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Stability analysis of engineering/physical dynamic systems using residual energy function

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In this article, an engineering/physical dynamic system including losses is analyzed in relation to the stability from an engineer's/physicist's point of view. Firstly, conditions for a Hamiltonian to be an energy function, time independent or not, is explained herein. To analyze stability of engineering system, Lyapunov-like energy function, called residual energy function is used. The residual function may contain, apart from external energies, negative losses as well. This function includes the sum of potential and kinetic energies, which are special forms and ready-made (weak) Lyapunov functions, and loss of energies (positive and/or negative) of a system described in different forms using tensorial variables. As the Lypunov function, residual energy function is defined as Hamiltonian energy function plus loss of energies and then associated weak and strong stability are proved through the first time-derivative of residual energy function. It is demonstrated how the stability analysis can be performed using the residual energy functions in different formulations and in generalized motion space when available. This novel approach is applied to RLC circuit, AC equivalent circuit of Gunn diode oscillator for autonomous, and a coupled (electromechanical) example for nonautonomous case. In the nonautonomous case, the stability criteria can not be proven for one type of formulation, however, it can be proven in the other type formulation.

Key words: stability analysis, residual energy function as Lyapunov function, physical dynamic systems, coupled engineering systems

1. Introduction

Energy functions to investigate the stability properties of dynamic systems are known for a relatively long time while a non-tensorial approach is used. When assuming the simplified theory, for which losses are ignored, it is necessary to consider system dynamics and the energies within the system. The Lyapunov's Direct Method (also called second method) has much in common with the theory of Hamiltonian systems. In such systems, the concept of energy function plays a central role, where the energy function consists of summation of available potential and kinetic energies. Here, Hamiltonians are of interest,

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which represent the sum of energies, constant or not with respect to time t, of the entire system together with loss of energies. However, not every Hamiltonian is the sum of energies. Moreover, Hamiltonian representing sum of energies does not need to be constant. Depending on constraints and variables, different Hamiltonians are obtained, two of which represent the sum of energies. One of them is constant with respect to the variable t, while the other is not. Also, one can not find many engineering/physical examples in the scientific literature.

Basic and detailed introduction to Lyapunov's second (or direct) method and stability of motion are given in [1–8]. A Hamiltonian-type Lyapunov function as a pure Hamiltonian consisting of the sum of potential and kinetic energies are investigated in [9–12], where dissipation may also be included [11–12] . However, the conditions to be met in order for a Hamiltonian to be the sum of energy functions (potential and kinetic energies) are ignored. In some of the works, e.g. [13–15], a stability analysis using the sum of potential and kinetic energies, scalar energy-like function of the state more general than quadratic forms or energy like Lyapunov functions is derived. Such a function may be viewed as generalized Lyapunov energy function or generalized distance function, without considering the concept as energy function in form of the Hamiltonian where losses are not taken into account. An example where friction is included is shown in [16], where the Lyapunov candidate is in the form of energy function. However, all the system energies (positive and negative) are not covered in this energy function. The approach considering losses for stability analysis using the sum of kinetic and potential energies including generalized velocity-proportional (Rayleigh) dissipation function (in tensorial forms) is explained in [17]. The proper conditions for a Hamiltonian to be the sum of energy functions are given in [18] and this work, together with [19] are some of the best sources on Lagrange and Hamiltonian formalisms and dissipative force and energy in a lossy case. Considering losses in examples in an obvious way is presented in [20] together with Lagrangian and Hamiltonian control systems.

In [21–22], detailed introduction to energy functions for coupled systems as Lagrangians and Hamiltonians are given. These works reveal how different forms of Lagrangians and Hamiltonians using tensor analysis are derived for an engineering (or a physical) system in an analytical way. Some applications and examples using Lagrangians of some multipoles and extended Hamiltonians, particularly in electrical engineering is presented in [22]. The extended Hamiltonian obtained through extended Lagrangian, both of which are in tensor analytic forms is shown in [23], where also equations of generalized motion in dissipative cases are derived directly in different tensorial forms and formulations containing an example with higher order elements. [24–28] give the theory of electrical engineering based on Lagrange and Hamilton formalisms including generalized motion space. Negative resistance oscillators are given in [29].

The existing approaches in literature suffer form a lack of clear formulation of the energy function which includes all energies (positive and negative). A typical method for derivation of a Lyapunov function is trial and error. However, there are also natural Lyapunov functions as energy functions in a form of Hamiltonian plus loss of energies which we call Residual Energy Function (REF) in autonomous case. It is still trial and error approach, when nonautonomous Lypunov candidate is to find using REF. To the best of our knowledge, a total energy function, called REF as a Lyapunov candidate including dissipation is originally presented in this paper to analyze an engineering/physical system treatment for stability. For a system to be stable, REF must be positive definite, while its first time-derivative has to be negative semidefinite. For asymptotic stability, it has to be negative definite. The author hereby investigates the stability properties of a system using REF with examples, time dependent or not that describes system dynamics of the entire system in various tensorial forms and formulations. In some cases, for example here in time dependent case, depending of formulation type a negative definite function fulfilling the stability criteria may be available while in other type of formulation may not.

2. Holonomic constraints and sum of energy functions as Hamiltonians

Relation between generalized coordinates $q^k(t)$, generalized velocities $\dot{q}^k(t)$ and time t is given in the following general vectorial form (the variables are defined in connection with tensors: superscript k means contravariant while subscript k means covariant form, $k \in \mathbf{N}^+$, $k = 1, \ldots, f$, f is the degree of freedom):

$$\begin{aligned} f_I(\widetilde{\mathbf{q}}, \dot{\widetilde{\mathbf{q}}}, t) &= 0, \\ \widetilde{\mathbf{q}} &= [q^1(t),\ q^2(t),\ \ldots,\ q^f(t)], \\ \dot{\widetilde{\mathbf{q}}} &= [\dot{q}^1(t),\ \dot{q}^2(t),\ \ldots,\ \dot{q}^f(t)], \end{aligned} \tag{1}$$

where subscript I: $1 \ldots K$, K – number of the equations, $K < 3f$, and whereas the functions f_I are continuous and differentiable with respect to their variables. The constraints must be formulated in such a way that the equations below

$$f_I = 0, \qquad \frac{\mathrm{d}\, f_I}{\mathrm{d}\, t} = 0 \tag{2}$$

of generalized motion to describe fulfill $2K$ limitations.

Holonomic constraints are special cases of the general form of (1), $f_I(\widetilde{\mathbf{q}}, t) = 0$: holonomic rheonomic, $f_I(\widetilde{\mathbf{q}}) = 0$: holonomic sclerenomic. Such equations are (holonomic) kinematic relations. In every holonomic case (also called integrable, but integrable does not mean holonomic), the degree of freedom can be reduced depending on the relations between the generalized coordinates.

The constraints which appear as velocity dependent are actually differential equations that can be integrated to give simply holonomic constraints, e.g. in electrical circuits, the constraints are integrable but not holonomic. However, they can be modified by integration to give a holonomic constraint. Otherwise, they are nonholonomic. Finally, a holonomic system is a dynamic system whose constraint equations, if any, are all of the holonomic form.

In this study, $H^+ = H^+(\mathbf{p}, \widetilde{\mathbf{q}}, t)$ form of Hamiltonian and $L^+ = L^+(\dot{\widetilde{\mathbf{q}}}, \widetilde{\mathbf{q}}, t)$ form of Lagrangian are used to obtain measurable results. Here in this paper, the Lagrangian L stands for the form of Lagrangian L^+.

A system is called a Hamiltonian system, if there exists a function $H^+(\mathbf{p}, \widetilde{\mathbf{q}}, t)$ such that

$$\dot{\widetilde{\mathbf{q}}} = \nabla_{\mathbf{p}} H^+(\mathbf{p}, \widetilde{\mathbf{q}}, t), \qquad \dot{\mathbf{p}} = -\nabla_{\widetilde{\mathbf{q}}} H^+(\mathbf{p}, \widetilde{\mathbf{q}}, t), \tag{3}$$

where $\nabla_{\widetilde{\mathbf{q}}} H^+ = \left(\dfrac{\partial H^+}{\partial q^1} \;\; \dfrac{\partial H^+}{\partial q^2} \;\; \cdots \;\; \dfrac{\partial H^+}{\partial q^f} \right)$ denotes the gradient of H^+ with respect to $\widetilde{\mathbf{q}}$, etc., and $\nabla_{\dot{\widetilde{\mathbf{q}}}} L = \left(\dfrac{\partial L}{\partial \dot{q}^1} \;\; \dfrac{\partial L}{\partial \dot{q}^2} \;\; \cdots \;\; \dfrac{\partial L}{\partial \dot{q}^f} \right) = [p_1(t) \;\; p_2(t) \;\cdots\; p_f(t)] = \mathbf{p}$ which is the generalized momentum vector. A Hamiltonian system consists of $2f$ differential equations of first order and necessarily has an even dimension $2f$. A necessary and sufficient condition for a function to be a Hamiltonian is

$$\nabla_{\widetilde{\mathbf{q}}}(\nabla_{\mathbf{p}} H^+) - \nabla_{\mathbf{p}}(\nabla_{\widetilde{\mathbf{q}}} H^+) = 0. \tag{4}$$

In dissipative (or lossy) cases, equations of generalized motion are derived either through (classical) Hamiltonian $H^+(\mathbf{p}, \widetilde{\mathbf{q}}, t)$ and generalized velocity proportional (Rayleigh) dissipative function $D(\dot{\widetilde{\mathbf{q}}})$ or directly through the extended Hamiltonian $\mathscr{H}^+(\mathbf{p}, \widetilde{\mathbf{q}}, t)$ as below:

$$\begin{aligned} \dot{\widetilde{\mathbf{q}}} &= \nabla_{\mathbf{p}} H^+(\mathbf{p}, \widetilde{\mathbf{q}}, t) & \dot{\mathbf{p}} &= -\nabla_{\widetilde{\mathbf{q}}} H^+(\mathbf{p}, \widetilde{\mathbf{q}}, t) - \nabla_{\dot{\widetilde{\mathbf{q}}}} D(\dot{\widetilde{\mathbf{q}}}) \\ &= \nabla_{\mathbf{p}} \mathscr{H}^+(\mathbf{p}, \widetilde{\mathbf{q}}, t), & &= -\nabla_{\dot{\widetilde{\mathbf{q}}}} \mathscr{H}^+(\mathbf{p}, \widetilde{\mathbf{q}}, t). \end{aligned} \tag{5}$$

The scalar functions $H^+(\mathbf{p}, \widetilde{\mathbf{q}}, t)$, $\mathscr{H}^+(\mathbf{p}, \widetilde{\mathbf{q}}, t)$ are assumed to have continuous second derivatives.

A Hamiltonian may include external force(s) as negative potential function(s), which will not be treated in this paper.

2.1. Conservative case

If the kinetic energy $T(\dot{\widetilde{\mathbf{q}}})$ of the Lagrangian L of a closed system, i.e. autonomouos, is the only part containing $\dot{\widetilde{\mathbf{q}}}$, that is

a) The potential energy is independent of generalized velocity, i.e. $V = V(\widetilde{\mathbf{q}})$;

b) The kinetic energy $T(\dot{\widetilde{\mathbf{q}}})$ is a homogeneous quadratic function of $\dot{\widetilde{\mathbf{q}}}$ (positive definite quadratic form);

c) The Lagrangian L is invariant with respect to a change in the coordinate time (homogeneity in time), $\dfrac{\partial L}{\partial t} = 0$.

Then the Lagrangian takes the form with prerequisites as shown:

$$L(\widetilde{\mathbf{q}}, \dot{\widetilde{\mathbf{q}}}) = T(\dot{\widetilde{\mathbf{q}}}) - V(\widetilde{\mathbf{q}}) \quad \text{whereas} \quad \det\left[\nabla_{\dot{\widetilde{\mathbf{q}}}}(\nabla_{\dot{\widetilde{\mathbf{q}}}} L)\right] \neq 0. \tag{6}$$

And if the holonomic constraints are scleronom, then the Hamiltonian H^+ can be obtained through Legendre transform which is a constant with respect to time t. And H^+ is equal to sum of potential and kinetic energies, the energy is an integral of generalized motion. With the following prerequisites in vector notation taken into account

$$f_I(\widetilde{\mathbf{q}}) = 0, \quad \frac{\partial L}{\partial t} = 0 \;\Leftrightarrow\; L = T(\dot{\widetilde{\mathbf{q}}}) - V(\widetilde{\mathbf{q}}), \quad \det\left[\nabla_{\dot{\widetilde{\mathbf{q}}}}(\nabla_{\dot{\widetilde{\mathbf{q}}}} L)\right] \neq 0. \tag{7}$$

Legendre transform leads to Hamiltonian given below:

$$\begin{aligned} H^+ &= T(\dot{\widetilde{\mathbf{q}}}) + V(\widetilde{\mathbf{q}}) \\ &= T(\mathbf{p}) + V(\widetilde{\mathbf{q}}) \\ &= \text{const}\Big|_t \;\Leftrightarrow\; \dot{H}^+ \equiv \frac{\mathrm{d}H^+}{\mathrm{d}t} = \frac{\partial H^+}{\partial t} = 0. \end{aligned} \tag{8}$$

A constant Hamiltonian $H^+ = H^+(\mathbf{p}, \widetilde{\mathbf{q}})$ is a Jacobian function.

2.2. Nonconservative case

Provided that the potential energy does not depend on the generalized velocities and the holonomic constraints are sclerenomic, in cases where Lagrangian L does depend explicitly on time t, the most general form of the Lagrangian in this case is

$$L(\widetilde{\mathbf{q}}, \dot{\widetilde{\mathbf{q}}}, t) = T(\dot{\widetilde{\mathbf{q}}}, t) - V(\widetilde{\mathbf{q}}, t) \quad \text{with} \quad \det\left[\nabla_{\dot{\widetilde{\mathbf{q}}}}(\nabla_{\dot{\widetilde{\mathbf{q}}}} L)\right] \neq 0. \tag{9}$$

If the form of the Hamiltonian is either generalized position and velocity dependent as

$$H^+(\widetilde{\mathbf{q}}, \dot{\widetilde{\mathbf{q}}}, t) = T(\dot{\widetilde{\mathbf{q}}}, t) + V(\widetilde{\mathbf{q}}, t) \tag{10}$$

or as is usual in generalized momentum and position dependent form

$$H^+(\mathbf{p}, \widetilde{\mathbf{q}}, t) = T(\mathbf{p}, t) + V(\widetilde{\mathbf{q}}, t) \tag{11}$$

then, the Hamiltonian H^+ obtained through Legendre transform is not a constant with respect to the variable time t, i.e. H^+ does depend explicitly on the time t. In this case H^+ is equal to the sum of potential and kinetic energies, but not constant. Symbolically, assuming the prerequsites shown below

$$f_I(\widetilde{\mathbf{q}}) = 0, \quad \frac{\partial L}{\partial t} \neq 0 \;\Leftrightarrow\; L = T(\dot{\widetilde{\mathbf{q}}}, t) - V(\widetilde{\mathbf{q}}, t), \quad \det\left[\nabla_{\dot{\widetilde{\mathbf{q}}}}(\nabla_{\dot{\widetilde{\mathbf{q}}}} L)\right] \neq 0. \tag{12}$$

Legendre transform yields Hamiltonian

$$\begin{aligned} H^+ &= T(\mathbf{p}, t) + V(\widetilde{\mathbf{q}}, t) \\ &\neq \text{const}\Big|_t \;\Leftrightarrow\; \frac{\mathrm{d}H^+}{\mathrm{d}t} = \frac{\partial H^+}{\partial t} \neq 0. \end{aligned} \tag{13}$$

In both cases, i.e. conservative and nonconservative ones, it is mathematically possible that kinetic energy may also appear in generalized position dependent form $T(\dot{\widetilde{\mathbf{q}}}, \widetilde{\mathbf{q}})$ without disturbing energy properties of Hamiltonian depending on the type of formulation.

3. Stability through residual energy function H^+

In some of the literature, for example [13, 14], authors assert incorrectly that the sum of kinetic and potential energies [assumed as (weak) Lyapunov function] of a system (Hamiltonian under certain conditions) as defined above in (7) and (12), will decrease in time. Their calculations do not cover any (generalized) velocity proportional dissipation function at all. In some other literature such as [15] and [16], authors mention losses (dissipation functions), so as the lost energy is shown while total energy functions including the dissipation are excluded. Total energy in this complete form is called **R**esidual **E**nergy **F**unction. The aim of this study is to complete the stability concept with dissipation function in a form of easy, practical and applicable manner so it is easy to follow. As it is known, the time derivative of a conservative Hamiltonian is always zero, i.e., marginal stability but not asymptotical stability is assured. A pure Hamiltonian system cannot be asymptotically stable. Hamiltonian only as the sum of energy functions without loss of energies is not sufficient. Instead, the sum of the potential and kinetic energy functions plus loss of energies must be used. Since dissipative force(s) derived from generalized velocity proportional (Rayleigh) dissipation function are negative in form

$$F_D = -\nabla_{\dot{\widetilde{\mathbf{q}}}} D(\dot{\widetilde{\mathbf{q}}}), \tag{14}$$

so the loss of energy in a related dissipative element (also in accordance with extended Euler-Lagrange differential equation) is negative as below:

$$-\int\left[\frac{\partial D(\dot{\widetilde{\mathbf{q}}})}{\partial \dot{q}^k}\dot{q}^k\right]\mathrm{d}t. \tag{15}$$

Then, the proposed REF is

$$\mathsf{H}^+ = H^+ - \int\left[\sum_{k=1}^{f}\frac{\partial D(\dot{\widetilde{\mathbf{q}}})}{\partial \dot{q}^k}\dot{q}^k\right]\mathrm{d}t, \qquad 0<\mathsf{H}^+<\infty\ \forall t\in\mathbf{R}_0^+ \tag{16}$$

which is the sum of all energies plus loss of energy in a system and through the relation $p=\dfrac{\partial L}{\partial \dot{q}}$, the dissipative function can be transformed into the generalized momentum dependent form. Such an REF is: $\mathsf{H}^+=\mathsf{H}^+(\mathbf{p},\widetilde{\mathbf{q}})$ in autonomous case and $\mathsf{H}^+=\mathsf{H}^+(\mathbf{p},\widetilde{\mathbf{q}},t)$ in nonautonomous case. Thus, one must use (classical) Hamiltonian plus loss of energies that is REF of a system. We must also distinguish between two cases of Hamiltonians as the sum of energy functions defined in (8) and (13). All the energies and losses (positive and negative) may be included in this term based on Hamiltonian H^+ and generalized velocity proportional Rayleigh dissipation function D, where $[H^+]=$ Watt.sec and $[D^+]=$ Watt. For an engineering/physical system, this energy is converted into another form (dissipation in heat, which is loss of energy) and this residual energy function of the system in all forms must be equal to or less than a finite constant as time changes (marginal stability) or decrease with time (asymptotic stability), i.e.

$$\mathsf{H}^+(\mathbf{p},\widetilde{\mathbf{q}})\leqslant \text{const}\Big|_t. \tag{17}$$

That means this energy never increases for the system to be in a stable state, although it may vary in limited boundaries. The function $\mathsf{H}^+(\mathbf{p},\widetilde{\mathbf{q}})$ can take zero value only when $\mathbf{p}=\widetilde{\mathbf{q}}=\mathbf{0}$. This, together with the fact that $H^+(\mathbf{p},\widetilde{\mathbf{q}})$ vanishes with increasing time which has only one meaning: asymptotic stability. Alternately, based on the behavior of the function $\mathsf{H}^+(\mathbf{p},\widetilde{\mathbf{q}})$, we can draw conclusions concerning the stability of the system. A Hamiltonian system, that is lossless, cannot be asymptotically stable as its stability is in a sense marginal. We apply Barbashin-Krasovskii-LaSalle (invariance) principle and its relation to Lyapunov theory to this approach and arrive to the conclusion that: $(\mathbf{p}=\widetilde{\mathbf{q}}=\mathbf{0})$ is an isolated critical point of an autonomous system, REF H^+ of which has the following properties:

$$\begin{aligned}
&\text{a)}\ \ \mathsf{H}^+(\mathbf{p},\widetilde{\mathbf{q}})>0\ \ \forall p_k\neq 0 \quad\text{and}\quad q^k\neq 0, \quad k\in\{1,\ldots,f\},\\
&\text{b)}\ \ \frac{\partial \mathsf{H}^+}{\partial q^k}\quad\text{and}\quad\frac{\partial \mathsf{H}^+}{\partial p_k}\ \ \text{exist}\ \ \forall q^k,\ p_k,\\
&\text{c)}\ \ \mathsf{H}^+(\mathbf{p},\widetilde{\mathbf{q}})=0,\quad \text{if}\ \ \mathbf{p}=\widetilde{\mathbf{q}}=\mathbf{0}.
\end{aligned} \tag{18}$$

Such a real valued scalar function $\mathsf{H}^+(\mathbf{p},\widetilde{\mathbf{q}})$ is positive definite. Hence, energy in a physical system is a positive definite function of the state and the rate of change of the sum of kinetic and potential energies plus loss of energy. The energy remains either constant or may oscillate between limited values in a marginal stable system and decreases always with (positive) dissipation, i.e.

$$\begin{aligned} &1.\ \mathsf{H}^+(\mathbf{p},\widetilde{\mathbf{q}}):\ \text{ positive definite,}\\ &2.\ \frac{\mathrm{d}\mathsf{H}^+}{\mathrm{d}t}\equiv\dot{\mathsf{H}}^+(\mathbf{p},\widetilde{\mathbf{q}})\leqslant 0\\ &\qquad\qquad \forall\ p_k\neq 0,\ \ q^k\neq 0\in\mathbf{R}^{2f}>\mathbf{0}:\ \text{ negative semidefinite.}\end{aligned} \tag{19}$$

That increases only if the system is unstable.

Similarly, we realize now that asymptotic stability could be predicted if the second property in (19) is substituted by

$$\dot{\mathsf{H}}^+(\mathbf{p},\widetilde{\mathbf{q}})<0\ \ \forall\ p_k,\ q^k\in\mathbf{R}^{2f}>\mathbf{0}. \tag{20}$$

Also note that the above property, namely, the scalar function vanishes with increasing time, is taken for guaranteed, provided that it is in negative definite form, that is, $\dot{\mathsf{H}}^+<0\ \ \forall t$. In that case it is a strong Lyapunov function (asymptotic stability). If the weaker condition in the form $\dot{\mathsf{H}}^+(\mathbf{p},\widetilde{\mathbf{q}})\leqslant 0$ is the case (weak Lyapunov function), the stability is guaranteed while 'asymptotic stability' may still be possible. For an engineering system to be marginally stable, the energy rate may either be zero or may oscillate between limited values. For a system to be asymptotically stable the energy rate is negative. The phase trajectories (loci) in $2f$-dimensional phase space, are $2f$-dimensional ellipses when $\mathsf{H}^+=\text{const}$. As time elapses, these ellipses either stay constant (marginal stability) or they must inevitably shrink (spirals towards the origin: asymptotic stability), provided that the system is stable.

If the REF depends explicitly on time t (nonautonomous case), by applying Barbalat's Lemma for the stability of time-varying systems, (19) and (20) are adjusted as

$$\begin{aligned}&\mathsf{H}(\mathbf{p},\widetilde{\mathbf{q}},t)\geqslant W(\mathbf{p},\widetilde{\mathbf{q}})\ \ \forall\, t\geqslant 0,\ \ \forall\mathbf{p},\ \widetilde{\mathbf{q}}\in\mathbf{R}^{2f}>\mathbf{0}\\ &\qquad\qquad\qquad\qquad\text{with}\ \ [\mathsf{H}(\mathbf{p},\widetilde{\mathbf{q}},t)]\Big|_{\mathbf{p},\widetilde{\mathbf{q}}=0}=0,\end{aligned} \tag{21}$$

where $W(\mathbf{p},\widetilde{\mathbf{q}})$ is a positive definite time independent function. This implies that $\mathsf{H}(\mathbf{p},\widetilde{\mathbf{q}},t)$ is also positive definite. For stability, the (first) time derivative of the REF must fulfill

$$\dot{\mathsf{H}}(\mathbf{p},\widetilde{\mathbf{q}},t)\leqslant\widehat{W}(\mathbf{p},\widetilde{\mathbf{q}}), \tag{22}$$

where $\widehat{W}(\mathbf{p},\widetilde{\mathbf{q}})$ is a negative semidefinite time independent function, while for asymptotic stability

$$\dot{\mathsf{H}}(\mathbf{p},\widetilde{\mathbf{q}},t) < \widehat{W}(\mathbf{p},\widetilde{\mathbf{q}}) \tag{23}$$

and $\widehat{W}(\mathbf{p},\widetilde{\mathbf{q}})$ has a negative definite form, which is sometimes not possible to find as will be shown using the nonautonomous example.

Time derivative of autonomous REF $\mathsf{H}(\mathbf{p},\widetilde{\mathbf{q}})$ can be interpreted geometrically as follows:

$$\begin{aligned}\dot{\mathsf{H}}(\mathbf{p},\widetilde{\mathbf{q}}) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}(\mathbf{p},\widetilde{\mathbf{q}}) &= \frac{\mathrm{d}}{\mathrm{d}t}\left\{H^+(\mathbf{p},\widetilde{\mathbf{q}}) - \int\left[\sum_{k=1}^{f}\frac{\partial D(\dot{\widetilde{\mathbf{q}}})}{\partial\dot{q}^k}\dot{q}^k\right]\mathrm{d}t\right\} \\ &= \sum_{k=1}^{f}\left(\frac{\partial H^+}{\partial p_k}\dot{p}_k + \frac{\partial H^+}{\partial q^k}\dot{q}^k\right) - \frac{\mathrm{d}}{\mathrm{d}t}\int\left[\sum_{k=1}^{f}\frac{\partial D(\dot{\widetilde{\mathbf{q}}})}{\partial\dot{q}^k}\dot{q}^k\right]\mathrm{d}t \\ &= -\sum_{k=1}^{f}\frac{\partial D(\dot{\widetilde{\mathbf{q}}})}{\partial\dot{q}^k}\dot{q}^k. \end{aligned} \tag{24}$$

With superscript T meanining transposed form, the gradient vector is introduced

$$\begin{aligned}\nabla_{\mathbf{p},\widetilde{\mathbf{q}}}H^+(\mathbf{p},\widetilde{\mathbf{q}}) &\equiv \operatorname{grad}H^+(\mathbf{p},\widetilde{\mathbf{q}}) = \\ &= \left(\frac{\partial H^+}{\partial p_1}\ \frac{\partial H^+}{\partial p_2}\ \cdots\ \frac{\partial H^+}{\partial p_f}\ \frac{\partial H^+}{\partial q^1}\ \frac{\partial H^+}{\partial q^2}\ \cdots\ \frac{\partial H^+}{\partial q^f}\right)^T \end{aligned} \tag{25}$$

which is a vector perpendicular to the H^+ surfaces. It is defined as positive along the direction where H^+ increases. Its magnitude is a measure of the rate of increase of H^+ with respect to the generalized coordinates and generalized momenta. Rewriting the first time derivative of an (autonomous) REF in (19) as the scalar product of the gradient vector $\nabla_{\mathbf{p},\widetilde{\mathbf{q}}}H^+$ and generalized force-velocity vector $(\dot{\mathbf{p}}\ \ \dot{\widetilde{\mathbf{q}}})$ one gets:

$$\begin{aligned}\dot{\mathsf{H}}(\mathbf{p},\widetilde{\mathbf{q}}) &= (\nabla_{\mathbf{p},\widetilde{\mathbf{q}}}H^+).(\dot{\mathbf{p}}\ \ \dot{\widetilde{\mathbf{q}}}) - \sum_{k=1}^{f}\frac{\partial D(\dot{\widetilde{\mathbf{q}}})}{\partial\dot{q}^k}\dot{q}^k \\ &\equiv (\operatorname{grad}H^+).(\dot{\mathbf{p}}\ \ \dot{\widetilde{\mathbf{q}}}) - \sum_{k=1}^{f}\frac{\partial D(\dot{\widetilde{\mathbf{q}}})}{\partial\dot{q}^k}\dot{q}^k = -\sum_{k=1}^{f}\frac{\partial D(\dot{\widetilde{\mathbf{q}}})}{\partial\dot{q}^k}\dot{q}^k \\ &= -\left\{\left[\nabla_{\dot{\widetilde{\mathbf{q}}}}D(\dot{\widetilde{\mathbf{q}}})\right]^T.\dot{\widetilde{\mathbf{q}}}\right\} \equiv -\left\{\left[\operatorname{grad}D(\dot{\widetilde{\mathbf{q}}})\right]^T.\dot{\widetilde{\mathbf{q}}}\right\} \end{aligned} \tag{26}$$

since $(\nabla_{\mathbf{p},\widetilde{\mathbf{q}}}H^+).(\dot{\mathbf{p}}\ \ \dot{\widetilde{\mathbf{q}}}) \equiv (\operatorname{grad}H^+).(\dot{\mathbf{p}}\ \ \dot{\widetilde{\mathbf{q}}}) = 0$, due to the properties of a Hamiltonian. That means: a lossless system is always marginally stable. On the other

hand, using a similar approach for a positive dissipation function, the term given below

$$\nabla_{\dot{\tilde{\mathbf{q}}}} D(\dot{\tilde{\mathbf{q}}}) \equiv \operatorname{grad} D(\dot{\tilde{\mathbf{q}}}) = \frac{\partial D(\dot{\tilde{\mathbf{q}}})}{\partial \dot{q}^k} \tag{27}$$

is a vector perpendicular to the D surfaces and in view of the (19) for time independent REF, one obtains

$$-\sum_{k=1}^{f} \frac{\partial D(\dot{\tilde{\mathbf{q}}})}{\partial \dot{q}^k} \dot{q}^k = -\left\{ \left[\nabla_{\dot{\tilde{\mathbf{q}}}} D(\dot{\tilde{\mathbf{q}}}) \right]^T . \dot{\tilde{\mathbf{q}}} \right\} \equiv -\left\{ \left[\operatorname{grad} D(\dot{\tilde{\mathbf{q}}}) \right]^T . \dot{\tilde{\mathbf{q}}}. \right\} \leqslant 0 \tag{28}$$

i.e., the angle θ between the gradient vector of Rayleigh dissipation function and the generalized velocity vector is $\dfrac{3\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}$, while the first time derivative for nonautonomous REF is as what follows

$$\begin{aligned} \dot{\mathsf{H}}(\mathbf{p}, \tilde{\mathbf{q}}, t) &= \frac{\partial H^+}{\partial t} - \left\{ \left[\nabla_{\dot{\tilde{\mathbf{q}}}} D(\dot{\tilde{\mathbf{q}}}) \right]^T . \dot{\tilde{\mathbf{q}}} \right\} \\ &\equiv \frac{\partial H^+}{\partial t} - \left\{ \left[\operatorname{grad} D(\dot{\tilde{\mathbf{q}}}) \right]^T . \dot{\tilde{\mathbf{q}}} \right\} \leqslant 0, \end{aligned} \tag{29}$$

where $-\nabla_{\dot{\tilde{\mathbf{q}}}} D(\dot{\tilde{\mathbf{q}}}) \equiv -\operatorname{grad} D(\dot{\tilde{\mathbf{q}}}) = -\dfrac{\partial D(\dot{\tilde{\mathbf{q}}})}{\partial \dot{q}^k}$ is taken from (or dissipated) and not added to the system for a positive Rayleigh dissipation function.

4. Stability analysis using different form of energy functions

The ideas and approach used above related to stability analysis can also be used exactly in the same manner, whenever different forms of the energy functions are available or derivable. If, for example, the forms $\mathsf{H}^-(\tilde{\mathbf{p}}, \mathbf{q})$ and $D(\dot{\mathbf{q}})$ are available where $\tilde{\mathbf{p}} = (p^1 \;\; p^2 \;\; \ldots \;\; p^f)$ and $\mathbf{q} = (q_1 \;\; q_2 \;\; \ldots \;\; q_f)$, which are derived using metric tensor $[g_{ij}]$ and its inverse $[g^{ij}]$, then the starting point is

$$\mathsf{H}^-(\tilde{\mathbf{p}}, \mathbf{q}) \leqslant \text{const}\Big|_t \tag{30}$$

which yields for autonomous case

$$\begin{aligned} \dot{\mathsf{H}}^-(\tilde{\mathbf{p}}, \mathbf{q}) &= \left(\nabla_{\tilde{\mathbf{p}}, \mathbf{q}} \mathsf{H}^- \right) . (\dot{\tilde{\mathbf{p}}} \;\; \dot{\mathbf{q}}) \equiv (\operatorname{grad} \mathsf{H}^-) . (\dot{\tilde{\mathbf{p}}} \;\; \dot{\mathbf{q}}) \\ &= -\sum_{k=1}^{f} \frac{\partial D(\dot{\mathbf{q}})}{\partial \dot{q}_k} \dot{q}_k = -\left\{ \left[\nabla_{\dot{\mathbf{q}}} D(\dot{\mathbf{q}}) \right] . \dot{\mathbf{q}} \right\} \equiv -\left\{ \left[\operatorname{grad} D(\dot{\mathbf{q}}) \right] . \dot{\mathbf{q}} \right\} \end{aligned} \tag{31}$$

and for nonautonomous case

$$\dot{\mathrm{H}}^{-}(\widetilde{\mathbf{p}},\mathbf{q},t)=\frac{\partial H^{-}}{\partial t}-\left\{\left[\nabla_{\dot{\mathbf{q}}}D(\dot{\mathbf{q}})\right].\dot{\mathbf{q}}\right\}\equiv\frac{\partial H^{-}}{\partial t}-\left\{\left[\operatorname{grad}D(\dot{\mathbf{q}})\right].\dot{\mathbf{q}}\right\}\leqslant 0 \tag{32}$$

when the generalized motion takes place in Euclidean Space. In Euclidean Space, all the elements of metric tensor are constants, i.e. whenever one substitutes p^k, q_k instead of p_k, q^k (the variable time t remains the same) it strictly implies that if a dynamic system is stable in a chosen coordinate system, then it must be stable in another coordinate system in Euclidean Space.

5. Application of the approach

5.1. RLC circuit

For the purpose of investigating the stability properties of a system, we will consider an example of a passive system in the form of an electric circuit consisting of RLC elements in series without any initial stored energy in energy storage elements. First, using system theoretical approach for the system, the stability properties will be shown. Then, the stability property will be determined using the method proposed in this paper.

Charge formulation

Assume that for a circuit consisting of a resistor $R>0$, inductor $L^*>0$ and a capacitor $C>0$ in series, the input of a system is the voltage $u(t)$ and the output is the current $i(t)$. Applying **K**irchoff **V**oltage **L**aw to the circuit, one obtains

$$u(t)=Ri(t)+L^*\frac{\mathrm{d}\,i(t)}{\mathrm{d}\,t}+\frac{1}{C}\int_0^t i(\tau)\mathrm{d}\,\tau. \tag{33}$$

The Laplace transform of the above equation is:

$$U(s)=\left[R+sL^*+\frac{1}{sC}\right]I(s). \tag{34}$$

Thus the transfer/system function, which is the transfer admittance in this case, is

$$H(s)=Y(s)=\frac{I(s)}{U(s)}=\frac{s}{s^2L^*+sR+\dfrac{1}{C}}. \tag{35}$$

The properties of the solutions of a second order equation yields

$$\left.\begin{aligned} s_1.s_2 &= \frac{1}{L^*C} \\ s_1+s_2 &= -\frac{R}{L^*} \end{aligned}\right\} \Rightarrow s_1<0, \quad s_2<0, \quad s_1\neq s_2 \tag{36}$$

which clearly reveals (asymptotic) stability. Such an electric circuit has a degree of freedom $f=1$, which is equal to the number of independent variables. The holonomic constraint is $\dot{q}=\dot{q}_L=\dot{q}_C \Rightarrow q=q_L=q_C$, so that the degree of freedom can not be reduced depending on the relations between the generalized coordinates. The Lagrange-dissipative or more concisely $\{L,D\}$-model in terms of charge formulation without external force(s) together with the prerequisite for Legendre transform of this system are as follows:

$$\begin{aligned} \{L,D\} &= \begin{cases} \dfrac{L^*(\dot{q})^2}{2}-\dfrac{(q)^2}{2C}, \\ R\dfrac{(\dot{q})^2}{2}, \end{cases} \\ \det\left[\frac{\partial^2 L}{\partial(\dot{q})^2}\right] &= \det(L^*)\neq 0 \ \ \forall L^*>0, \end{aligned} \tag{37}$$

where q is the charge, $\dot{q}$ is the current. So, the related REF in generalized position and velocity dependent form is

$$\mathsf{H}(\dot{q},q)=\frac{L^*(\dot{q})^2}{2}+\frac{(q)^2}{2C}-\int R(\dot{q})^2\mathrm{d}t. \tag{38}$$

The REF in generalized momentum and generalized position dependent form for our case is

$$\mathsf{H}(p,q)=\frac{(p)^2}{2L^*}+\frac{(q)^2}{2C}-\int \frac{R}{(L^*)^2}(p)^2\mathrm{d}t \tag{39}$$

and it fulfills all the properties of positive definiteness for generalized motion to take place, where the Hamilton function in both cases, $H=H(\dot{q},q)$ and $H=H(p,q)$, are available as Jacobian integrals. The first time derivative of the REF is as follows:

$$\dot{\mathsf{H}}=-R(\dot{q})^2=-\frac{R}{(L^*)^2}(p)^2 \tag{40}$$

which is in negative definite form. It means that the system is asymptotically stable.

Flux formulation

When the variables are defined in terms of flux formulation, the variables in charge formulation must be substituted in this case as shown below:

$$u(t) \to i(t), \qquad q \to \psi, \qquad \dot{q} \to \dot{\psi}, \qquad L^* \to C, \qquad C \to L^*, \qquad R \to G.$$

The degree of freedom in flux formulation is $f = 1$ again. And the holonomic constraint is as given above. The $\{L, D\}$-model with the prerequisit for Legendre transform yields in what follows:

$$\{L, D\} = \begin{cases} \dfrac{C(\dot{\psi})^2}{2} - \dfrac{(\psi)^2}{2L^*}, \\ G\dfrac{(\dot{\psi})^2}{2}, \end{cases} \qquad \det\left[\frac{\partial^2 L}{\partial(\dot{\psi})^2}\right] = \det(C) \neq 0 \ \ \forall\, C > 0, \tag{41}$$

where ψ is the flux, $\dot{\psi}$ is the voltage. So the related REF in generalized momentum and position dependent form for this case is

$$\mathsf{H}(\dot{\psi}, \psi) = \frac{C(\dot{\psi})^2}{2} + \frac{(\psi)^2}{2L^*} - \int G(\dot{\psi})^2 \mathrm{d}t \tag{42}$$

and in generalized momentum and position dependent form for this case is

$$\mathsf{H}(p, \psi) = \frac{(p)^2}{2C} + \frac{(\psi)^2}{2L^*} - \int \frac{G}{(C)^2}(p)^2 \mathrm{d}t. \tag{43}$$

That fulfills again all the properties of positive definiteness, where the Hamiltonian in both cases, $H = H(\dot{\psi}, \psi)$ and $H = H(p, \psi)$. The first time derivative of the REF is

$$\dot{\mathsf{H}} = -G(\dot{\psi})^2 = -\frac{G}{(C)^2}(p)^2 \tag{44}$$

which is in negative definite form that in turn means that the system is (asymptotically) stable.

5.2. AC equivalent circuit of Gunn diode oscillator

For a Gunn diode oscillator, an equivalent circuit is obtained when adding as a nonlinear element, negative (differential) resistance $-r$ in series to the RLC elements assuming current $i(t)$ flowing in the mesh and the voltage source is removed. Using KVL one obtains for AC current the following differential equation

$$\frac{\mathrm{d}^2 i}{\mathrm{d}t^2} + \frac{R - r}{L^*}\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{1}{L^* C} i = 0. \tag{45}$$

The solution of this equation is given below

$$i(t) = i(0)\mathrm{e}^{\alpha t}\cos(\omega t + \varphi), \qquad i(0) \neq 0;$$
$$\alpha = \frac{r-R}{2L^*}, \qquad \omega = \sqrt{\frac{1}{L^*C} - \left(\frac{r-R}{2L^*}\right)^2}. \tag{46}$$

As can be seen, the stability is given when

$$\alpha = \frac{r-R}{2L^*} \leqslant 0 \;\Rightarrow\; r \leqslant R \;\Rightarrow \begin{cases} r = R: & \text{marginal stability,} \\ r < R: & \text{asymptotic, stability.} \end{cases} \tag{47}$$

Charge formulation

For such a circuit, REF in charge formulation is

$$\mathsf{H}(p,q) = \frac{(p)^2}{2L^*} + \frac{(q)^2}{2C} - \int \frac{(R-r)}{(L^*)^2}(p)^2 \mathrm{d}t \tag{48}$$

which is positive definite. And the first time derivative of REF is negative semidefinite as below:

$$\dot{\mathsf{H}} = -\frac{(R-r)}{(L^*)^2}(p)^2 \leqslant 0 \;\Rightarrow\; r \leqslant R \tag{49}$$

that overlaps with the result found through the solution of the differential equation.

Flux formulation

In this kind of formulation, the variables are substituted as follows:

$$i(t) \to u(t), \quad L^* \to C, \quad C \to L^*, \quad R \to G,$$
$$r \to g = \frac{1}{r}: \text{ negative (differential) conductance.}$$

And this differential equation shows exactly the same stability properties for

$$g \leqslant G. \tag{50}$$

For this kind of a circuit, REF in flux formulation is

$$\mathsf{H}(p,\psi) = \frac{(p)^2}{2C} + \frac{(\psi)^2}{2L^*} - \int \frac{G-g}{(C)^2}(p)^2 \mathrm{d}t \tag{51}$$

which is positive definite. And the first time derivative of REF is negative semidefinite as below:

$$\dot{\mathsf{H}} = -\frac{G-g}{(C)^2}(p)^2 \leqslant 0 \;\Rightarrow\; g \leqslant G \tag{52}$$

that overlaps with the result found through the solution of the differential equation.

5.3. Electromechanical system

Another example is an electromechanical time dependent system as a coupled one presented in the Fig. 1.

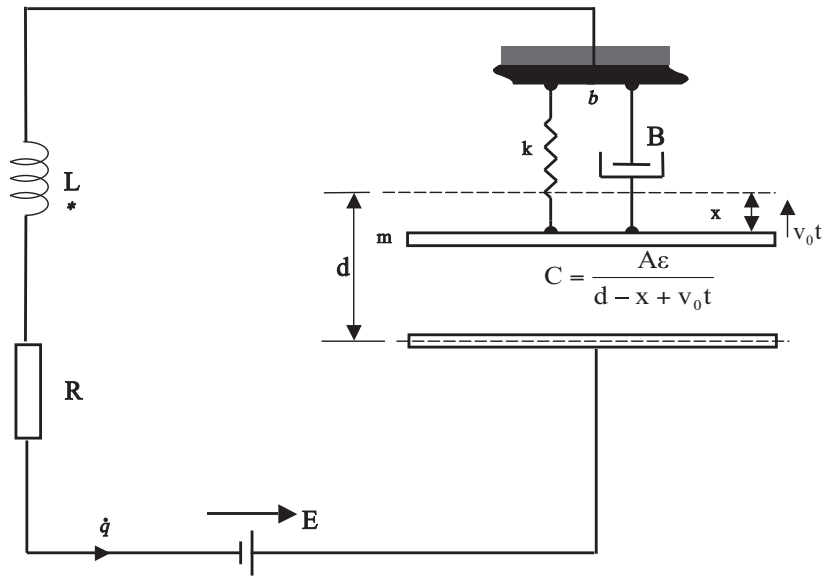

Figure 1: An electromechanical system: The upper plate of the variable condenser is free to move along a straight line with constant velocity v_0 under the action of a spring, damper and the electric field between them. B: damper, k: spring, L^*: inductor, R: resistor

Charge formulation

The generalized coordinates here are: $q^1 = \text{distance}\, x$, $q^2 = \text{charge}\, q$. The degree of freedom for this system is $f = 2$ and the constraint is holonomic scleronomic, $f(\dot{\tilde{\mathbf{q}}}) = 0$ since generalized velocities (velocity and current) are proportional with each other. So are the generalized coordinates: $\dot{q}^1 \propto \dot{q}^2 \Rightarrow q^1 \propto q^2$. The potential function is independent of generalized velocity, $V \neq V(\dot{\tilde{\mathbf{q}}})$. The $\{L,D\}$-model with prerequisites for Hamiltonian in translational-charge formulation is

$$\{L,D\} = \begin{cases} \dfrac{m(\dot{q}^1)^2 + L^*(\dot{q}^2)^2}{2} - \dfrac{k(q^1 - v_0 t)^2 + \dfrac{(q^2)^2(d - q^1 + v_0 t)}{A\varepsilon}}{2}, \\ B\dfrac{(\dot{q}^1)^2}{2} + R\dfrac{(\dot{q}^2)^2}{2}, \end{cases} \tag{53}$$

where

$$\det\begin{pmatrix} m & 0 \\ 0 & L^* \end{pmatrix} \neq 0 \ \ \forall\, m \neq 0 \quad \text{and/or} \quad L^* \neq 0. \tag{54}$$

The Hamiltonian derived by means of Legendre transform fulfilling the property (4), consists of the sum of energies but it is not constant in this case since it is dependent on time t in an explicit way. The dissipation energy can be calculated easily and the sum of the kinetic and potential energy functions of quadratic forms (which is Hamiltonian) plus the dissipation energy is REF given as below

$$\mathsf{H}^+(\mathbf{p},\widetilde{\mathbf{q}},t) = \frac{(p_1)^2}{2m} + \frac{(p_2)^2}{2L^*} + \frac{k(q^1 - v_0 t)^2 + \dfrac{(q^2)^2(d - q^1 + v_0 t)}{A\varepsilon}}{2} - \int \left[B\left(\frac{p_1}{m}\right)^2 + R\left(\frac{p_2}{L^*}\right)^2 \right] \mathrm{d}t, \tag{55}$$

where $d - q^1 > 0$, $0 \leqslant v_0 t \leqslant q^1$, satisfying the condition given in (21) with a positive definite time independent function,

$$W_1(\mathbf{p},\widetilde{\mathbf{q}}) = \frac{1}{2}\left[\frac{(p_1)^2}{m} + \frac{(p_2)^2}{L^*} + \frac{(q^2)^2(d - q^1)}{A\varepsilon} \right] - \int \left[B\left(\frac{p_1}{m}\right)^2 + R\left(\frac{p_2}{L^*}\right)^2 \right] \mathrm{d}t \ \ \forall\ A,\ d \in \mathbf{R}^+. \tag{56}$$

The first time derivative of the REF is as follows:

$$\dot{\mathsf{H}}^+(\mathbf{p},\widetilde{\mathbf{q}},t) = v_0 \left[\frac{(q^2)^2}{2A\varepsilon} - k(q^1 - v_0 t) \right] - \left[B\left(\frac{p_1}{m}\right)^2 + R\left(\frac{p_2}{L^*}\right)^2 \right]. \tag{57}$$

Since $0 \leqslant v_0 t \leqslant q^1$, a negative (semi)definite form satisfying stability criteria for this term can not be found.

The metric tensor and its inverse for this case are

$$[g_{ij}] = \frac{1}{K}\begin{pmatrix} m & 0 \\ 0 & L^* \end{pmatrix}, \qquad [g^{ij}] = K\begin{pmatrix} \dfrac{1}{m} & 0 \\ 0 & \dfrac{1}{L^*} \end{pmatrix} \tag{58}$$

which means that the generalized motion takes place in Euclidean space, where the elements of the metric tensor are constants.

Using the other form of Hamiltonian and velocity proportional (Rayleigh) dissipation function, i.e. H^- and $D(\dot{q}_k, q_k)$ one gets

$$\mathsf{H}^{-}(\dot{\mathbf{q}},\mathbf{q},t)=\frac{K^2}{2}\left[\frac{(\dot{q}_1)^2}{m}+\frac{(\dot{q}_2)^2}{L^*}\right]+\frac{1}{2}\left[k\left(\frac{K}{m}q_1-v_0t\right)^2+\frac{\left(\frac{K}{L^*}q_2\right)^2\left(d-\frac{K}{m}q_1+v_0t\right)}{A\varepsilon}\right]-\int\left[B\frac{K^2}{m^2}(\dot{q}_1)^2+R\frac{K^2}{(L^*)^2}(\dot{q}_2)^2\right]\mathrm{d}t, \tag{59}$$

$$\mathsf{H}^{-}(\widetilde{\mathbf{p}},\mathbf{q},t)=\frac{1}{K^2}\left[m(p^1)^2+L^*(p^2)^2\right]+\frac{1}{2}\left[k\left(\frac{K}{m}q_1-v_0t\right)^2+\frac{\left(\frac{K}{L^*}q_2\right)^2\left(d-\frac{K}{m}q_1+v_0t\right)}{A\varepsilon}\right]-\int\left[B\left(\frac{p^1}{K}\right)^2+R\left(\frac{p^2}{K}\right)^2\right]\mathrm{d}t. \tag{60}$$

For this form, a negative (semi) definite relationship for stability criteria can not be found, which was already explained for the other form.

Flux formulation

Using flux formulation instead of charge formulation, i.e. $q^2=\psi$, one obtains: the degree of freedom is also $f=2$ and the constraint is holonomic scleronomic, $f(\dot{\widetilde{\mathbf{q}}})=0$, since generalized velocities (velocity and voltage) are proportional again and so are the generalized coordinates, $\dot{q}^1\propto\dot{q}^2=\dot{\psi}\Rightarrow q^1\propto q^2=\psi$, and the potential function is independent of generalized velocity, $V\neq V(\dot{\widetilde{\mathbf{q}}})$. The $\{L,D\}$-model with prerequisites for Hamiltonian is

$$\{L,D\}=\begin{cases}\dfrac{1}{2}\left[m(\dot{q}^1)^2+\dfrac{A\varepsilon}{d-q^1+v_0t}(\dot{q}^2)^2\right]-\dfrac{1}{2}\left[k(q^1)^2+\dfrac{(q^2)^2}{L^*}\right],\\[2ex] B\dfrac{(\dot{q}^1)^2}{2}+G\dfrac{(\dot{q}^2)^2}{2},\end{cases} \tag{61}$$

where

$$\det\begin{pmatrix} m & 0\\ 0 & \dfrac{A\varepsilon}{d-q^1+v_0t}\end{pmatrix}\neq 0\ \ \forall\, m\neq 0\ \text{ and/or }\ A\neq 0. \tag{62}$$

Thus the related REF is as follows:

$$\mathsf{H}^{+}(\mathbf{p},\widetilde{\mathbf{q}},t)=\frac{1}{2}\left[\frac{(p_1)^2}{m}+\frac{d-q^1+v_0t}{A\varepsilon}(p_2)^2\right]+\frac{1}{2}\left[k(q^1)^2+\frac{(q^2)^2}{L^*}\right]$$
$$-\int\left[\left(\frac{p_1}{m}\right)^2B+\left(\frac{d-q^1+v_0t}{A\varepsilon}p_2\right)^2G\right]\mathrm{d}t. \tag{63}$$

As can be seen, beside of time variable, the kinetic energy is also in generalized position dependent form $T(\dot{\widetilde{\mathbf{q}}},\widetilde{\mathbf{q}},t)$ or $T(\mathbf{p},\widetilde{\mathbf{q}},t)$ which is responsible for a (generalized) position dependent metric tensor, i.e.

$$[g_{ij}]=\begin{pmatrix} m & 0 \\ 0 & \dfrac{A\varepsilon}{(d-q^1+v_0t)} \end{pmatrix}. \tag{64}$$

That means the generalized motion takes place in noneuclidean space. This problem will not be handled any further. Instead, the literatures [24–26] are to be referred. The condition in (21) with a positive definite quadratic form

$$W_2(\mathbf{p},\widetilde{\mathbf{q}})=\frac{1}{2}\left[\frac{(p_1)^2}{m}+\frac{(d-q^1)}{A\varepsilon}(p_2)^2+k(q^1)^2+\frac{(q^2)^2}{L^*}\right]$$
$$-\int\left[\left(\frac{p_1}{m}\right)^2B+\left(\frac{d-q^1}{A\varepsilon}p_2\right)^2G\right]\mathrm{d}t \tag{65}$$

is satisfied. The first time derivative of the REF in this case is

$$\dot{\mathsf{H}}^{+}(\mathbf{p},\widetilde{\mathbf{q}},t)=\frac{v_0}{2A\varepsilon}(p_2)^2-\left[\left(\frac{p_1}{m}\right)^2B+\left(\frac{d-q^1+v_0t}{A\varepsilon}p_2\right)^2G\right] \tag{66}$$

which is a constant related the variable $q^2=\psi$. Equation (23) is fulfilled with the negative definite term given by

$$\widehat{W}_2(\mathbf{p},\widetilde{\mathbf{q}})=\frac{v_0}{2A\varepsilon}(p_2)^2-\left[\left(\frac{p_1}{m}\right)^2B+\left(\frac{d-q^1}{A\varepsilon}\right)^2(p_2)^2\right] \tag{67}$$

prerequest that the following inequality is valid:

$$\left[\left(\frac{p_1}{m}\right)^2B+\left(\frac{d-q^1}{A\varepsilon}\right)^2(p_2)^2\right]>\frac{v_0}{2A\varepsilon}(p_2)^2. \tag{68}$$

That is, the system must be (asymptotic) stable.

It is not surprising that for this system, the origin is asymptotically stable, provided that the input goes to zero, since it should be expected that:

a) in translational-charge formulation, the distance and the current (so the time dependent charge),

b) in translational-flux formulation, the distance and the voltage (so the time dependent flux).

decay to zero with time, due to consumption of energy in the resistor and the damper.

6. Conclusions

In this research paper, another stability analysis approach for engineering/physical systems is presented. The method is based on REF concept which consists of Hamiltonian energy function plus loss of energy. The conditions for a Hamiltonian, whether time dependent or not, as the sum of energy functions of an engineering/physical system, is also explained. It is shown that stability analysis can be achieved using REF consisting of Hamiltonian (representing the sum of the potential and the kinetic energies of an engineering/physical system which is not always the case) plus loss of energies. For an engineering/physical system Hamiltonian of which represents the sum of energies of a system, stability analysis can be carried out very efficiently using such a Hamiltonian and generalized velocity proportional (Rayleigh) dissipation function. As can be seen, the losses play a significant role in the equations and are never to be ignored for a complete stability analysis, since such a Hamiltonian is responsible for marginal stability while the REF stands for stability in general. The same approach is applicable, if the other forms of Hamiltonian and (Rayleigh) dissipation function, e.g. H^- and $D(q_k)$ are used alternatively.

In virtue of examples given herein, how to apply the approach was demonstrated. A stable system in a formulation type and generalized motion space remains also stable in the other type of formulation and generalized motion space (when available). Additionally, using an electromechanical example in different formulations, it was shown that negative definite relation satisfying stability criteria may be obtained or not. Application of formulation type depending on problem type and further applications of the method to find the stability of an engineering/physical problem need further studies.

References

[1] A.M. LYAPUNOV: *The General Problem of the Stability of Motion*, Taylor & Francis, London, 1992.

[2] N. ROUCHE, P. HABETS and M. LALOY: *Stability Theory by Liapunov's Direct Method*, Springer-Verlag, New York, 1977.

[3] T. YOSHIZAWA: *Stability Theory by Liapunov's Second Method*, Mathematical Society of Japan, Tokyoi, 1966.

[4] W. HAHN: *Stability of Motion*, Springer-Verlag, Berlin, 1967.

[5] E.A. BARBASHIN and N.N. KRASOVSKII: *On the stability of motion as a whole*, Doklady Akademii Nauk SSSR 86: pp. 453–456, 1952. (in Russian).

[6] N.N. KRASOVSKII: *Problems of the Theory of Stability of Motion* (in Russian), 1959. English translation: *Stanford University Press*, Stanford, CA, 1963.

[7] J.P. LASALLE: Some extensions of Liapunov's second method, *IRE Transactions on Circuit Theory*, CT-7, (1960), 520–527.

[8] A. BACCIOTTI and L. ROSIER: Liapunov Functions and Stability in Control Theory, *Lecture Notes in Control and Information Sciences*, 267. *Appl. Mech. Rev.*, 55, B88, 2002.

[9] R.I. MCLACHLAN, G.R.W. QUISPEL and N. ROBIDOUX: Unified Approach to Hamiltonian Systems, Poisson Systems, Gradient Systems, and Systems with Lyapunov Functions or First Integrals, *Physical Review Letters*, **81**(12), (1998), 2399–2403.

[10] R. MARINO and S. NICOSIA: Hamiltonian–Type Lyapunov Functions, *IEEE Transactions on Automatic Control*, **AC-28**(1), (1983).

[11] A.J. VAN DER SCHAFT and B. MASCHKE: The Hamiltonian formulation of energy-conserving physical systems with external ports, *Archive für Elektronik und Übertragungstechnik*, **49** (1995), 362–371.

[12] B. MASCHKE, R. ORTEGA and A.J. VAN DER SCHAFT: Energy-Based Lyapunov Functions for Forced Hamiltonian Systems with Dissipation, *IEEE Transactions on Automatic Control*, **45**(8), (2000).

[13] O. ELGERD: *Control Systems Theory*, McGraw-Hill Kogakusha LTD, Tokyo, 1967.

[14] H. HEUSER: *Gewöhnliche Differentialgleichungen*, BG Teubner, Stuttgart, 1989.

[15] K. DUTTON, S. THOMSON and B. BARRACLOUGH: *The Art of Control Engineering*, Addison-Wesley Publishing Company, Harword, 1997.

[16] K.H. KHALIL: *Nonlinear Systems*, Macmillan Publishing Company, New York, 1997.

[17] C. CIVELEK and U. DIEMAR: Stability Analysis using Energy Functions, *48. Internationales wissenschaftliches Kolloquim der Technischen Universität Ilmenau*, Tagungsband, 427, 2003.

[18] T.L. CHOW: Classical Mechanics. John Wiley & Sons, New York, 1995.

[19] H. GOLDSTEIN: *Classical Mechanics*, 2nd ed. Addison Wesley Publishing Company, Massachusets, 1980.

[20] B. BROGLIA, R. LOZANO, B. MASCHKE and O. EGELAND: Dissipative Systems Analysis and Control, 2nd ed. Springer, Heidelberg, 2007.

[21] R. SUESSE and C. CIVELEK: Analysis of Engineering Systems by means of Lagrange and Hamilton Formalisms depending on contravariant, covariant tensorial variables, *Forschung im Ingenieurwesen – Engineering Research*, **68**(1), (2003), 66–74.

[22] C. CIVELEK: Berechnung von elektrotechnischen Systemen mit Wandlern mittels erweitertem Lagrange – bzw. Hamilton-Formalismus, Wissenschaftsverlag Ilmenau, Ilmenau, 2001.

[23] R. SUESSE and C. CIVELEK: Analysis of coupled dissipative dynamic systems of engineering using extended Hamiltonian H for classical and non-conservative Hamiltonian H^{*n} for higher order Lagrangian, *Forschung im Ingenieurwesen-Engineering Research*, **77**(1-2), (2012), 1–11.

[24] B. MARX and R. SUESSE: Theoretische Elektrotechnik (in fünf Bänden) – Band 1: Netzwerke und Elemente hoeherer Ordnung. VDI Verlag, Düsseldorf, 2012.

[25] R. SUESSE, U. DIEMAR and P. BURGER: Theoretische Elektrotechnik (in fünf Bänden) – Band 2: Elektrische Netzwerke und Elemente höherer Ordnung, Wissenschaftsverlag Thüringen, Langewiesen, 2013.

[26] R. SUESSE and T. STROEHLA: Theoretische Elektrotechnik (in fünf Bänden) – Band 3: Analyse und Synthese elektrotechnischer Systeme, 2. Überarbeitete und erweiterte Auflage, Wissenschaftsverlag Thüringen, Langewiesen, 2014.

[27] R. SUESSE and U. DIEMAR: Theoretische Elektrotechnik (in fünf Bänden) – Band 4: Beschreibung, Berechnung und Synthese von Feldern, Wissenschaftsverlag Ilmenau, Ilmenau, 2000.

[28] R. SUESSE and B. MARX: Theoretische Elektrotechnik (in fünf Bänden) – Band 5: Netzwerke-Berechnung und Synthese für vorgegebenes Bifurkationsverhalten. Wissenschaftsverlag Ilmenau, Ilmenau, 2000.

[29] J. LESURF: Negative Resistance Oscillators, The Scots Guide to Electronics, School of Physics and Astronomy, Univ. of St. Andrews, Retrieved August 20, 2012.