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Two relations for generalized discrete Fourier transform coefficients

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Abstract. A new generalized discrete Fourier transform DFT that allows for sample shift $\delta \in [0, T/N]$ in time-domain is defined. Two relations are proved for the sum of errors between generalized DFT coefficients and theirs theoretical values. The first is the equation for samples received for continuous and piecewise–smooth functions. The second relation is the inequality for samples generated by discontinuous functions. Moreover, the influence of samples shift on generalized DFT coefficients values, which leads to aliasing phenomenon, is presented.

Key words: generalized discrete Fourier transform, coefficient errors, aliasing.

1. Introduction

Numerical approaches to generalized discrete Fourier transform (DFT), allowing for sample shift, are presented. First, two relations are presented: equality and inequality for sum of modulus of generalized DFT Fourier coefficients versus theirs theoretical values. Subsequently, some aliasing examples for shifted samples sets are investigated.

2. Discrete Fourier transform for shifted samples

Fourier series for signal $f(\,)$ for $t \in \Delta = [a, b] = [a, a + T]$ is given as follows

$$S_N(t)\frac{a_0}{2} + \sum_{h=1}^{N}\left\{a_h\cos(h\omega t) + b_h\sin(h\omega t)\right\}, \tag{1}$$

where the coefficients are equal to

$$\begin{aligned} a_h &= \frac{2}{T}\int_a^b f(t)\cos(h\omega t)\mathrm{d}t \\ b_h &= \frac{2}{T}\int_a^b f(t)\sin(h\omega t)\mathrm{d}t, \end{aligned} \tag{2}$$

and angular speed

$$\omega = \frac{2\pi}{T}. \tag{3}$$

The complex Fourier series coefficients c_h are defined as given

$$c_h = \frac{1}{T}\int_a^b f(t)\exp(-\mathrm{j}h\omega t)\mathrm{d}t = \frac{1}{2}(a_h - \mathrm{j}b_h) = c^*_{-h}, \tag{4}$$

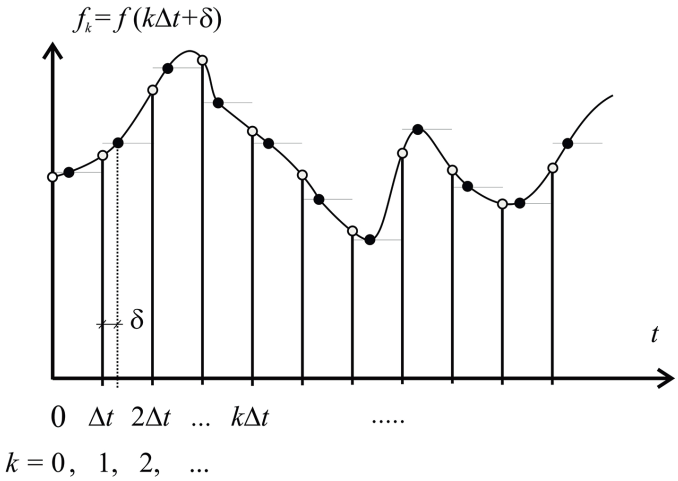

Fig. 1. Indexation of signal samples (black points) in time-domain for shift $\delta \in [0, T/N]$

and they enable introduce the complex form for Fourier series in the form of

$$f(t) = \lim_{N\to\infty} S_N(t) = \lim_{N\to\infty} \sum_{h=0,\pm1,\pm2,\ldots}^{\pm N} c_h\exp(\mathrm{j}h\omega t). \tag{5}$$

Usually, the integrals (4) are approximated by means of formula “value of each sample is multiplied by subinterval length”. This formula defines discrete Fourier transform DFT [2–4]. Let us obtain the sequence of N samples as particular values of signal $f(\,)$ on N subsequent subintervals of length $\Delta t = T/N$. The samples are distanced equally, shifted of $\delta \in [0, T/N]$, i.e. each of them is placed on closed subinterval $[0, \Delta t]$ as shown in Fig. 1.

Basing on shifted samples set generalized DFT coefficients are now defined as follows

$$\underset{(\mathrm{DFT})}{c_h} \overset{\mathrm{df}}{=} \frac{1}{T}\sum_{k=0}^{N-1} f(k\Delta t + \delta)\exp(-\mathrm{j}h\omega k\Delta t)\Delta t, \tag{6}$$

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or equivalently

$$\underset{(\mathrm{DFT})}{c_h} = \frac{1}{N}\sum_{k=0}^{N-1} f_k W_N^{kh}, \tag{7}$$

where it is denoted as

$$W_N = \exp\left(-\mathrm{j}\frac{2\pi}{N}\right). \tag{8}$$

The samples can be placed either at the beginnings or in the middles or at others places of each subintervals, i.e.

$$f_k = f\left(k\frac{T}{N} + \delta\right) = f(k\Delta t + \delta), \tag{9}$$

for $k = 0, 1, \ldots, N-1$. From the mathematical point of view, the sample shift $\delta \in [0, \Delta t] = [0, T/N]$ defines a generalized formula of DFT defined below by (6). For classical DFT it is set $\delta = 0$. From the technical point of view, the process of sample acquisition is not often free from the shift $\delta \neq 0$, which is caused by many miscellaneous reasons [6, 7] particularly for the non-periodic samples.

3. Error analysis of generalized DFT coefficients

The main theoretical problem is to calculate the errors e_h between values $\underset{(\mathrm{DFT})}{c_h}$ given by generalized discrete Fourier transform DFT (6) and appropriate theoretical values

$$\underset{(\mathrm{theor})}{c_h} = c_h = \frac{1}{2}(a_h - \mathrm{j}b_h), \tag{10}$$

defined as follows

$$e_h = \underset{(\mathrm{DFT})}{c_h} - \underset{(\mathrm{theor})}{c_h} = \underset{(\mathrm{DFT})}{c_h} - c_h. \tag{11}$$

Let us assume that the function $f(\ )$ is continuous and piecewise–smooth (piecewise of the class C^1 i.e. has a bounded derivative which is continuous everywhere except a finite number of points at which left- and right-sided derivatives exist). The finite number of points at which the derivative does not exist are denoted by t_k. Furthermore, a finite number of points at which the first derivative is not continuous may appear on each subinterval $[t_k, t_k + \Delta t]$. These points are called irregular points. For error analysis of coefficients (6) purpose, on each subinterval having at least one irregular point, the function it is replaced by a secant line (Fig. 2). Function $g(\ )$ equals to $f(\ )$ times either $\cos(\ )$ or $\sin(\ )$ for error analysis of either real or imaginary parts of (6), subsequently. Furthermore, functions e.g. $f(\ )$, $g(\ )$ mean the modified (i.e. replaced) by secants functions. For the modified functions the discontinuity points of first derivative could appear only at the ends of subintervals $[t_k, t_k + \Delta t]$. This replacement does not change functions values at the beginnings of each subinterval.

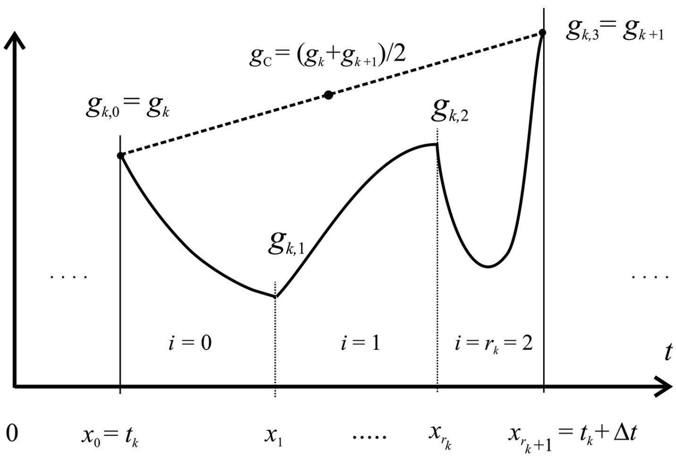


Fig. 2. Error analysis of coefficients $\underset{(\mathrm{DFT})}{c_h}$ given by DFT – points and subintervals notations

Definition (6) for shifted and periodic samples set leads to the same results as the classical DFT with $\delta = 0$ i.e. the coefficients modules $|c_h|$ are the same but the arguments are shifted. However, for non-periodic samples set the definition (6) is essentially different from the classical one.

The real part of complex Fourier coefficients error e_k on k^{th} subinterval for h^{th} harmonic (h and N are not denoted explicitly) is equal to

$$\begin{aligned}\mathrm{Re}\{e_k\} = \frac{1}{T} f(k\Delta t + \delta)\cos(h\omega k\Delta t)\Delta t - \\ - \frac{1}{T}\int_{t_k}^{t_k+\Delta t} f(t)\cos(h\omega t)\mathrm{d}t.\end{aligned} \tag{12}$$

The multiplication of function $f(\ )$ of class C^1 and function $\cos(\ )$ is the function $g_{\mathrm{c}}(\ )$ of class C^1, too. For the imaginary part of error analysis the function is multiplied by $\sin(\ )$ and gives function $g_{\mathrm{s}}(\ )$ of class C^1.

The well-known theorem about mean value on the closed interval [1] firstly for the integral of $g_{\mathrm{c}}(t)$, and secondly for the difference $g_{\mathrm{c}}(k\Delta t + \delta) - g_{\mathrm{c}}(\tilde{t}) = g_{\mathrm{c}}'(\tilde{\tilde{t}})\Delta t$ leads to the equalities

$$\mathrm{Re}\{e_k\} = [g_{\mathrm{c}}(k\Delta t + \delta) - g_{\mathrm{c}}(\tilde{t})]/N = g_{\mathrm{c}}'(\tilde{\tilde{t}})\Delta t/N, \tag{13}$$

where $\tilde{t}$ is a point on $[t_k, t_k + \Delta t]$, $\tilde{\tilde{t}}$ is a certain point placed between points $k\Delta t + \delta$ and $\tilde{t}$ (is on $[t_k, t_k + \Delta t]$, too). According to function $g_{\mathrm{c}}(\ )$ periodicity is satisfied for the whole time-period $T[t_0, t_0 + T]$ implication

$$\sum_{k=0}^{N-1} \Delta g_{\mathrm{c}k} = 0 \;\Rightarrow\; \sum_{k=0}^{N-1} g_{\mathrm{c}}'(\tilde{t}_k)\Delta t = 0, \tag{14}$$

where $\tilde{t}_k$ denotes a certain point on k^{th} subintervals. The sum of real parts of errors from (13) – for harmonic h – is equal

$$\operatorname{Re}\{\underset{(\mathrm{DFT})}{c_h} - c_h\} = \sum_{k=0}^{N-1} \operatorname{Re}\{e_k\} = T\sum_{k=0}^{N-1} g_c'(\tilde{t}_k)/N^2 + \\ + T\sum_{k=0}^{N-1}\left(g_c'(\tilde{\tilde{t}}) - g_c'(\tilde{t}_k)\right)/N^2. \tag{15}$$

Due to (14) and continuity of derivative $g'(\,)$ one obtains

$$\left|\operatorname{Re}\{\underset{(\mathrm{DFT})}{c_h} - c_h\}\right| \le T\sum_{k=0}^{N-1} M_k \left|\tilde{\tilde{t}} - \tilde{t}_k\right|/N^2, \tag{16}$$

thus

$$\left|\operatorname{Re}\{\underset{(\mathrm{DFT})}{c_h} - c_h\}\right| \le T^2\sum_{k=0}^{N-1} M_k/N^3 = M_{\mathrm{av}}T^2/N^2, \tag{17}$$

where M_{av} is the arithmetic mean value of Lipschitz constants M_k for subinterval $[t_k, t_k + \Delta t]$ (is a limited number).

Analogous inequality is satisfied for imaginary parts of errors. Hence, for complex coefficients the following inequality holds

$$\left|\underset{(\mathrm{DFT})}{c_h} - c_h\right| \le MT^2/N^2. \tag{18}$$

On the limited number of subintervals $[t_k, t_k + \Delta t]$ the function $f(\,)$ was replaced by the secant line, for which the real part of error is limited by $TM_k|\tilde{\tilde{t}} - \tilde{t}_k|/N^2$. The corrections denoted by Δe_k for real part are graphically represented by the area between the function and the secant (Fig. 2; r_k denotes the number of points with discontinuous first derivative) and they are equal to the sum of $(r_k + 1)$ differences as follows

$$\begin{aligned}\Delta e_k &= \sum_{i=0}^{r_k} g_c(\xi_i)(x_{i+1} - x_i) - 0.5(g_{ck} + g_{ck+1})\Delta t = \\ &= \sum_{i=0}^{r_k}\left(g_c(\xi_i) - g_{ck,i}\right)(x_{i+1} - x_i) - \\ &- \sum_{i=0}^{r_k}(g_{ck,i} - g_{c0})(x_{i+1} - x_i) - \\ &- \sum_{i=0}^{r_k}(g_{ck\mathrm{av}} - g_{k,0})(x_{i+1} - x_i) = \\ &= \sum_{i=0}^{r_k}\left(g_{ck}(\xi_i) - g_{ck,i}\right)(x_{i+1} - x_i) - \\ &= \sum_{i=0}^{r_k}(x_{i+1} - x_i)\sum_{j=0}^{i-1}\left(g_{ck,j+1} - g_{ck,j}\right) - \\ &- (g_{ck\mathrm{av}} - g_{k,0})\Delta t.\end{aligned} \tag{19}$$

where $g_{ck\mathrm{av}} = 0.5(g_{ck} + g_{ck+1})$ is the mean value of the function on k^{th} subinterval, ξ_i denotes a certain point on $[x_i, x_{i+1}]$. Each difference appearing in (19) is bounded by module either $|\Delta t|$ or $|g_c'(\,)||\Delta t|$, thus the correction Δe_k of error the inequality holds

$$|\Delta e_k| \le A(\Delta t)^2 = \frac{AT^2}{N^2}. \tag{20}$$

Similar inequality can be written for imaginary part of error correction.

Hence, the inequality (18) is satisfied for continuous and piecewise–smooth functions (now the replacement by the secant is recalled). Finally, from the relations (18) and (20) for continuous and piecewise–smooth functions results relation 1 i.e. equality

$$\lim_{N\to\infty}\sum_{h=0}^{N/2}\left|\underset{(\mathrm{DFT})}{c_h} - c_h\right| = 0. \tag{21}$$

It should be pointed out that the decreasing rule of the order $1/N^2$ given by inequality (18) is faster than the decreasing rule of the order $1/N$ proved for integral piecewise–constant approximation [2, 4]. That fact results directly from the relation (14) valid for periodic and piecewise–smooth functions. The equality (26) yields the coefficients convergence

$$\lim_{N\to\infty}\underset{(\mathrm{DFT})}{c_h} = c_h. \tag{22}$$

On the contrary, when on k^{th} subinterval appears discontinuity of the function the error formula (18) is not valid, i.e. the error can not be bounded with the help of decreasing rule of the order $1/N^2$. Subsequently, the real part of error (13) is bounded by the step–change Δ_{ck} as follows

$$\left|\operatorname{Re}\{e_k\}\right| = \left|g_c(k\Delta t + \delta) - g_c(\tilde{t})\right|/N \le \left|\Delta_{ck}\right|/N, \tag{23}$$

where the errors bounded by decreasing rule of $1/N^2$ are omitted because they do not contribute to the error sum, finally. Hence, for all discontinuity points

$$\left|\operatorname{Re}\{\underset{(\mathrm{DFT})}{c_h} - c_h\}\right| \le \sum_{k=0}^{s}\left|\Delta_{ck}\right|/N, \tag{24}$$

where s is the number of all discontinuity points (steps) on interval $[t_0, t_0 + T]$.

Analogously, for imaginary parts

$$\left|\operatorname{Im}\{\underset{(\mathrm{DFT})}{c_h} - c_h\}\right| \le \sum_{k=0}^{s}\left|\Delta_{sk}\right|/N, \tag{25}$$

where Δ_{sk} denotes the step-change of function $f(\,)$ multiplied by $\sin(\,)$ i.e. $g_s(\,)$ – see comments below (12). The inequalities (24) and (25) lead to the coefficients convergence in the form of (22), too. In this case in spite of relation (21) for the function having s discontinuity points is valid the following relation 2 i.e. inequality

$$\sum_{h=0}^{N/2}\left|\underset{(\mathrm{DFT})}{c_h}-c_h\right| \leq \sum_{k=0}^{s}\left(0.5+N^{-1}\right)\left(\left|\Delta_{ck}\right|+\left|\Delta_{sk}\right|\right), \quad (26)$$

because the sum of errors bounded according to formula (17) is omitted as the term decreasing stronger then $1/N$. The sum is convergent (as bounded and not decreasing [1]), thus exists the limit

$$\lim_{N\to\infty}\sum_{h=0}^{N/2}\left|\underset{(\mathrm{DFT})}{c_h}-c_h\right| \leq \frac{1}{2}\sum_{k=0}^{s}\left(\left|\Delta_{ck}\right|+\left|\Delta_{sk}\right|\right). \quad (27)$$

The analysis conclusions and properties of error for generalized DFT coefficients are presented in Table 1 and [8–11].

In order to present the two new formulated properties for generalized DFT coefficients (the last two rows in Table 1) a global error E_G of the sum of modulus is defined in the form of

$$E_G \overset{\mathrm{df}}{=} \sum_{h=0}^{N/2}\left|\mathrm{Re}\left\{\underset{(\mathrm{DFT})}{c_h}-c_h\right\}\right| + \mathrm{j}\sum_{h=0}^{N/2}\left|\mathrm{Im}\left\{\underset{(\mathrm{DFT})}{c_h}-c_h\right\}\right|. \quad (28)$$

Exemplary, the generated sample series are considered. The first example is basing on two-pulse (full-wave) function (Fig. 3). For Relation 1 (equality) it is satisfied and the global error E_G vanishes, respectively (Table 2). The second example is for step-changes function (Fig. 4) – Relation 2 (inequality) is satisfied and global error E_G does not vanish (Table 3). The

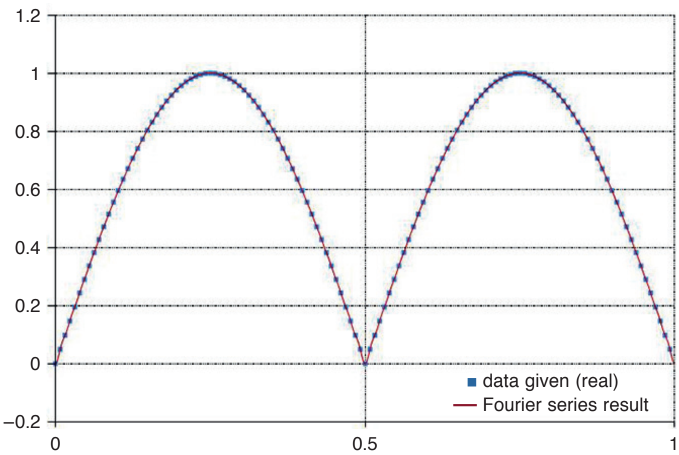


Fig. 3. Example of continuous function – two-pulse (full-wave) curve (points – samples, line – Fourier series)

Table 1
Generalized Fourier coefficients analysis

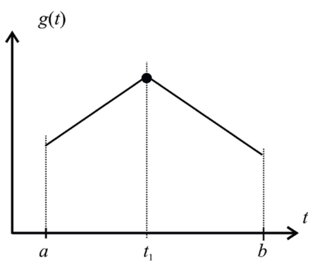
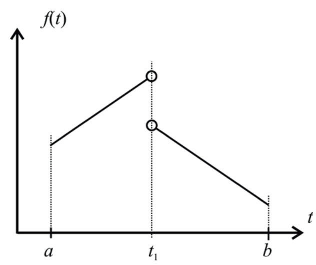
Assumptions satisfied for	theorem I	theorem II
The sample function		
Convergence of Fourier series	Fourier series converges absolutely $\lim_{N\to\infty}\left\|S_N(t)\right\| < \infty$ and uniformly on the interval Δ $\lim_{N\to\infty} S_N(t) \mapsto f(t)$	Fourier series converges at each point $t\in\Delta$ $\lim_{N\to\infty} S_N(t) = \frac{1}{2}\left(f(t_-)+f(t_+)\right)$
Gibbs phenomenon	Does not appear	Appears at discontinuity point
Coefficients convergence $\lim_{N\to\infty}\underset{(\mathrm{DFT})}{c_h}$	$\lim_{N\to\infty}\underset{(\mathrm{DFT})}{c_h} = c_h$	$\lim_{N\to\infty}\underset{(\mathrm{DFT})}{c_h} = c_h$
The two new relations for generalized DFT coefficients modules sum $\lim_{N\to\infty}\sum_{h=0}^{N/2}\left\|\underset{(\mathrm{DFT})}{c_h}-c_h\right\|$	Relation 1: Equality $\lim_{N\to\infty}\sum_{h=0}^{N/2}\left\|\underset{(\mathrm{DFT})}{c_h}-c_h\right\| = 0$	Relation 2: Inequality $\lim_{N\to\infty}\sum_{h=0}^{N/2}\left\|\underset{(\mathrm{DFT})}{c_h}-c_h\right\| \leq \frac{1}{2}\sum_{k=0}^{s}\left(\left\|\Delta_{ck}\right\|+\left\|\Delta_{sk}\right\|\right)$

Table 2
Global error EG between DFT and theoretical values of coefficients – continuous function

Number of samples N ($h_{max} = N/2$)	Samples of two-pulse (full-wave) function – Fig. 3	
	Real part	Imaginary part
$\delta = 0$		
$N = 2^5 = 128$	0.0228252788779130	0.0000000000000002
$N = 2^{10} = 1\,024$	0.0006244818592422	0.0000000000000214
$N = 2^{15} = 32\,768$	0.0000194308084530	0.0000000000020136
$N = 2^{20} = 1\,048\,576$	0.0000006071617167	0.0000000001370209
$N = 2^{25} = 33\,554\,432$	0.0000000193399901	0.0000000016686864
$\delta = \Delta t/2$		
$N = 2^5 = 128$	0.0067326078091597	0.0093678784334546
$N = 2^{10} = 1\,024$	0.0001958442739652	0.0002533303009525
$N = 2^{15} = 32\,768$	0.0000061050462541	0.0000078797609253
$N = 2^{20} = 1\,048\,576$	0.0000001907461125	0.0000002463437141
$N = 2^{25} = 33\,554\,432$	0.0000000060411680	0.0000000093625601

Table 3
Global error E_G between DFT and theoretical values of coefficients – step function

Number of samples N ($h_{max} = N/2$)	Samples of step-changes function – Fig. 4	
	Real part	Imaginary part
$\delta = 0$		
$N = 2^5 = 128$	0.4999999999999990	0.1434132784778930
$N = 2^{10} = 1\,024$	0.4999999999999470	0.1437429132295530
$N = 2^{15} = 32\,768$	0.5000000000063420	0.1437432391969320
$N = 2^{20} = 1\,048\,576$	0.4999999993747650	0.1437432400165020
$N = 2^{25} = 33\,554\,432$	0.4999999902712390	0.1437432454268770
$\delta = \Delta t/2$		
$N = 2^5 = 128$	0.0631536391042182	0.0341652413292065
$N = 2^{10} = 1\,024$	0.0636723443287259	0.0334916406441515
$N = 2^{15} = 32\,768$	0.0636728497689771	0.0334909796137696
$N = 2^{20} = 1\,048\,576$	0.0636728508794668	0.0334909784892709
N = 225 = 33 554 432	0.0636728600809515	0.0334909739728020

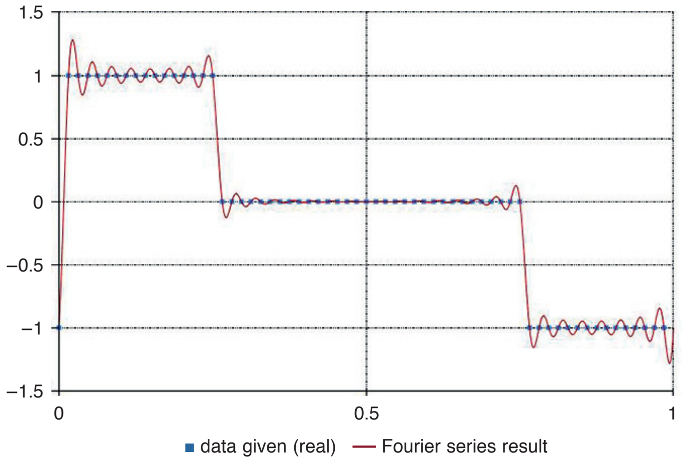


Fig. 4. Example of discontinuous function – step-changes curve (points – samples, line – Fourier series)

shift δ does not change these two relations. The global error E_G either vanishes to zero (Relation 1) or is limited (Relation 2) while N increases infinitely.

Moreover, in Fig. 5 the Fourier coefficients theoretical values for $N = 64$ are presented by crosses and DFT coefficients values by columns, respectively. One can see the small difference between real parts of DFT coefficients and theoretical values, respectively.

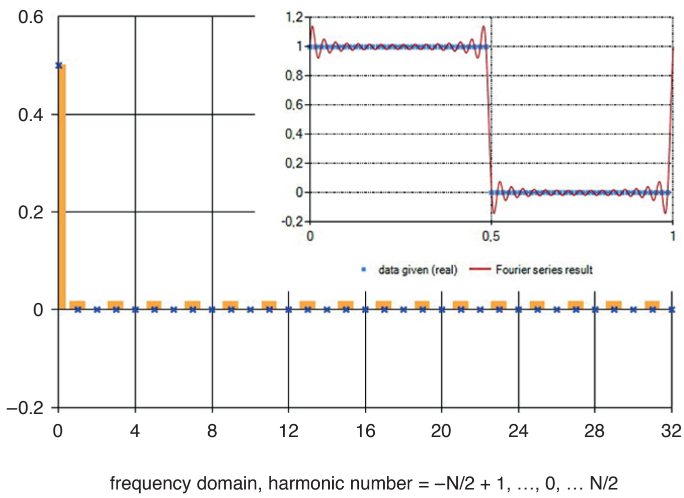


Fig. 5. Fourier coefficients real parts: theoretical values (crosses) and DFT values (columns) for step-change rectangular function (points – samples, line – Fourier series)

4. Aliasing examples for discrete Fourier transform at Shannon frequency

The samples shift δ in (6) change DFT coefficients values. In order to show this influence on generalized DFT coefficients values is considered for an trigonometric polynomial as follows

$$f_k = 1\cos(2\pi f k\Delta t) + 5\sin(2\pi f 5k\Delta t) + \\ + 2\sin(2\pi f 16k\Delta t). \tag{29}$$

The N samples f_k are gathered at even-distanced points tk where $\Delta t = T/N = 1/(Nf)$.

The generalized DFT yields the Fourier harmonics coefficients. The magnitude of highest harmonic depends on samples shift $\delta \in [0, \Delta t]$. Exemplary, there are presented two cases showing the aliasing phenomenon for shift $\delta = 0$ (Fig. 6) and $\delta = \Delta t/2$ (Fig. 7). The highest harmonics value strongly depends on the shift δ. It may vanish (Fig. 6) or it takes the particular value (Fig. 7).

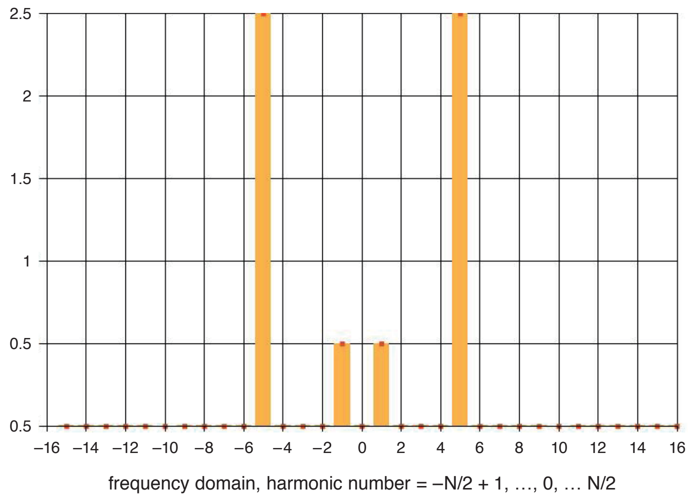


Fig. 6. DFT $N = 32$ – limit case; samples satisfy Shannon theorem assumption $\Delta t = T/N = 1/(Nf) = 1/(32f) \le 1/(2f_{max}) = 1/(2 \cdot 16f)$ and are placed at the beginnings of each interval $[0, \Delta t]$ i.e. $\delta = 0$. 16^{th} harmonics do not appear at all – aliasing

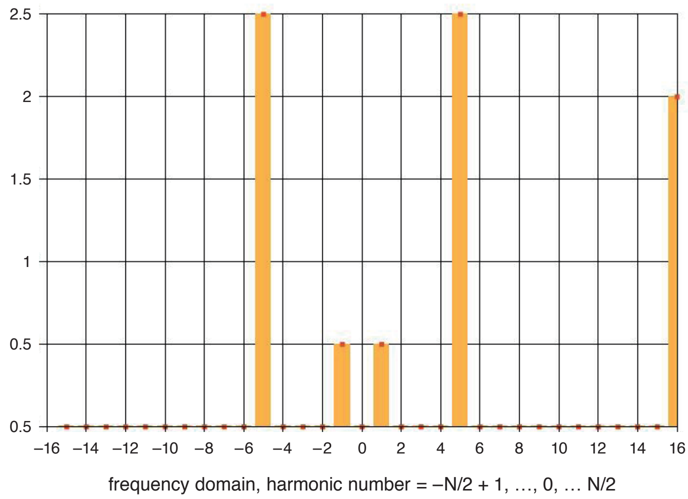


Fig. 7. DFT $N = 32$ – limit case; samples satisfy Shannon theorem assumption $\Delta t = T/N = 1/(Nf) = 1/(32f) \le 1/(2f_{max}) = 1/(2 \cdot 16f)$ and are placed in the middle of each interval $[0, \Delta t]$ i.e. $\delta = \Delta t/2$. The value of 16^{th} harmonic magnitude is as double as it is given by (29) – aliasing

5. Conclusions

Two new relations for generalized discrete Fourier transform DFT are presented, allowing for the samples shift $\delta \in [0, T/N]$ as defined by (6).

Two relations are proved for the sum of errors between generalized DFT coefficients and theirs theoretical values (evaluated analytically) – (28).

The first relation is the equality (21) for samples of continuous and piecewise–smooth functions.

The second is the inequality (27) for samples generated by discontinuous functions.

Moreover, the influence of sample shift on generalized DFT coefficients values is presented. The samples shift δ leads to aliasing phenomenon for highest harmonic (Figs. 6–7).

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Appendix

Theorem I. For periodic function $f(\,)$ that satisfies the Dirichlet condition, i.e. the quotient

$$g(t, u) \overset{\text{df}}{=} \frac{f(t+u) - f(t_{\pm})}{u}, \tag{31}$$

is absolutely integrable for $u > 0$ (and $u < 0$) where $t \in \Delta = [a, b] = [a, a + T]$, the series $S_N(\,)$.

$$S_N(t)\frac{a_0}{2} + \sum_{h=1}^{N} \{a_h \cos(h\omega t) + b_h \sin(h\omega t)\}, \tag{32}$$

is convergent to arithmetic mean of left- and right-sided limits at point t

$$S_N(t) \longrightarrow \frac{f(t_-) + f(t_+)}{2}, \tag{33}$$

having the coefficients 0.

Theorem II. If the periodic function $f(\,)$ is continuous and piecewise–smooth (i.e. piecewise of the class C^1 has a bounded derivative which is continuous everywhere except at a finite number of points at which left- and right-sided derivatives exist), then Fourier series $S_N(\,)$ is uniformly and absolutely convergent on interval Δ.