{vmatrix} -s+\frac{1}{T_L}+K_{11} & C\omega_0^2+K_{12} & K_{13} & K_{14} & \cdots & K_{1n} \\ -L\omega_0^2 & -s+\frac{1}{T_C} & L\omega_0^2 & 0 & \cdots & 0 \\ 0 & -C\omega_0^2 & -s+\frac{1}{T_L} & C\omega_0^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & -C\omega_0^2 & -s+\frac{1}{T_L} & C\omega_0^2 \\ 0 & 0 & 0 & 0 & -L\omega_0^2 & -s+\frac{1}{T_C} \end{vmatrix} = 0. \quad (73)$$

Comparing (69) and (73) we obtain the relations for optimal coefficients K_i.

In the simple case, for $n = 1$ we have (in the case of using Kalman equation)

$$M_c(s)M_c(-s) = \Pi_{j=1}^{1}\left(s^2 - \delta_j^2\right) =$$
$$= s^4 + \left[\frac{\left(8\omega_0^2 \sin^2\frac{\pi}{6}\right)T_L^2 T_C^2 - T_L^2 - T_C^2}{T_L^2 T_C^2}\right]s^2 + \quad (74)$$
$$+ \frac{8\omega_0^2 \sin^2\frac{\pi}{6}}{T_L T_C} + 16\omega_0^4 \sin^4\frac{\pi}{6} + \frac{1}{T_L^2 T_C^2} + 2 = 0.$$

From other side using equation (73)

$$M_c(s)M_c(-s) = \left\|\begin{bmatrix} s+\frac{1}{T_L} & C\omega_0^2 \\ -L\omega_0^2 & s+\frac{1}{T_C} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}[1 \;\; 0]\begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}\right\|$$
$$\left\|\begin{bmatrix} -s+\frac{1}{T_L} & C\omega_0^2 \\ -L\omega_0^2 & -s+\frac{1}{T_C} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}[1 \;\; 0]\begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}\right\| = 0 \quad (75)$$

$$M_c(s)M_c(-s)=\left\|\begin{bmatrix} s+\frac{1}{T_L} & C\omega_0^2 \\ -L\omega_0^2 & s+\frac{1}{T_C}\end{bmatrix}+\begin{bmatrix}K_{11} & K_{12}\\ 0 & 0\end{bmatrix}\right\|$$
$$\left\|\begin{bmatrix} -s+\frac{1}{T_L} & C\omega_0^2 \\ -L\omega_0^2 & -s+\frac{1}{T_C}\end{bmatrix}+\begin{bmatrix}K_{11} & K_{12}\\ 0 & 0\end{bmatrix}\right\|=$$
$$\begin{vmatrix} s+\frac{1}{T_L}+K_{11} & C\omega_0^2+K_{12} \\ -L\omega_0^2 & s+\frac{1}{T_C}\end{vmatrix}\begin{vmatrix} -s+\frac{1}{T_L}+K_{11} & C\omega_0^2+K_{12} \\ -L\omega_0^2 & -s+\frac{1}{T_C}\end{vmatrix}= \tag{76}$$
$$s^4+\left\{\frac{2\left(T_L^2T_C^2\omega_0^2+T_L^2T_C^2L\omega_0^2K_{12}-K_{11}T_LT_C^2\right)-T_c^2-T_L^2-K_{11}^2T_L^2T_C^2}{T_L^2T_C^2}\right\}s^2+$$
$$\frac{\left(1+\omega_0^2T_LT_c+L\omega_0^2T_LT_cK_{12}+K_{11}T_L\right)^2}{T_L^2T_C^2}=0.$$

Comparing (74) and (76) we obtain the relations for calculation K_{11} and K_{12}.

$$\begin{aligned}&\frac{\left(8\omega_0^2\sin^2\frac{\pi}{6}\right)T_L^2T_C^2-T_L^2-T_C^2}{T_L^2T_C^2}=\left[2\left(T_L^2T_C^2\omega_0^2+\right.\right.\\&+T_L^2T_C^2L\omega_0^2K_{12}-K_{11}T_LT_C^2\left.\right)-T_c^2-T_L^2-\\&-K_{11}^2T_L^2T_C^2\left.\right]/\left(T_L^2T_C^2\right)\end{aligned} \tag{77}$$

$$\begin{aligned}&\frac{8\omega_0^2\sin^2\frac{\pi}{6}}{T_LT_C}+16\omega_0^4\sin^4\frac{\pi}{6}+\frac{1}{T_L^2T_C^2}+2=\\&=\left[\left(1+\omega_0^2T_LT_c+L\omega_0^2T_LT_cK_{12}+K_{11}T_L\right)^2\right]/\left(T_L^2T_C^2\right).\end{aligned} \tag{78}$$

From relations (77) and (78) we have the linear equation for K_{12}

$$K_{12}=\left(\frac{8\omega_0^2T_L\sin^2\frac{\pi}{6}-2\omega_0^2T_L+2K_{11}+T_LK_{11}^2}{2L\omega_0^2T_L}\right) \tag{79}$$

and the equation of 2-nd degree for K_{11}

$$\begin{aligned}&\sqrt{\frac{8\omega_0^2\sin^2\frac{\pi}{6}}{T_LT_C}+16\omega_0^4\sin^4\frac{\pi}{6}+\frac{1}{T_L^2T_C^2}+1}=\frac{1}{2}K_{11}^2+\\&+\frac{(T_L+T_C)K_{11}}{T_LT_C}+\\&+\frac{1+\omega_0^2T_LT_C+\frac{1}{2}T_C\left(8T_L\omega_0^2\sin^2\frac{\pi}{6}-2L\omega_0^4T_LT_C\right)}{T_LT_C}.\end{aligned} \tag{80}$$

In the case of using Riccati equation we have in this case the relations (81 ÷ 90) from which we obtain for K_{11} the equation 4-th degree (90) which is not usefull for practice.

$$A=\begin{bmatrix}-\frac{1}{T_L} & -C\omega_0^2\\ L\omega_0^2 & -\frac{1}{T_C}\end{bmatrix} \tag{81}$$

$$B=\begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix} \tag{82}$$

$$R=[1] \tag{83}$$

$$Q=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \tag{84}$$

$$K=\begin{bmatrix}K_{11} & K_{12}\\ K_{12} & K_{22}\end{bmatrix} \tag{85}$$

From (4) we have

$$\begin{bmatrix}\frac{K_{11}^2}{L^2}+\frac{2K_{11}}{T_L}-2L\omega_0^2K_{12} & E\\ E & \frac{K_{12}^2}{L^2}+2C\omega_0^2K_{12}+\frac{2K_{22}}{T_C}-1\end{bmatrix}=0 \tag{86}$$

where

$$E=\frac{K_{11}K_{12}}{L^2}+\frac{K_{12}}{T_L}-L\omega_0^2K_{22}+C\omega_0^2K_{11}+\frac{K_{12}}{T_C}.$$

From (86) we have

$$\frac{K_{11}^2}{L^2}+\frac{2K_{11}}{T_L}-2L\omega_0^2K_{12}=0 \tag{87}$$

$$E=\frac{K_{11}K_{12}}{L^2}+\frac{K_{12}}{T_L}-L\omega_0^2K_{22}+C\omega_0^2K_{11}+\frac{K_{12}}{T_C}=0 \tag{88}$$

$$\frac{K_{12}^2}{L^2}+2C\omega_0^2K_{12}+\frac{2K_{22}}{T_C}-1=0. \tag{89}$$

Solving (87), (88) and (89) we obtain

$$K_{22}=-\frac{T_C}{2}\left[\frac{K_{12}(K_{12}+2L)-L^2}{L^2}\right]$$

$$K_{12}=\frac{1}{2}\left(\frac{K_{11}^2T_L+2K_{11}L^2}{L^3T_L\omega_0^2}\right)$$

$$\begin{aligned}&\frac{1}{4}\left[-T_C^2K_{11}^4T_L^2-4T_C^2K_{11}^3T_LL^2+\left(-4T_LT_CL^4-4L^4T_L^2-\right.\right.\\&-4T_C^2L^4-4L^4\omega_0^2T_L^2T_C^2\left.\right)K_{11}^2+\left(-8L^6\omega_0^2T_LT_C^2-\right.\\&-8L^6T_C-8L^6T_L\left.\right)K_{11}+4L^8\omega_0^4T_L^2T_C^2\left.\right]/\\&/\left[L^2T_LT_C\left(2\omega_0^2L^4T_L+K_{11}^2T_L+2K_{11}L^2-L^2T_L\right)\right]=0.\end{aligned} \tag{90}$$

4. Numerical example

Let

$$T_L = 0.5, \qquad T_C = 1, \qquad \omega_0 = 1, \qquad L = 1, \qquad C = 1.$$

In the case of using Kalman method we have from (74)

$$M_C(s)M_C(-s) = s^4 - 3s^2 + 10$$

and from (76)

$$M_C(s)M_C(-s) = s^4 + \left[6 + 2K_{12} + 2K_{11} - 4(1.5 + 0.5K_{11})^2\right]s^2 + 4(1.5 + 0.5K_{12} + 0.5K_{11})^2.$$

Comparing coefficients we have relations for calculation K_{11} and K_{12}

$$\begin{cases} 6 + 2K_{12} + 2K_{11} - 4(1.5 + 0.5K_{11})^2 = -3 \\ 4(1.5 + 0.5K_{12} + 0.5K_{11})^2 = 10. \end{cases}$$

and solving we obtain

$$K_{11} = 0.053613 \qquad K_{12} = 0.108664$$

After using Riccati equation we have from (87), (88) and (89) folowing relations

$$\begin{cases} K_{11}^2 + 4K_{11} - 2K_{12} = 0 \\ K_{11}K_{12} + 3K_{12} - K_{22} + K_{11} = 0 \\ K_{12}^2 + 2K_{12} + 2K_{22} - 1 = 0 \end{cases}$$

and solving we obtain

$$K_{11} = 0.053613 \qquad K_{12} = 0.108664 \qquad K_{22} = 0.385432$$

where K_{22} is not needed for control (3).

5. Conclusions

1. For n-order system with a scalar control the proposed Kalman procedure needs only n equations for calculation n optimal gains.
2. The Riccati method requires $1/2\, n(n+1)$ equations for calculation optimal gains.

Some extensions for MIMO systems are possible. Authors are working on such generalizations.

Remark. In [6] the following theorem is proposed: the analytic solution of the differential Riccati equation can be expressed as a linear function of the known algebraic Riccati equation.

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