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INVITED PAPER

Behaviour of fractional discrete-time consensus models with delays for summator dynamics

E. GIREJKO, D. MOZYRSKA, and M. WYRWAS*

Faculty of Computer Science, Bialystok University of Technology, 45A Wiejska St., 15-351 Białystok, Poland

Abstract. The leader-following consensus problem of fractional-order multi-agent discrete-time systems with delays is considered. In the systems, interactions between agents are defined like in Krause and Cucker-Smale models, but the memory is included by taking both the fractional-order discrete-time operator on the left hand side of the nonlinear systems and the delays. Since in practical problems only bounded number of delays can be considered, we study the fractional order discrete-time models with a finite number of delays. The models of opinions under consideration are investigated for single- and double-summator dynamics of discrete-time by means of analytical methods as well as computer simulations.

Key words: fractional calculus, consensus, single-summator, double-summator, discrete-time models with delays.

1. Introduction

Recently, consensus, i.e. a typical collective behaviour, has been drawing considerable research interest, due to the versatility of its applications in biological systems, satellite formation, sensor networks and others, see for example [1] and the references therein. In general, there are two kinds of consensus: with and without leader. When examining the case with a leader, there are still two approaches to the problem: without and with control. For the latter case a critical problem is how to design appropriate protocols and make all the agents converge to a common value by following the leader. In the literature many positions can be found concerning continuous- and discrete-time multi-agent systems on leaderless consensus problems, see for example [2–5]. It is clear that group consensus without a leader is useful in many cases, however there are many other applications that require a dynamic leader. As a well known example we can mention formation control, where the agents regulate their states according to their state deviations and attain the expected formation. Theoretical results on leader following consensus with multi-agent fractional systems were presented including first- and second-order integrator (summator) in [6–8] and in the references therein. However, let us note that research on the consensus problem with more practical dynamics with delays is still ongoing today.

Motivated by the above, we investigate the group consensus problems in network of agents with a dynamic leader in single- and double-summator dynamics for fractional discrete-time consensus models with delays. Using stability tools of fractional linear discrete-time systems with delays, we consider the case where the state of leader is available only to a subset of followers. We propose a protocol for multi-agent systems with arbitrary delay. The control input for each agent relies on its own state and its neighbours' states. Some necessary and sufficient conditions for consensus are established in terms of stability of fractional systems with delays. We analyse the convergence of the protocol and obtain the conditions for the control parameter to ensure that the tracking process leads the system to consensus. In order to validate the consensus control, some simulations are carried out.

The paper is an extension of the results given in [8] for fractional order difference systems. Observe that in [8] systems with $k_0 = 1$ were considered while in this paper we take into account arbitrary delay $k_0 \geq 1$. The paper is organized in the following manner. In Sec. 2 we gather some basic definitions and facts that are needed in the follow-up study. The main results are presented in Sec. 3, where the conditions under which the leader-following consensus is achieved for fractional-order discrete-time systems with delays are given. In Sec. 2 and 3 we consider a discrete case with step $h = 1$ while in Sec. 4, step $h > 0$ is taken into account. In the last section our consideration is extended to the case with arbitrary step $h > 0$, and conditions that guarantee the leader-following consensus for fractional-order h-difference systems with delays are formulated. Numerical examples that validate the results obtained show the effectiveness of the design method.

2. Preliminaries

Let us recall some definitions and facts known from fractional discrete calculus. Let $c \in \mathbb{R}$, $\mathbb{N}_c := \{c, c+1, c+2, \ldots\}$ and $\alpha \in \mathbb{R}$.

The following sequence is defined as follows:

$$a^{(\alpha)}(k) := \begin{cases} 1 & \text{for } \ k = 0, \\ (-1)^k \dfrac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} & \text{for ; } k \in \mathbb{N}_1. \end{cases} \tag{1}$$

Using sequence (1) one can define the following difference operator.

*e-mail: m.wyrwas@pb.edu.pl

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Definition 1. Let $\alpha \in \mathbb{R}$. The Grünwald-Letnikov difference operator Δ^α of order α of function $x : \mathbb{N}_0 \to \mathbb{R}$ is defined by $(\Delta^\alpha x)(k) := \sum_{s=0}^{k} a^{(\alpha)}(s)x(k-s)$, where $a^{(\alpha)}(s)$ is the sequence given by (1).

The extension of the Grünwald-Letnikov difference operator to a vector valued function is made in the componentwise manner, i.e. for $x = (x_1, \ldots, x_n) : \mathbb{N}_0 \to \mathbb{R}^n$ we have $\Delta^\alpha x = (\Delta^\alpha x_1, \ldots, \Delta^\alpha x_n)$.

Note that for $\alpha = 0$ we get $(\Delta^0 x)(k) = x(k)$, while for $\alpha = 1$ we have $\left(\Delta^1 x\right)(k) := x(k) - x(k-1)$.

Let us consider the following fractional-order systems of order $\alpha \in (0,1]$ with the Grünwald-Letnikov difference operator:

$$(\Delta^\alpha x)(k) = F(x(k-k_0)), \quad k \geq k_0, \quad k_0 \in \mathbb{N}_1, \tag{2}$$

with initial conditions $x(0) = x_0, x(1) = x_1, \ldots, x(k_0-1) = x_{k_0-1} \in \mathbb{R}^n$, where $x = (x_1, \ldots, x_n)^{\mathrm{T}} : \mathbb{N}_0 \to \mathbb{R}^n$ is a vector function and $F : \mathbb{R}^n \to \mathbb{R}^n$. The solutions of system (2) exist according to given initial conditions. System (2) takes the recursive form

$$x(k) = -\sum_{s=1}^{k} a^{(\alpha)}(s)x(k-s) + F(x(k-k_0)) \quad \text{for} \quad k \geq k_0$$

with given initial values $x(0), x(1), \ldots, x(k_0-1)$, $k_0 \in \mathbb{N}_1$.

Let us define, for $\varphi \in [0, 2\pi)$, $\alpha \in (0,1]$ and $k_0 \geq 1$, the following function

$$w(\varphi, \alpha, k_0) :=$$
$$\begin{cases} \left(2\left|\sin\dfrac{2\varphi - \alpha\pi}{2(2k_0-\alpha)}\right|\right)^\alpha, & \varphi \in [0,\pi], \\ \left(2\left|\sin\dfrac{2\varphi - \alpha\pi + 4(k_0-1)\pi}{2(2k_0-\alpha)}\right|\right)^\alpha, & \varphi \in (\pi, 2\pi). \end{cases}$$

In the follow-up study, we are interested in the solutions of linear systems of the form

$$(\Delta^\alpha x)(k) = Ax(k-k_0), \tag{3}$$

where $k \geq k_0$, $k_0 \in \mathbb{N}_1$, $x(\cdot) = (x_1(\cdot), \ldots, x_n(\cdot))^{\mathrm{T}} : \mathbb{N}_0 \to \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Observe that in order to determine the solution of (3) one has to know the initial values $x(0)$, ..., $x(k_0-1)$ of $x(\cdot)$ that are assumed to be given.

In [9] the following characterization of the asymptotic stability of (3) is given.

Proposition 2. If for all $i = 1, \ldots, n$

$$\arg\lambda_i \in \left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right] \tag{4}$$

and

$$|\lambda_i| < w(\arg\lambda_i, \alpha, k_0), \tag{5}$$

where $\arg\lambda_i$ and $|\lambda_i|$ are, respectively, the main argument and modulus of $\lambda_i \in Spec(A)$, then system (3) (with $\alpha \in (0,1]$ and delay k_0) is asymptotically stable.

Additionally, in [9] the condition for the instability of (3) is given.

Proposition 3. If there exists $\lambda_i \in Spec(A)$ such that $|\lambda_i| > w(\arg\lambda_i, \alpha, k_0)$, then system (3) (with $\alpha \in (0,1]$ and delay k_0) is not stable.

Finally, we define the following set:

$$R_{\alpha,k_0} := \Big\{z \in \mathbb{C} : \varphi = \arg z \in \left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right] \wedge |z| < w(\varphi, \alpha, k_0)\Big\}, \tag{6}$$

which is related to the asymptotical stability of the considered systems. Note that the condition that all eigenvalues of the matrix A belong to R_{α,k_0} guarantees the asymptotical stability of system (3).

3. The leader-following consensus for models with summator dynamics

In this section the conditions under which the leader-following consensus is achieved for fractional-order discrete-time systems with delays and with constant adjacency matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ are stated. Additionally, a consensus control law for tracking the virtual leader by use of stability analysis of fractional order systems with delays are given.

Let $N := \{1, \ldots, n\}$ and consider the system with n agents where the single-summator dynamics of each agent is given by

$$(\Delta^\alpha x_i)(k) = \sum_{j=1}^{n} a_{ij}\left[x_j(k-k_0) - x_i(k-k_0)\right] + u_i(k), \tag{7}$$

where $k \geq k_0$, $k_0 \in \mathbb{N}_1$, $i \in N$, a_{ij} $(i,j = 1,2,\ldots,n)$ is the (i,j)-th entry of the adjacency matrix A, and $\alpha \in (0,1]$, $x_i : \mathbb{N}_0 \to \mathbb{R}$ is the state function for the i-th agent, $u_i : \mathbb{N}_0 \to \mathbb{R}$ is the control input function for the i-th agent.

For the simplicity of presentation we assume that all agents are in a one-dimensional space.

The virtual leader for multiagent system (7) is an isolated agent such that

$$(\Delta^\alpha x_r)(k) = f(k), \tag{8}$$

where $k \geq 0$, $x_r : \mathbb{N}_0 \to \mathbb{R}$ is the state function of the virtual leader.

Let $\ell = 0, 1, \ldots, k_0-1$ and $x(\ell) = (x_1(\ell), \ldots, x_n(\ell))^{\mathrm{T}} \in \mathbb{R}^n$ be the given vector of initial values. Now, let us formulate the definition where all agents can track the virtual leader with local interaction by designing control laws u_i.

Definition 4. Multiagent system (7) is said to achieve leader-following consensus if its solution satisfies $\lim_{k\to+\infty} |x_i(k) - x_r(k)| = 0$ for any initial values $x(0), x(1), \ldots, x(k_0-1)$ and for all $i \in N$.

Let $k \geq k_0$ and $k_0 \in \mathbb{N}_1$. The consensus control law is proposed as:

$$\begin{aligned} u_i(k) &= f(k) \\ &+ (1-\beta)\sum_{j=1}^{n} a_{ij}\left(x_i(k-k_0) - x_j(k-k_0)\right) \\ &- \beta b_i\left[x_i(k-k_0) - x_r(k-k_0)\right], \end{aligned} \tag{9}$$

where $f(k)$ is the nonlinear dynamics, $b_i = 1$ if the dynamics of the virtual leader is available to agent i and $b_i = 0$ otherwise, $\beta > 0$ is a constant that is called control parameter.

Let $B := \operatorname{diag}\{b_1, \ldots, b_n\} \in \mathbb{R}^{n\times n}$, where $b_i \in \{0,1\}$, be a diagonal matrix with nonzero trace. Let $L = (l_{ij})$, $l_{ii} = \sum_{j\neq i} a_{ij}$ with $l_{ij} = -a_{ij}$, $i \neq j$ be the (nonsymmetric) Laplacian matrix. Let us define $M := L + B$ and $M_\beta := \beta M$.

Definition 5. A matrix $M \in \mathbb{R}^{n\times n}$ is called fractional discrete-time stable if all eigenvalues of the matrix M satisfy the conditions (4) and (5), i.e. all eigenvalues of the matrix M belong to the set R_{α,k_0} defined by (6).

Theorem 6. If the matrix $(-M_\beta)$ is fractional discrete-time stable, i.e. $Spec\,(-M_\beta) \subset R_{\alpha,k_0}$, where the set R_{α,k_0} is defined by (6), then control law (9) solves the consensus problem for single-summator system (7) with the time-varying dynamics of the virtual leader given by (8).

Proof. Similarly as in [8] one can show that by using control law (9), system (7) can be written as follows:

$$(\Delta^\alpha x)(k) = f(k)\mathbb{1} - M_\beta x(k-k_0) + \beta B x_r(k-k_0)\mathbb{1},$$

where $x(k-k_0) = (x_1(k-k_0), \ldots, x_n(k-k_0))^{\mathrm{T}} \in \mathbb{R}^n$ and $\mathbb{1} = (1,1,\ldots,1)^{\mathrm{T}} \in \mathbb{R}^n$. Let $\widetilde{x}_i(k-k_0) := x_i(k-k_0) - x_r(k-k_0)$. Then one gets

$$\begin{aligned}(\Delta^\alpha \widetilde{x})(k) &= (\Delta^\alpha x)(k) - (\Delta^\alpha x_r)(k)\mathbb{1} \\ &= f(k)\mathbb{1} - M_\beta x(k-k_0) \\ &\quad + \beta B x_r(k-k_0)\mathbb{1} - (\Delta^\alpha x_r)(k)\mathbb{1} \\ &= f(k)\mathbb{1} - M_\beta(x(k-k_0) - x_r(k-k_0)\mathbb{1}) \\ &\quad - M_\beta x_r(k-k_0)\mathbb{1} + \beta B x_r(k-k_0)\mathbb{1} \\ &\quad - (\Delta^\alpha x_r)(k)\mathbb{1} \\ &= f(k)\mathbb{1} - M_\beta \widetilde{x}(k-k_0) - L x_r(k-k_0)\mathbb{1} \\ &\quad - (\Delta^\alpha x_r)(k)\mathbb{1}.\end{aligned}$$

where $\widetilde{x}(k-k_0) := (\widetilde{x}_1(k-k_0), \ldots, \widetilde{x}_n(k-k_0))^{\mathrm{T}} = (x_1(k-k_0) - x_r(k-k_0), \ldots, x_n(k-k_0) - x_r(k-k_0))^{\mathrm{T}}$. Since $L x_r(k-k_0)\mathbb{1} = 0$ and $(\Delta^\alpha x_r)(k) = f(k)$, we get the following system

$$(\Delta^\alpha \widetilde{x})(k) = -\,M_\beta \widetilde{x}(k-k_0). \tag{10}$$

By assumption, the matrix $(-M_\beta)$ is fractional discrete-time stable, thus by Proposition 2 we claim that $\lim_{k\to+\infty} \widetilde{x}_i(k) = 0$, $i \in N$. Consequently, $\lim_{k\to+\infty} |x_i(k) - x_r(k)| = 0$, for $i \in N$. It means that consensus with the time-varying dynamics of the virtual leader is achieved by control law (9). □

Remark 7. Let us notice that we can choose matrix B in such a way that we put $b_i = 1$ for those i for which Jordan block of matrix L is made of zeros. Then, $Spec(M) \subset \mathbb{R}_+ \times \mathbb{R}$, as by Gershgorin's Theorem is known that all eigenvalues of L has their real part larger or equal to 0 (see for example [10]). Moreover, in the case when $Spec(M_\beta) \subset \mathbb{R}_+$, to have $Spec\,(-M_\beta) \subset (-w(\pi,\alpha,k_0), 0)$ it is enough to take $\beta \in \left(0, \frac{w(\pi,\alpha,k_0)}{\max_i \lambda_i}\right)$, $\lambda_i \in Spec(M)$.

Example 8. Let us consider system (7) with 6 agents and the following Laplacian matrix L_6:

$$L_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

and with

$$B_6 = \operatorname{diag}\{1,0,0,0,0,0,0\}.$$

Let us also consider the dynamics of the virtual leader given by (8) with $f(k) = \sin(k/6)$ and the delay $k_0 = 2$. Then the consensus is reached for $\beta = 0.3814457866 = 0.95w(\pi,0.9,1)/\max_i \lambda_i$ where $\lambda_i \in Spec(M) = \{2.618033988, 0.381966012, 1\}$, see Fig. 1. When we change the delay for $k_0 = 3$ we need to choose smaller β, for example $\beta = 0.2510678301 = 0.95w(\pi,0.9,3)/\max_i \lambda_i$, where $\lambda_i \in Spec(M) = \{2.618033988, 0.381966012, 1\}$, to receive consensus, see Fig. 2. Note that for better visualisation the points which correspond to the solutions of the considered systems are connected.

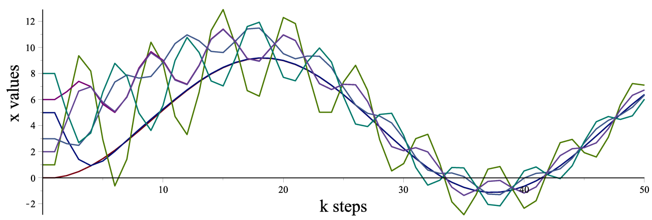


Fig. 1. Solutions of system of equations (7) with the control law given by (9) and the virtual leader; $\alpha = 0.9$; $f(k) = \sin(k/6)$, $T = 50$ steps, $k_0 = 2$, $\beta = 0.3814457866 = 0.95w(\pi,0.9,2)/\max_i \lambda_i$ where $\lambda_i \in Spec(M) = \{2.618033988, 0.381966012, 1\}$

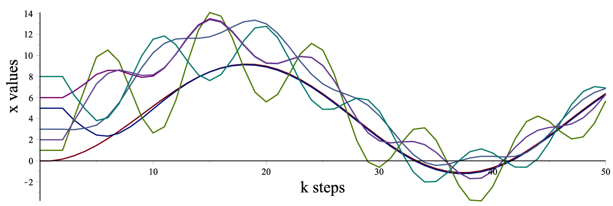


Fig. 2. Solutions of system of equations (7) with the control law given by (9) and the virtual leader; $\alpha = 0.9$; $f(k) = \sin(k/6)$, $T = 50$ steps, $k_0 = 3$, $\beta = 0.2510678301 = 0.95w(\pi,0.9,3)/\max_i \lambda_i$ where $\lambda_i \in Spec(M) = \{2.618033988, 0.381966012, 1\}$

3.1. Models for double-summator dynamics. In this section, similarly as for single-summator dynamics case, we consider the multiagent system consisting of n agents such that dynamics of each agent is given by

$$\begin{aligned}(\Delta^\alpha x_i)(k) &= v_i(k-k_0),\\ (\Delta^\alpha v_i)(k) &= \sum_{j=1}^n a_{ij}\left[(x_j(k-k_0)-x_i(k-k_0))\right.\\ &\quad \left.+(v_j(k-k_0)-v_i(k-k_0))\right]+u_i(k),\end{aligned}\tag{11}$$

where $k \geq k_0$, $k_0 \in \mathbb{N}_1$, a_{ij} is the (i,j)-th entry of the adjacency matrix $A \in \mathbb{R}^{n\times n}$ associated with graph $\mathcal{G}$, $\alpha \in (0,1]$, $x_i : \mathbb{N}_0 \to \mathbb{R}$ is the state function of the i-th agent, $v_i : \mathbb{N}_0 \to \mathbb{R}$ is the velocity function of the i-th agent, $u_i : \mathbb{N}_0 \to \mathbb{R}$ is the control input function for the i-th agent, $i = 1, 2, \ldots, n$.

The virtual leader for fractional multiagent system (11) is an isolated agent described by

$$\begin{aligned}(\Delta^\alpha x_r)(k) &= v_r(k-k_0),\\ (\Delta^\alpha v_r)(k) &= f(k),\end{aligned}\tag{12}$$

where x_r is the state of the virtual leader, v_r is the velocity of the virtual leader and f is a continuous function.
Now, let us give the definition of reaching a consensus by system (11).

Definition 9. The multiagent system (11) is said to achieve leader-following consensus if its solution satisfies

$$\lim_{k\to\infty} |x_i(k)-x_r(k)| = 0 \tag{13}$$

and

$$\lim_{k\to\infty} |v_i(k)-v_r(k)| = 0 \tag{14}$$

for any initial values $x(\ell) = (x_1(\ell),\ldots,x_n(\ell))^T$, $v(\ell) = (v_1(\ell),\ldots,v_n(\ell))^T$, $x_r(\ell)$ and $v_r(\ell)$, $\ell = 0,1,\ldots,k_0-1$, $k_0 \in \mathbb{N}_1$.

The following control input is considered to achieve leader-following consensus in multiagent system (11):

$$\begin{aligned}u_i(k) = f(k)+(1-\beta)&\sum_{j=1}^n a_{ij}(v_i(k-k_0)-v_j(k-k_0))\\ &-b_i\left[(x_i(k-k_0)-x_r(k-k_0))\right.\\ &\left.+\beta(v_i(k-k_0)-v_r(k-k_0))\right],\end{aligned}\tag{15}$$

where $b_i = 1$ if the virtual leader state is available to agent i and $b_i = 0$ otherwise, $\beta > 0$ is a constant parameter.

Similarly, as for the single-summator dynamics let $B = \mathrm{diag}\{b_1, b_2, \ldots, b_n\}$, $M = L + B$, where $L = (l_{ij})$, $l_{ii} = \sum_{j\neq i} a_{ij}$ with $l_{ij} = -a_{ij}$, $i \neq j$ is the (nonsymmetric) Laplacian matrix. Then using (15), system (11) can be rewritten in the matrix form as follows

$$\begin{aligned}(\Delta^\alpha x)(k) &= v(k-k_0),\\ (\Delta^\alpha v)(k) &= -M(x(k-k_0)+\beta v(k-k_0))\\ &\quad + B[x_r(k-k_0)+\beta v_r(k-k_0)]\mathbb{1}+f(k)\mathbb{1},\end{aligned}\tag{16}$$

where $x(k-k_0) = (x_1(k-k_0),\ldots,x_n(k-k_0))^T$, $v(k-k_0) = (v_1(k-k_0),\ldots,v_n(k-k_0))^T$ and $\mathbb{1} = (1,1,\ldots,1)^T$ belong to $\mathbb{R}^n$.

Now let us define

$$\mathbb{M} := \begin{pmatrix} \mathbf{0}_{n\times n} & -I_{n\times n}\\ M & M_\beta\end{pmatrix} \in \mathbb{R}^{2n\times 2n}, \tag{17}$$

where $I_{n\times n}$ is the $n\times n$ identity matrix, $\mathbf{0}_{n\times n}$ denotes the $n\times n$ zero matrix, and put

$$y(k-k_0) := (x(k-k_0), v(k-k_0))^T = \begin{pmatrix} x(k-k_0)\\ v(k-k_0)\end{pmatrix}.$$

Then multiagent system (16) can be rewritten as

$$\begin{aligned}(\Delta^\alpha y)(k) &= -\mathbb{M}y(k-k_0)\\ &+\begin{pmatrix}\mathbf{0}_{n\times 1}\\ B(x_r(k-k_0)+\beta v_r(k-k_0))\mathbb{1}\end{pmatrix}+\begin{pmatrix}\mathbf{0}_{n\times 1}\\ f(k)\mathbb{1}\end{pmatrix}.\end{aligned}$$

Theorem 10. If the matrix $(-\mathbb{M})$ is fractional discrete-time stable, i.e. $Spec\,(-\mathbb{M}) \subset R_{\alpha,k_0}$, where the set R_{α,k_0} is defined by (6), then control law (15) solves the consensus problem for double-summator system (11) with the virtual leader given by (15).

Proof. Similarly as in [8] one can introduce the following changes of coordinations: $\widetilde{x}_i(k-k_0) := x_i(k-k_0)-x_r(k-k_0)$, $k \geq k_0+1$ and $\widetilde{v}_i(k-k_0) = v_i(k-k_0)-v_r(k-k_0)$, $k \geq k_0$, then $\widetilde{x} = (\widetilde{x}_1,\ldots,\widetilde{x}_n)^T \in \mathbb{R}^n$ and $\widetilde{v} = (\widetilde{v}_1,\ldots,\widetilde{v}_n)^T \in \mathbb{R}^n$. Now putting $\widetilde{y}(k-k_0) := (\widetilde{x}(k-k_0), \widetilde{v}(k-k_0))^T = (x(k-k_0)-x_r(k-k_0)\mathbb{1}, v(k-k_0)-v_r(k-k_0)\mathbb{1})^T \in \mathbb{R}^{2n}$ in system (11), using (12) and the facts that $Lv_r(k-k_0)\mathbb{1} = 0$, $Lx_r(k-k_0)\mathbb{1} = 0$ and $(\Delta^\alpha v_r)(k) = f(k)$, one gets:

$$\begin{aligned}(\Delta^\alpha \widetilde{y})(k) &= \begin{pmatrix}(\Delta^\alpha x)(k)-(\Delta^\alpha x_r)(k)\cdot\mathbb{1}\\ (\Delta^\alpha v)(k)-(\Delta^\alpha v_r)(k)\cdot\mathbb{1}\end{pmatrix}\\ &= -\mathbb{M}\begin{pmatrix}x(k-k_0)\\ v(k-k_0)\end{pmatrix}\\ &\quad+\begin{pmatrix}\mathbf{0}_{n\times 1}\\ B(x_r(k-k_0)+\beta v_r(k-k_0))\mathbb{1}\end{pmatrix}\\ &\quad+\begin{pmatrix}\mathbf{0}_{n\times 1}\\ f(k)\mathbb{1}\end{pmatrix}-\begin{pmatrix}v_r(k-k_0)\mathbb{1}\\ f(k)\mathbb{1}\end{pmatrix}\\ &= -\mathbb{M}\begin{pmatrix}x(k-k_0)-x_r(k-k_0)\mathbb{1}\\ v(k-k_0)-v_r(k-k_0)\mathbb{1}\end{pmatrix}\\ &\quad-\begin{pmatrix}\mathbf{0}_{n\times n} & -I_{n\times n}\\ L+B & \beta(L+B)\end{pmatrix}\begin{pmatrix}x_r(k-k_0)\mathbb{1}\\ v_r(k-k_0)\mathbb{1}\end{pmatrix}\\ &\quad+\begin{pmatrix}\mathbf{0}_{n\times 1}\\ Bx_r(k-k_0)\mathbb{1}+\beta Bv_r(k-k_0)\mathbb{1}\end{pmatrix}\\ &\quad-\begin{pmatrix}v_r(k-k_0)\mathbb{1}\\ 0\mathbb{1}\end{pmatrix}\\ &= -\mathbb{M}\begin{pmatrix}\widetilde{x}(k-k_0)\\ \widetilde{v}(k-k_0)\end{pmatrix} = -\mathbb{M}\widetilde{y}(k-k_0).\end{aligned}$$

Therefore in the coordinates $\widetilde{y}$ we get the following linear system

$$(\Delta^\alpha \widetilde{y})(k) = -\mathbb{M}\widetilde{y}(k-k_0).$$

By assumption $Spec\,(-\mathbb{M}) \subset R_{\alpha,k_0}$, thus by Proposition 2 we get $\lim_{k\to+\infty} \|\widetilde{y}(k)\| = 0$. Consequently, conditions (13) and (14) are satisfied. Therefore the consensus with the virtual leader is achieved by control law (15).

□

Now let us assume that all eigenvalues of the matrix M are positive and formulate the conditions for β that give the leader following consensus in network of agents with a double-summator dynamics for fractional discrete-time models.

Proposition 11. Let $Spec(M) = \{\lambda_i,\ i = 1,\dots,k,\ k \le n\}$ be the subset of $\mathbb{R}_+ := (0,+\infty)$, $\mathfrak{m} := \frac{2}{\sqrt{\min_i \lambda_i}}$, $\mathfrak{u}_i := \min\left\{\frac{w(\pi,\alpha,k_0)}{\lambda_i} + \frac{1}{w(\pi,\alpha,k_0)}, 2\frac{w(\pi,\alpha,k_0)}{\lambda_i}\right\}$ and $\mathfrak{u} := \min_i \mathfrak{u}_i$. If the parameter β satisfies the following inequalities:

$$\mathfrak{m} \le \beta < \mathfrak{u}, \tag{18}$$

then control law (15) solves the consensus problem for double-summator system (11) with the virtual leader.

Proof. Let λ_i, $i = 1,\dots,k$, $k \le n$ be the eigenvalues of the matrix M. Note that one can choose diagonal matrix B in such a way that zero is not the eigenvalue of the Laplacian matrix L. Then $Spec(M) \subset \mathbb{R}_+ = (0,+\infty)$. Therefore $\lambda_i \ne 0$ for all $i = 1,\dots,k$.

Observe that $\mathfrak{m} = \frac{2}{\sqrt{\min_i \lambda_i}} \ge \frac{2}{\sqrt{\lambda_i}}$ for $i = 1,\dots,k$. Consequently, from $\frac{2}{\sqrt{\min_i \lambda_i}} \le \beta$ we have $\frac{2}{\sqrt{\lambda_i}} \le \beta$ and using the fact that $\beta, \lambda_i > 0$ one gets $\lambda_i^2\beta^2 - 4\lambda_i \ge 0$ for all $i = 1,\dots,k$. Let w_{ij}, $j = 1,2$ be the eigenvalues of the matrix $\mathbb{M}$. Note that if $\lambda_i^2\beta^2 - 4\lambda_i \ge 0$ for all $i = 1,\dots,k$, then all eigenvalues of the matrix $\mathbb{M}$ are positive real numbers and by [8, Lemma 2.4] the following relation between the eigenvalues of matrices M and $\mathbb{M}$ holds:

$$w_{i1} = \frac{\lambda_i\beta + \sqrt{\lambda_i^2\beta^2 - 4\lambda_i}}{2} \wedge w_{i2} = \frac{\lambda_i\beta - \sqrt{\lambda_i^2\beta^2 - 4\lambda_i}}{2}. \tag{19}$$

Since $\lambda_i > 0$ and $\beta > 0$, we get $0 < w_{i2} < w_{i1}$ for all $i = 1,\dots,k$. Taking into account $\mathfrak{u} = \min_i \mathfrak{u}_i$ for $i = 1,\dots,k$, from $\beta < \mathfrak{u}$, we get $\beta < \frac{w(\pi,\alpha,k_0)}{\lambda_i} + \frac{1}{w(\pi,\alpha,k_0)}$ and consequently, $4w(\pi,\alpha,k_0)\lambda_i\beta < 4w^2(\pi,\alpha,k_0) + 4\lambda_i$, where $w^2(\pi,\alpha,k_0) := \left(2\sin\frac{(2-\alpha)\pi}{2(2k_0-\alpha)}\right)^{2\alpha}$. Then we have

$$\begin{aligned}\lambda_i^2\beta^2 - 4\lambda_i &< 4w^2(\pi,\alpha,k_0) - 4w(\pi,\alpha,k_0)\lambda_i\beta + \lambda_i^2\beta^2 \\ &= \left(2w(\pi,\alpha,k_0) - \lambda_i\beta\right)^2,\end{aligned}$$

for all $i = 1,\dots,k$. Moreover, $\beta < \mathfrak{u} \le \mathfrak{u}_i = 2\frac{w(\pi,\alpha,k_0)}{\lambda_i}$ for $i = 1,\dots,k$, so $2w(\pi,\alpha,k_0) - \lambda_i\beta \ge 0$. Taking into account $\lambda_i^2\beta^2 - 4\lambda_i \ge 0$ and $2w(\pi,\alpha,k_0) - \lambda_i\beta \ge 0$, one gets $\sqrt{\lambda_i^2\beta^2 - 4\lambda_i} < 2w(\pi,\alpha,k_0) - \lambda_i\beta$. Consequently, from (18) we get

$$w_{i1} = \frac{\lambda_i\beta + \sqrt{\lambda_i^2\beta^2 - 4\lambda_i}}{2} < w(\pi,\alpha,k_0)\,.$$

Hence all eigenvalues of the matrix $(-\mathbb{M})$ belong to the set $R_{\alpha,k_0} = (-w(\pi,\alpha,k_0), 0)$ and from Theorem 10 the thesis holds.

□

Proposition 12. Let $Spec(M) = \{\lambda_i,\ i = 1,\dots,k,\ k \le n\}$ be the subset of $\mathbb{R}_+$, $\mathfrak{m}_i := \frac{2}{\sqrt{\lambda_i}}$ and $\eta(\lambda_i) := -\frac{2}{\sqrt{\lambda_i}}\cos\left(\alpha\frac{\pi}{2} + (2k_0-\alpha)\arcsin\left(\frac{1}{2}\lambda_i^{\frac{1}{2\alpha}}\right)\right)$.

If $\lambda_i < \left(2\sin\frac{(2-\alpha)\pi}{2(2k_0-\alpha)}\right)^{2\alpha}$ for all $i = 1,\dots,k$, and

$$\max_{1\le i\le k} \eta(\lambda_i) < \beta < \min_{1\le i\le k} \mathfrak{m}_i,$$

then the control law (15) solves the consensus problem for double-summator system (11) with the virtual leader.

Proof. Let λ_i, $i = 1,\dots,k$, $k \le n$ be the eigenvalues of the matrix M. Observe that one can choose diagonal matrix B in such a way that zero is not the eigenvalue of the Laplacian matrix L. Then $Spec(M) \subset \mathbb{R}_+ = (0,+\infty)$ and $\lambda_i \ne 0$ for all $i = 1,\dots,k$.

Since $\beta < \min_i \mathfrak{m}_i = \min_i \frac{2}{\sqrt{\lambda_i}}$ and obviously, $\min_i \frac{2}{\sqrt{\lambda_i}} < \frac{2}{\sqrt{\lambda_i}}$ for all $i = 1,\dots,k$, one gets $\beta < \frac{2}{\sqrt{\lambda_i}}$ for all $i = 1,\dots,k$. Then $\lambda_i^2\beta^2 - 4\lambda_i < 0$. Let s_{ij}, $j = 1,2$ be the eigenvalues of the matrix $(-\mathbb{M})$. Since $\lambda_i^2\beta^2 - 4\lambda_i < 0$ for all $i = 1,\dots,k$, we see that all eigenvalues of the matrix $(-\mathbb{M})$ are complex with nonzero imaginary part and given by the following pairs of values

$$\begin{aligned} s_{i1} &= \frac{-\lambda_i\beta + \sqrt{4\lambda_i - \lambda_i^2\beta^2}\,i}{2}, \\ s_{i2} &= \frac{-\lambda_i\beta - \sqrt{4\lambda_i - \lambda_i^2\beta^2}\,i}{2}. \end{aligned} \tag{20}$$

Then, for $i = 1,\dots,k$, $j = 1,2$, we have that $|s_{ij}| = \sqrt{\lambda_i}$ and $\cos\varphi_{ij} = -\frac{\beta\sqrt{\lambda_i}}{2} < 0$, $\sin\varphi_{ij} = \pm\frac{\sqrt{4-\beta^2\lambda_i}}{2}$, where $\varphi_{ij} := \arg s_{ij}$. Then, for each $i = 1,\dots,k$, $j = 1,2$, we have $\varphi_{ij} \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, so condition (4) given in Proposition 2 is satisfied. Considering condition (5) given in Proposition 2 we claim that $|s_{ij}| < w(\varphi_{ij},\alpha,k_0)$. Since values given by (20) are conjugate, we can consider only the cases with $\sin\varphi_{i1} = \frac{\sqrt{4-\beta^2\lambda}}{2} > 0$, $\cos\varphi_{i1} < 0$. Hence $\frac{\pi}{2} \le \varphi_{i1} \le \pi$. Then, in this case

$$\frac{1-\alpha}{2k_0-\alpha}\cdot\frac{\pi}{2} \le \frac{\varphi_{i1} - \alpha\frac{\pi}{2}}{2k_0-\alpha} \le \frac{2-\alpha}{2k_0-\alpha}\cdot\frac{\pi}{2} \le \frac{\pi}{2}. \tag{21}$$

Observe that $\lambda_i < \left(2\sin\frac{(2-\alpha)\pi}{2(2k_0-\alpha)}\right)^{2\alpha}$ implies that $\lambda_i^{\frac{1}{2\alpha}} < 2\sin\frac{(2-\alpha)\pi}{2(2k_0-\alpha)} \le 2$ for $i = 1,\dots,k$. From $\eta(\lambda_i) < \beta$ and $\beta = -\frac{2\cos\varphi_{i1}}{\sqrt{\lambda_i}}$ we easily derive that

$$\cos\varphi_{i1} < \cos\left(\alpha\frac{\pi}{2} + (2k_0-\alpha)\arcsin\left(\frac{1}{2}\lambda_i^{\frac{1}{2\alpha}}\right)\right)$$

and consequently,

$$\varphi_{i1} > \alpha\frac{\pi}{2} + (2k_0-\alpha)\arcsin\left(\frac{1}{2}\lambda_i^{\frac{1}{2\alpha}}\right)$$

and

$$\frac{\varphi_{i1} - \alpha\frac{\pi}{2}}{2k_0-\alpha} > \arcsin\left(\frac{1}{2}\lambda_i^{\frac{1}{2\alpha}}\right)$$

as $\alpha \in (0,1]$. Moreover, by (21) we have

$$\frac{\lambda_i^{\frac{1}{2\alpha}}}{2} < \sin\frac{\varphi_{i1} - \alpha\frac{\pi}{2}}{2k_0 - \alpha} \Leftrightarrow \sqrt{\lambda_i} < \left(2\sin\frac{\varphi_{i1} - \alpha\frac{\pi}{2}}{2k_0 - \alpha}\right)^{\alpha}.$$

Taking into account conjugate value of s_{ij} we get that $\sqrt{\lambda_i} = |s_{ij}| < \left|2\sin\frac{\varphi_{ij}-\alpha\frac{\pi}{2}}{2k_0-\alpha}\right|^{\alpha}$, and it is required for $s_{ij} \in R_{\alpha,k_0}$, $i = 1,\ldots,k$, $j = 1,2$. Hence, from Theorem 10 the thesis holds.

□

Proposition 13. Let $Spec(M) = \{\lambda_i,\ i = 1,\ldots,k,\ k \leq n\}$ be the subset of $\mathbb{R}_+$. If there exists λ_i such that $\lambda_i > \left(2\sin\frac{(2-\alpha)\pi}{2(2k_0-\alpha)}\right)^{2\alpha}$, then the control law (15) does not solve the consensus problem for double-summator system (11) with the virtual leader.

Proof. Let $\lambda_i \in Spec(M)$. Assume that $\lambda_i > \left(2\sin\frac{(2-\alpha)\pi}{2(2k_0-\alpha)}\right)^{2\alpha}$. In the case when $\beta < \frac{2}{\sqrt{\lambda_i}}$ we get that the eigenvalues for all s_{i1} and s_{i2} of the matrix $(-\mathbb{M})$ are complex with nonzero imaginary part and given by (20). Then for $j = 1,2$ we have $|s_{ij}| = \sqrt{\lambda_i}$. Similarly, as in the proof of Proposition 12 we use the fact that s_{i1} and s_{i2} are conjugate and one can consider only the cases with $\sin\varphi_{i1} = \frac{\sqrt{4-\beta^2\lambda}}{2} > 0$, $\cos\varphi_{i1} < 0$. Hence $\frac{\pi}{2} \leq \varphi_{i1} \leq \pi$. Observe that for $\varphi_{i1} = \arg s_{i1} \in \left[\frac{\pi}{2},\pi\right]$ one gets

$$\sin\frac{2\varphi_{i1} - \alpha\pi}{2(2k_0 - \alpha)} \leq \sin\frac{(2-\alpha)\pi}{2(2k_0 - \alpha)}. \tag{22}$$

Hence from the assumption and (22) we get

$$|s_{i1}| = \sqrt{\lambda_i} > \left(2\sin\frac{2\varphi_{i1} - \alpha\pi}{2(2k_0 - \alpha)}\right)^{\alpha}.$$

Then by Proposition 3 the thesis holds for $\beta < \frac{2}{\sqrt{\lambda_i}}$.

Now, let us consider the case when $\beta \geq \frac{2}{\sqrt{\lambda_i}}$. Then $\lambda_i^2\beta^2 - 4\lambda_i \geq 0$ and the eigenvalues w_{ij}, $j = 1,2$ of the matrix $\mathbb{M}$ are positive real numbers given by (19). Note that

$$w_{i1} = \frac{\lambda_i\beta + \sqrt{\lambda_i^2\beta^2 - 4\lambda_i}}{2} \geq \frac{\lambda_i\beta}{2} \geq \frac{\lambda_i\frac{2}{\sqrt{\lambda_i}}}{2} = \sqrt{\lambda_i}.$$

Therefore using the assumption one gets $w_{i1} \geq \sqrt{\lambda_i} > \left(2\sin\frac{(2-\alpha)\pi}{2(2k_0-\alpha)}\right)^{\alpha}$. Then by Proposition 3 the thesis holds for $\beta \geq \frac{2}{\sqrt{\lambda_i}}$.

□

From Proposition 13 we get the following necessary condition for reaching the consensus by double-summator system (11) with the virtual leader.

Corollary 14. Let $Spec(M) = \{\lambda_i,\ i = 1,\ldots,k,\ k \leq n\}$ be the subset of $\mathbb{R}_+$. If the control law (15) solves the consensus problem for double-summator system (11) with the virtual leader, then $\lambda_i \leq \left(2\left|\sin\frac{(2-\alpha)\pi}{2(2k_0-\alpha)}\right|\right)^{2\alpha}$, for all $\lambda_i \in Spec(M)$.

Now let $\ell_i := \eta(\lambda_i)$, where η is defined in Proposition 12 by

$$\eta(\lambda_i) = -\frac{2}{\sqrt{\lambda_i}}\cos\left(\alpha\frac{\pi}{2} + (2k_0 - \alpha)\arcsin\left(\frac{1}{2}\lambda_i^{\frac{1}{2\alpha}}\right)\right)$$

and $\mathfrak{u}_i$ is defined in Proposition 11 as follows

$$\mathfrak{u}_i = \min\left\{\frac{w(\pi,\alpha,k_0)}{\lambda_i} + \frac{1}{w(\pi,\alpha,k_0)}, 2\frac{w(\pi,\alpha,k_0)}{\lambda_i}\right\}.$$

Theorem 15. Let $Spec(M) = \{\lambda_i,\ i = 1,\ldots,k,\ k \leq n\} \subset \mathbb{R}_+$ and $\lambda_i < \left(2\sin\frac{(2-\alpha)\pi}{2(2k_0-\alpha)}\right)^{2\alpha}$, for all $\lambda_i \in Spec(M)$. If $\max_{1\leq i\leq k}\ell_i < \beta < \min_{1\leq i\leq k}\mathfrak{u}_i$, then control law (15) solves the consensus problem for double-summator system (11) with the virtual leader.

Proof. Let $0 < \lambda_i \in Spec(M)$. The proof is divided onto two cases, namely 1° $\frac{2}{\sqrt{\lambda_i}} \leq \beta$ and 2° $\frac{2}{\sqrt{\lambda_i}} > \beta$.

1° If $\frac{2}{\sqrt{\lambda_i}} \leq \beta$, then using the fact that $\beta, \lambda_i > 0$ one gets $\lambda_i^2\beta^2 - 4\lambda_i \geq 0$. Then the eigenvalues w_{ij}, $j = 1,2$, of the matrix $\mathbb{M}$ are positive real numbers. Since $\beta < \min_{1\leq i\leq k}\mathfrak{u}_i$, one can use the same arguments as in the proof of Proposition 11 and show that all eigenvalues of the matrix $(-\mathbb{M})$ belong to the set $R_{\alpha,k_0} = (-w(\pi,\alpha,k_0), 0)$ and from Theorem 10 the control law (15) solves the consensus problem for double-summator system (11) with the virtual leader.

2° In the case when $\frac{2}{\sqrt{\lambda_i}} > \beta$ one gets the eigenvalues s_{ij}, $j = 1,2$, of the matrix $(-\mathbb{M})$ are complex with nonzero imaginary part given by (20). Since $\max_{1\leq i\leq k}\ell_i < \beta$, one can use the same arguments as in the proof of Proposition 12 and show that $s_{ij} \in R_{\alpha,k_0}$, $i = 1,\ldots,k$, $j = 1,2$ and from Theorem 10 the control law (15) solves the consensus problem for double-summator system (11) with the virtual leader. Therefore the theorem holds.

□

4. Models with the Grünwald-Letnikov h-difference operator

Now let us define the Grünwald-Letnikov h-difference operator Δ_h^α of order α of function $x : (h\mathbb{N})_0 \to \mathbb{R}$ with the step $h > 0$ defined by

$$\left(\Delta_h^\alpha x\right)(kh) := h^{-\alpha}\sum_{s=0}^{k} a^{(\alpha)}(s)x((k-s)h).$$

Consider the following fractional-order systems of order $\alpha \in (0,1]$ with the Grünwald-Letnikov h-difference operator:

$$\left(\Delta_h^\alpha x\right)(kh) = F(x((k-k_0)h)), \quad k \geq k_0, \quad k_0 \in \mathbb{N}_1, \tag{23}$$

with initial conditions $x(0) = x_0, x(h) = x_1, \ldots, x((k_0-1)h) = x_{k_0-1} \in \mathbb{R}^n$, where $x = (x_1,\ldots,x_n)^{\mathrm{T}} : (h\mathbb{N})_0 \to \mathbb{R}^n$ is a vector function and $F : \mathbb{R}^n \to \mathbb{R}^n$.

It is easy to see that (23) is equivalent to

$$\left(\Delta^\alpha \overline{x}\right)(k) = h^\alpha F(\overline{x}(k-k_0)), \quad k \geq k_0, \ k_0 \in \mathbb{N}_1,$$

where $\overline{x}(k) := x(kh)$ for $k \in \mathbb{N}_0$.

Then, obviously, the set of asymptotical stability of the following linear systems:

$$\left(\Delta_h^\alpha x\right)(kh) = Ax((k-k_0)h), \tag{24}$$

depends on h and it is defined by

$$R_{\alpha,k_0,h} := \left\{z \in \mathbb{C} : \varphi = \arg z \in \left[\alpha\frac{\pi}{2}, 2\pi - \alpha\frac{\pi}{2}\right] \wedge |z| < h^{-\alpha} w(\varphi,\alpha,k_0)\right\}. \tag{25}$$

Note that the condition that all eigenvalues of the matrix A belong to $R_{\alpha,k_0,h}$ implies the asymptotical stability of system (24).

Now, let us formulate the conditions for reaching the consensus by Grünwald-Letnikov h-difference systems with the virtual leader. Since the proof of the presented below statements are similar to the ones given in the previous section, we are will not repeat them. The reader will be referred to appropriative earlier statements given in the previous section. Let $k_0 \in \mathbb{N}_1$, $k \geq k_0$, $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, $B := \mathrm{diag}\{b_1,\ldots,b_n\} \in \mathbb{R}^{n\times n}$, $b_i \in \{0,1\}$, $L = (l_{ij})$, M and M_β be defined like in the previous section, see Sec. 3.

Observe that for the systems with n agents with the single-summator dynamics we have the following results:

Theorem 16. If $Spec(-M_\beta) \subset R_{\alpha,k_0,h}$, where the set $R_{\alpha,k_0,h}$ is defined by (25), then control law of the following form

$$\begin{aligned} u_i(kh) &= f(kh) \\ &+(1-\beta)\sum_{j=1}^{n} a_{ij}\left(x_i((k-k_0)h) - x_j((k-k_0)h)\right) \\ &-\beta b_i\left[x_i((k-k_0)h) - x_r((k-k_0)h)\right], \end{aligned} \tag{26}$$

solves the consensus problem for single-summator system of the form

$$\begin{aligned} \left(\Delta^\alpha x_i\right)(kh) &= \sum_{j=1}^{n} a_{ij}\left[x_j((k-k_0)h)\right. \\ &\left.-x_i((k-k_0)h)\right] + u_i(kh), \end{aligned} \tag{27}$$

with the time-varying dynamics of the virtual leader given by

$$\left(\Delta^\alpha x_r\right)(kh) = f(kh). \tag{28}$$

Proof. The proof is the same as the proof of Theorem 6. □

Remark 17. Note that in the case when $Spec(M_\beta) = \{\lambda_i,\ i = 1,\ldots,k,\ k \leq n\} \subset \mathbb{R}_+$, the condition $0 < \beta < \frac{h^{-\alpha}w(\pi,\alpha,k_0)}{\max_i \lambda_i}$ implies the fact that control law (26) solves the consensus problem for single-summator system (27) with the virtual leader given by (28).

Now, let us consider double-summator systems with the Grünwald-Letnikov h-difference operator Δ_h^α.

Like in Subsec. 3.1 let a_{ij} be the (i,j)-th entry of the adjacency matrix $A \in \mathbb{R}^{n\times n}$ associated with graph $\mathcal{G}$, $\alpha \in (0,1]$, $x_i : (h\mathbb{N})_0 \to \mathbb{R}$ be the state function of the i-th agent, $v_i : (h\mathbb{N})_0 \to \mathbb{R}$ be the velocity function of the i-th agent, $u_i : \mathbb{N}_0 \to \mathbb{R}$ be the control input function for the i-th agent, $i = 1,2,\ldots,n$. Moreover, let x_r be the state of the virtual leader, v_r be the velocity of the virtual leader and f be a continuous function. Similarly as for the systems with the Grünwald-Letnikov difference operator Δ^α one can prove the following necessary condition for reaching the consensus by double-summator system with the virtual leader and with the Grünwald-Letnikov h-difference operator Δ_h^α.

Proposition 18. Let $Spec(M) = \{\lambda_i,\ i = 1,\ldots,k,\ k \leq n\}$ be the subset of $\mathbb{R}_+$. If the control law

$$\begin{aligned} u_i(kh) &= f(kh) \\ &+(1-\beta)\sum_{j=1}^{n} a_{ij}\left(v_i((k-k_0)h) - v_j((k-k_0)h)\right) \\ &-b_i\left[\left(x_i((k-k_0)h) - x_r((k-k_0)h)\right)\right. \\ &\left.+\beta\left(v_i((k-k_0)h) - v_r((k-k_0)h)\right)\right], \end{aligned} \tag{29}$$

solves the consensus problem for double-summator system

$$\begin{aligned} \left(\Delta^\alpha x_i\right)(kh) &= v_i((k-k_0)h), \\ \left(\Delta^\alpha v_i\right)(kh) &= \sum_{j=1}^{n} a_{ij}\left[\left(x_j((k-k_0)h) - x_i((k-k_0)h)\right)\right. \\ &\left.+\left(v_j((k-k_0)h) - v_i((k-k_0)h)\right)\right] + u_i(kh), \end{aligned} \tag{30}$$

with the virtual leader given by:

$$\left(\Delta^\alpha x_r\right)(kh) = v_r((k-k_0)h),$$

$$\left(\Delta^\alpha v_r\right)(kh) = f(kh),$$

then $\lambda_i \leq \left(\frac{2}{h}\left|\sin\frac{(2-\alpha)\pi}{2(2k_0-\alpha)}\right|\right)^{2\alpha}$, for all $\lambda_i \in Spec(M)$.

Proof. The proof can be done by contradiction in the similar way as the proof of Proposition 13 and $h^{-\alpha}w(\pi,\alpha,k_0)$ should be taken instead of $w(\pi,\alpha,k_0)$. □

Let $\ell_i := \eta(\lambda_i)$, where

$$\eta(\lambda_i) = -\frac{2}{\sqrt{\lambda_i}}\cos\left(\alpha\frac{\pi}{2} + (2k_0-\alpha)\arcsin\left(\frac{h^\alpha}{2}\lambda_i^{\frac{1}{2\alpha}}\right)\right)$$

and $\mathfrak{u}_i$ is defined as follows

$$\mathfrak{u}_i = \min\left\{\frac{w(\pi,\alpha,k_0)}{h^\alpha\lambda_i} + \frac{h^\alpha}{w(\pi,\alpha,k_0)}, 2\frac{w(\pi,\alpha,k_0)}{h^\alpha\lambda_i}\right\}.$$

Theorem 19. Let $Spec(M) = \{\lambda_i,\ i = 1,\ldots,k,\ k \leq n\} \subset \mathbb{R}_+$ and $\lambda_i < \left(\frac{2}{h}\sin\frac{(2-\alpha)\pi}{2(2k_0-\alpha)}\right)^{2\alpha}$, for all $\lambda_i \in Spec(M)$. If $\max_{1\leq i\leq k}\ell_i < \beta < \min_{1\leq i\leq k}\mathfrak{u}_i$, then the control law (29) solves the consensus problem for double-summator system (30) with the virtual leader.

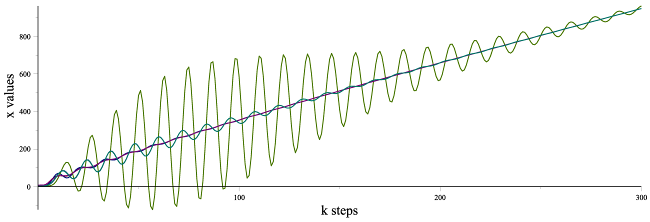
Proof. The proof is the same as the proof of Theorem 15 and $h^{-\alpha}w(\pi,\alpha,k_0)$ should be taken instead of $w(\pi,\alpha,k_0)$. □

Remark 20. Let $\mathbb{M}$ be defined by (17). Observe that in the case when $Spec(M) \subset \mathbb{R}_+$ in order to guarantee the consensus it is enough to take β such that $Spec(-h^\alpha M_\beta) \subset (-w(\pi,\alpha,k_0),0)$ and $Spec(-h^\alpha\mathbb{M}) \subset (-w(\pi,\alpha,k_0),0)$ for the single- and double-summator systems, respectively.

Example 21. Let us consider system (30) with 6 agents and the Laplacian matrix L_6 and the matrix B like in Example 8. Then after calculating eigenvalues of $L+B$ we obtain

$Spec(L+B) = \{2,1,1,1,1,1\}$. Let the leader have constant velocity $v_r = 10$.
Considering the system with delay $k_0 = 2$ and $h = 1$, any range of coefficients β does not exist for stability. However, if we change to $h = 0.5$ with $k_0 = 2$ we have the interval $(1.197066379; 1.490943053)$. The limits were calculated by Maple program. For $\alpha = 0.9$ and for the range $(1.197066379; 1.414213562)$ we have only pairs of complex eigenvalues of matrix $\mathbb{M}$ and for β from interval $(1.414213562; 1.490943053)$ we have on pair of complex eigenvalues and one with a real pair of eigenvalues. In both cases we have consensus, as shown in Figs. 3 and 4.

a) x values



b) v values

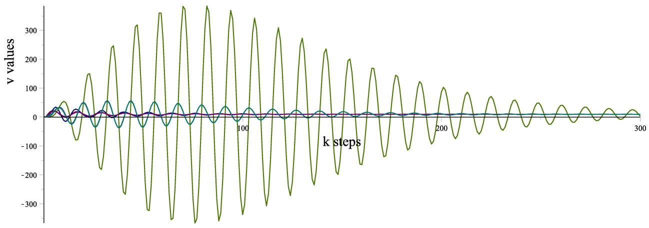
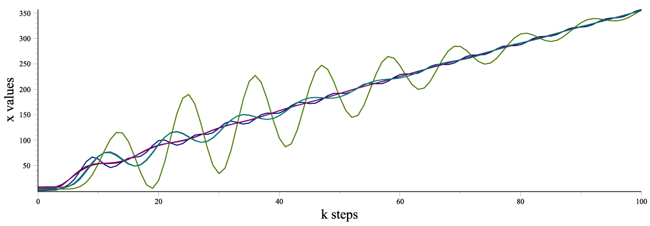


Fig. 3. Values of x and v for double summator model with matrix L_6, given in Example 8, with initial condition $x(0) = x(1) = (5,1,3,6,8,2)$, $v(0) = v(1) = [1,1,1,1,1,1]$, $v_r = 10$, $\underline{\alpha = 0.9}$, $T = 300$ steps, $n = 6$ agents, $k_0 = 2$, $\underline{\beta = 1.3}$, $h = 0.5$

a) x values



b) v values

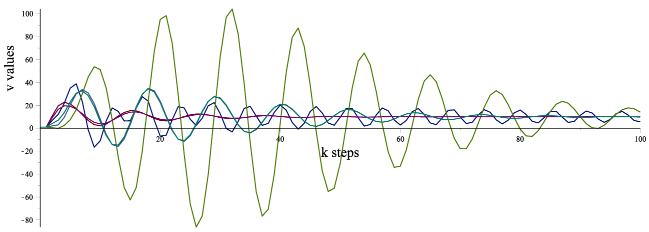


Fig. 4. Values of x and v for double summator model with matrix L_6, given in Example 8, with initial condition $x(0) = x(1) = (5,1,3,6,8,2)$, $v(0) = v(1) = [1,1,1,1,1,1]$, $v_r = 10$, $\underline{\alpha = 0.9}$, $T = 100$ steps, $n = 6$ agents, $k_0 = 2$, $\underline{\beta = 1.48}$, $h = 0.5$

Considering the system with delay $k_0 = 3$ and $h = 0.5$, any range of coefficients β does not exist for stability. Then, we choose smaller $h = 0.1$ with $k_0 = 3$. Then the range of β to reach the consensus is $(0.3059187677; 2.931854052)$. Graphs of trajectories are similar to the previous item.

5. Conclusions

In the paper, the leader-following consensus problem of fractional-order multi-agent discrete-time system with delays was considered. We included the memory to the system by taking both the fractional-order discrete-time operator on the left hand side of the nonlinear systems and the delays associated with the system. Models for the single- and double-summator dynamics of discrete-time fractional order opinions were investigated in both ways: by analytical methods and by computer simulations. Considered systems are an extension of the classical ones with the forms similar to the Krause and the Cucker-Smale models.

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