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Responses of positive standard and fractional linear systems and electrical circuits with derivatives of their inputs

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Abstract. Responses of positive standard and fractional continuous-time and discrete-time linear systems with derivatives of their inputs are presented herein. It is shown that the formulae for state vectors and outputs are also valid for their derivatives if the inputs and outputs and their derivatives of suitable order are zero for $t = 0$. Similar results are also shown for positive standard and fractional discrete-time linear systems.

Key words: positive fractional linear system, continuous-time, discrete-time, response formula.

1. Introduction

A dynamic system is referred to as positive if its trajectory starting from any nonnegative initial condition state remains forever in the positive orthant for all nonnegative inputs. An overview of the state of the art in positive system theory is provided in monographs [1, 2] and in papers [3–8]. Models manifesting positive behavior can be found in engineering, economics, social sciences, biology, medicine, etc.

Positive standard and descriptor systems and their stability have been analyzed in [2, 4, 5, 7–9]. Positive linear systems with different fractional orders have been addressed in [6, 9] and descriptor discrete-time linear systems – in [5]. Descriptor positive discrete-time and continuous-time nonlinear systems have been analyzed in [3] while the positivity and linearization of nonlinear discrete-time systems by state-feedbacks was discussed in [4]. New stability tests of positive standard and fractional linear systems have been investigated in [24]. The stability and robust stabilization of discrete-time switched systems have been analyzed in [11, 12]. Recently a new definition of the fractional derivative without singular kernel has been proposed in [13].

Derivation of the response formulae for linear systems is a classical problem of linear systems theory and it has been addressed in numerous books and papers [9–12, 14–17, 25]. Mathematical fundamentals of fractional calculus and some of its applications are discussed in monographs [18–21]. Some problems of fractional systems theory and its applications have been considered in [9].

In this paper the following problem is addressed: under which conditions are the well-known formulae for solutions of the equations of state and their outputs valid additionally for derivatives of their inputs for positive standard and fractional continuous-time and discrete-time linear systems?

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The paper is organized in the below-described manner. In Sec. 2, the problem is analyzed for positive standard continuous-time linear systems and electrical circuits, and in Sec. 3 – for positive standard discrete-time linear systems. The problem for fractional positive continuous-time linear systems is addressed in Sec. 4, and for fractional positive discrete-time linear systems – in Sec. 5. Concluding remarks are provided in Sec. 6.

The following notation will be used: $\Re$ – set of real numbers, $\Re^{n\times m}$ – set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – set of $n \times n$ Metzler matrices, Z_+ – set of nonnegative integers, I_n – $n \times n$ identity matrix.

2. Continuous-time positive linear systems

Consider the continuous-time linear system shown in Fig. 1, with the following impulse response matrix: $g(t) = \mathcal{L}^{-1}[G(s)]$, $G(s) = \mathcal{L}[g(t)] = \int_0^\infty g(t)e^{-st}dt$, where $G(s) \in \Re^{p\times m}(s)$ is the transfer matrix, $\mathcal{L}^{-1}$ is the inverse Laplace transform and $\Re^{p\times m}(s)$ is the set of $p\times m$ rational matrices in s.

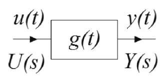

Fig. 1. Continuous-time linear system

Output $y(t) \in \Re^p$ of the system for input $u(t) \in \Re^m$ and zero initial conditions $x(0) = 0$ is obtained by applying the following formula:

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau. \tag{1}$$

Definition 1. [2] The continuous-time linear system is called externally positive if $y(t) \in \Re_+^p$, $t \in [0, +\infty)$ for every $u(t) \in \Re_+^m$, $t \in [0, +\infty)$.

Theorem 1. [2] The continuous-time linear system is externally positive if and only if

$$g(t) \in \Re_+^{p\times m} \quad \text{for} \quad t \in [0, +\infty). \tag{2}$$

In [22], it was shown that for standard systems (1) implies the following:

$$\dot{y}(t) = \int_0^t g(t-\tau)\dot{u}(\tau)d\tau \tag{3}$$

if and only if $u(0) = 0$ implies $y(0) = 0$.

Theorem 2. Assuming the following for the positive system:

$$u^{(q)}(t) \in \Re_+^m, \qquad t \in [0, +\infty) \quad \text{for} \quad q = 1, 2, ... \tag{4a}$$

implies:

$$y^{(q)}(t) \in \Re_+^p, \qquad t \in [0, +\infty) \quad \text{for} \quad q = 1, 2, ... \tag{4b}$$

And then equality (1) implies:

$$y^{(q)}(t) = \int_0^t g(t-\tau)u^{(q)}(\tau)d\tau \quad \text{for} \quad q = 1, 2, ... \tag{5}$$

if and only if

$$\begin{aligned} u^{(k)}(0) &= \left.\frac{d^k u(t)}{dt^k}\right|_{t=0} = 0, \\ y^{(k)}(0) &= \left.\frac{d^k y(t)}{dt^k}\right|_{t=0} = 0 \end{aligned} \tag{6}$$

for $k = 1, ..., q-1$.

Proof. Applying Laplace transform and the convolution theorem to (5) we obtain:

$$\begin{aligned} \mathcal{L}[y^{(q)}(t)] &= s^q Y(s) - \sum_{j=1}^{q} s^{q-j} y^{(j-1)}(0) = G(s)\mathcal{L}[u^{(q)}(t)] \\ &= G(s)\left[s^q U(s) - \sum_{j=1}^{q} s^{q-j} u^{(j-1)}(0)\right]. \end{aligned} \tag{7}$$

For zero initial conditions, we have:

$$y^{(k)}(0) = 0 \quad \text{for} \quad k = 1, ..., q-1, \tag{8}$$

$$s^q Y(s) = G(s) s^q U(s) = s^q G(s) U(s) \tag{9}$$

and (5) holds if and only if the conditions from (6) are satisfied. □

Example 1. Consider the electrical circuit shown in Fig. 2, with given resistance R, capacitance C and source voltage $u(t)$.

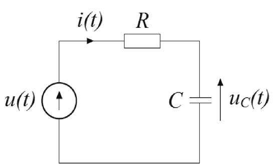

Fig. 2. Electrical circuit

By applying Kirchhoff's laws and Laplace transform to the electrical circuit, we obtain:

$$U(s) = sTU_C(s) + U_C(s) \quad \text{for} \quad u_C(0) = 0, \tag{10}$$

where $T = RC$, $U(s) = \mathcal{L}[u(t)]$, $U_C(s) = \mathcal{L}[u_C(t)]$.

The transfer function has the following form:

$$G(s) = \frac{U_C(s)}{U(s)} = \frac{1}{sT+1} \tag{11}$$

and the impulse response is:

$$g(t) = \mathcal{L}^{-1}[G(s)] = \frac{1}{T}e^{\frac{-t}{T}}. \tag{12}$$

Therefore, the electrical circuit is externally positive since $g(t) \in \Re_+$ for $t \geq 0$.

From (10), we get the following:

$$U_C(s) = \frac{U(s)}{T}\frac{1}{s+\frac{1}{T}} = \frac{1}{T}U(s)\mathcal{L}\left[e^{\frac{-t}{T}}\right]. \tag{13}$$

By applying the convolution theorem and inverse Laplace transform to (13), we obtain:

$$u_C(t) = \frac{1}{T}\int_0^t e^{-\frac{t-\tau}{T}}u(\tau)d\tau \tag{14}$$

and

$$\dot{u}_C(t) = \frac{1}{T}\int_0^t e^{-\frac{t-\tau}{T}}\dot{u}(\tau)d\tau \quad \text{for} \quad u(0) = 0. \tag{15}$$

Note that for:

$$u(t) = U\sin t \tag{16}$$

$u(0) = 0$, but for:

$$u(t) = U\cos t \tag{17}$$

$u(0) = U \neq 0$.

Using (15) for (16), we obtain:

$$i(t) = C\frac{du_C(t)}{dt} = \frac{U}{R}\int_0^t e^{-\frac{t-\tau}{T}}\cos\tau d\tau. \tag{18}$$

Consider the linear continuous-time system described by the following equations of state:

$$\dot{x} = Ax + Bu, \tag{19a}$$

$$y = Cx + Du, \tag{19b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors, respectively, and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

The solution to equation (19a) for zero initial conditions $x(0) = x_0 = 0$ has the following form:

$$x(t) = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{20}$$

Substitution of (20) into (19a) yields:

$$y(t) = C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t). \tag{21}$$

Definition 2. [1, 2] Continuous-time system (19) is referred to as internally positive if the state vector $x(t) \in \Re_+^n$ and the output vector $y(t) \in \Re_+^p$, $t \geq 0$ for all initial conditions $x_0 \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \geq 0$.

Definition 3. [2] Real matrix $A = [a_{ij}] \in \Re^{n\times n}$ is referred to as Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij} \geq 0$ for $i \neq j$, $i, j = 1, ..., n$.

Theorem 3. [1, 2] Continuous-time system (19) is internally positive if and only if:

$$A \in M_n, \quad B \in \Re_+^{n\times m}, \quad C \in \Re_+^{p\times n}, \quad D \in \Re_+^{p\times m}. \tag{22}$$

From Definitions 1 and 2 it follows that every internally positive system is always externally positive. However, there exists a large class of externally positive systems which are not internally positive [2].

Theorem 4. Equalities (20) and (21) imply, respectively:

$$x^{(q)}(t) = \int_0^t e^{A(t-\tau)} B u^{(q)}(\tau) d\tau, \qquad q = 1, 2, ... \tag{23}$$

and

$$y^{(q)}(t) = C \int_0^t e^{A(t-\tau)} B u^{(q)}(\tau) d\tau + D u^{(q)}(t), \tag{24}$$

if and only if the conditions of (6) are satisfied.

Proof. The proof is similar to the proof of Theorem 2.

Note that $x^{(q)}(t) \in \Re_+^n$ and $y^{(q)}(t) \in \Re_+^p$ for $t \in [0, +\infty)$ if and only if the conditions of (22) are satisfied and $u^{(q)}(t) \in \Re_+^m$, $t \in [0, +\infty)$.

Example 2. Consider the electrical circuit shown in Fig. 3, with given resistances R_1, R_2, R_3, inductances L_1, L_2 and source voltages e_1, e_2.

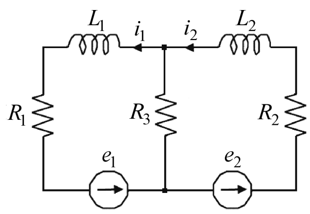

Fig. 3. Electrical circuit from Example 2

Using the Kirchhoff's laws, we may write the following equations:

$$e_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + R_3(i_1 - i_2), \tag{25a}$$

$$e_2 = R_2 i_2 + L_2 \frac{di_2}{dt} + R_3(i_2 - i_1) \tag{25b}$$

which can also be written in the following form:

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A_1 \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B_1 \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{26a}$$

where

$$A_1 = \begin{bmatrix} -\frac{R_1 + R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2 + R_3}{L_2} \end{bmatrix}, \quad B_1 = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \tag{26b}$$

As the output we choose:

$$y = i_1 + i_2 = C_1 \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, \qquad C_1 = [\,1 \quad 1\,]. \tag{27}$$

The electrical circuit is thus positive since $A_1 \in M_2$, $B_2 \in \Re_+^{2\times 2}$ and $C \in \Re_+^{1\times 2}$ for all values of resistances $R_k > 0$, $k = 1, 2, 3$ and inductances $L_k > 0$, $k = 1, 2$. The electrical circuit is asymptotically stable and the eigenvalues of matrix A_1 are real and negative since:

$$\det[I_2 s - A_1] = \begin{vmatrix} s + \frac{R_1 + R_3}{L_1} & -\frac{R_3}{L_1} \\ -\frac{R_3}{L_2} & s + \frac{R_2 + R_3}{L_2} \end{vmatrix} = s^2 + a_1 s + a_0, \tag{28a}$$

where

$$a_1 = \frac{R_1 + R_3}{L_1} + \frac{R_2 + R_3}{L_2} > 0, \quad a_0 = \frac{R_1(R_2 + R_3) + R_2 R_3}{L_1 L_2} > 0 \tag{28b}$$

for all $R_k > 0$, $k = 1, 2, 3$ and $L_k > 0$, $k = 1, 2$.

Using the Lagrange-Sylvester formula or Cayley-Hamilton theorem [1], we obtain:

$$e^{A_1 t} = c_0(t) I_2 + c_1(t) A_1 \in \Re_+^{2\times 2} \quad \text{for} \quad t \geq 0, \tag{29a}$$

where

$$c_0(t) = \frac{s_2 e^{s_1 t} - s_1 e^{s_2 t}}{s_2 - s_1} \geq 0, \quad c_1(t) = \frac{e^{s_2 t} - e^{s_1 t}}{s_2 - s_1} \geq 0, \qquad t \geq 0. \tag{29b}$$

From (20), (21), (26) and (29) for $u(0) = 0$ and $i_k(0) = 0$, $k = 1, 2$, we have:

$$\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \int_0^t e^{A_1(t-\tau)} B_1 u(\tau) d\tau \in \Re_+^2 \quad \text{for} \quad t \geq 0 \quad \text{and} \quad u(t) \in \Re_+^2 \tag{30a}$$

and

$$y(t) = C_1 \int_0^t e^{A_1(t-\tau)} B_1 u(\tau) d\tau \in \Re_+ \quad \text{for} \quad t \geq 0 \quad \text{and} \quad u(t) \in \Re_+^2. \tag{30b}$$

Note that:

$$\frac{d}{dt}\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = \int_0^t e^{A_1(t-\tau)} B_1 \dot{u}(\tau) d\tau \in \Re_+^2 \quad \text{and} \quad t \geq 0 \tag{31a}$$

and

$$\dot{y}(t) = C_1 \int_0^t e^{A_1(t-\tau)} B_1 \dot{u}(\tau) d\tau \in \Re_+ \quad \text{for} \quad t \geq 0 \tag{31b}$$

if and only if $\dot{u}(t) \in \Re_+^2$ for $t \geq 0$.

3. Discrete-time positive linear systems

Consider the discrete-time linear system shown in Fig. 4, with the given impulse response matrix $g(i) = \mathcal{Z}^{-1}[G(z)]$, $i \in Z_+$, $G(z) = \mathcal{Z}[g(i)] = \sum_{i=0}^{\infty} g(i) z^{-i}$, where $G(z) \in \Re^{p \times m}(z)$ is the transfer matrix of the discrete-time system and $\Re^{p \times m}(z)$ is the set of $p \times m$ rational matrices in z.

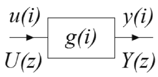

Fig. 4. Discrete-time linear system

Output $y(i) \in \Re^p$ of the system for input $u(i) \in \Re^m$ and zero initial conditions $x(0) = 0$ is obtained by applying the following formula:

$$y(i) = \sum_{j=0}^{i} g(i-j) u(j), \qquad i \in Z_+. \tag{32}$$

Definition 4. The discrete-time linear system is referred to as externally positive if $y(i) \in \Re_+^p$, $i \in Z_+$ for any $u(i) \in \Re_+^m$, $i \in Z_+$.

Theorem 4. [1, 2] The discrete-time linear system is externally positive if and only if:

$$g(i) \in \Re_+^{p \times m} \qquad \text{for} \qquad i \in Z_+. \tag{33}$$

It was shown in [22] that (32) implies the following:

$$\Delta y(i) = \sum_{j=0}^{i} g(i-j) \Delta u(j), \qquad i \in Z_+, \tag{34}$$

where $\Delta y(i) = y(i+1) - y(i)$ and $\Delta u(j) = u(j+1) - u(j)$ if and only if $u(0) = 0$ implies $y(0) = 0$.

In a general case we have the theorem described below.

Theorem 5. Equality (32) implies:

$$\Delta^{(q)} y(i) = \sum_{j=0}^{i} q(i-j) \Delta^{(q)} u(j), \qquad q = 1, 2, \ldots \tag{35}$$

if and only if

$$\Delta^{(k)} u(0) = \sum_{j=0}^{k} (-1)^j \frac{k!}{(k-j)!j!} u(k-j) = 0,$$

$$\Delta^{(k)} y(0) = \sum_{j=0}^{k} (-1)^j \frac{k!}{(k-j)!j!} y(k-j) = 0 \tag{36a}$$

$$\text{for} \qquad k = 1, 2, \ldots, q-1$$

or

$$u(i) = 0, \quad y(i) = 0 \quad \text{for} \quad i = 0, 1, \ldots, q-1. \tag{36b}$$

Proof. The proof is similar to the proof of Theorem 2.

Consider the linear discrete-time system described by the following equations of state:

$$x(i+1) = Ax(i) + Bu(i), \qquad i \in Z_+ = \{0, 1, \ldots\} \tag{37a}$$

$$y(i) = Cx(i) + Du(i), \tag{37b}$$

where $x(i) \in \Re^n$, $u(i) \in \Re^m$, $y(i) \in \Re^p$, $i \in Z_+$ are the state, input and output vectors, respectively, and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$, $D \in \Re^{p \times m}$.

The solution to equation (37a) for zero initial conditions $x(0) = 0$ has the following form:

$$x(i) = \sum_{j=0}^{i-1} A^{i-j-1} Bu(j), \qquad i \in Z_+. \tag{38}$$

Substitution of (38) into (37b) yields:

$$y(i) = C \sum_{j=0}^{i-1} A^{i-j-1} Bu(j) + Du(i), \qquad i \in Z_+. \tag{39}$$

Definition 5. Discrete-time system (6) is referred to as internally positive if $x(i) \in \Re_+^n$ and $y(i) \in \Re_+^p$, $i \in Z_+$ for all initial conditions $x(0) \in \Re_+^n$ and every $u(i) \in \Re_+^m$, $i \in Z_+$.

Theorem 6. [1, 2] Discrete-time system (6) is internally positive if and only if:

$$A \in \Re_+^{n \times n}, \qquad B \in \Re_+^{n \times m},$$
$$C \in \Re_+^{p \times n}, \qquad D \in \Re_+^{p \times m}. \tag{40}$$

Theorem 7. Equalities (38) and (39) imply, respectively:

$$\Delta^{(q)} x(i) = \sum_{j=0}^{i-1} A^{i-j-1} B \Delta^{(q)} u(j),$$
$$i \in Z_+, \quad q = 1, 2, \ldots \tag{41}$$

and

$$\Delta^{(q)} y(i) = C \sum_{j=0}^{i-1} A^{i-j-1} B \Delta^{(q)} u(j) + D \Delta^{(q)} u(i),$$
$$q = 1, 2, \ldots \tag{42}$$

if and only if the conditions of (36) are satisfied.

Proof. The proof is similar to the proof of Theorem 5.

Example 3. Consider discrete-time linear system (37) with the following matrices:

$$A = \begin{bmatrix} 0.2 & 1 \\ 0 & 0.3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = [\ 1 \quad 0\ ], \quad D = 0, \tag{43}$$

input:

$$u(i) = 2(1 - e^{-i}) \tag{44}$$

and zero initial conditions.

The system is positive since (43) satisfies the conditions of (40).

The transfer function of the system is equal to:

$$G(z) = C[Iz - A]^{-1}B + D = [\ 1 \quad 0\ ]\begin{bmatrix} z-0.2 & -1 \\ 0 & z-0.3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{z-0.2} \tag{45}$$

and

$$g(i) = \mathcal{Z}^{-1}[G(z)] = \mathcal{Z}^{-1}\left[\frac{1}{z-0.2}\right] = (0.2)^{i-1}, \quad i \in Z_+. \tag{46}$$

Using (32) and (46), we obtain:

$$y(i) = \sum_{j=0}^{i} g(i-j)u(j) = \sum_{j=0}^{i} (0.2)^{i-j-1} \cdot 2(1-e^{-j}), \quad i \in Z_+. \tag{47}$$

Note that (44) satisfies the condition $u(0) = 0$, but $u(1) = 2(1-e^{-1}) \neq 0$. Therefore, equalities (41) and (42) are satisfied only for $q = 1$ but are not satisfied for $q = 2, 3, \ldots$. From (42) and (47) for $q = 1$, we have:

$$\Delta y(i) = \sum_{j=0}^{i} g(i-j)\Delta u(j) = \sum_{j=0}^{i} (0.2)^{i-j-1} \cdot 2e^{-j}(1-e^{-1}), \qquad i \in Z_+. \tag{48}$$

Using (38), (43) and (44), we obtain:

$$x(i) = \sum_{j=0}^{i-1} \begin{bmatrix} 0.2 & 1 \\ 0 & 0.3 \end{bmatrix}^{i-j-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} 2(1-e^{-j}) \in \Re_+^2 \quad \text{for} \quad i = 1, 2, \ldots \tag{49}$$

and

$$\Delta x(i) = \sum_{j=0}^{i-1} \begin{bmatrix} 0.2 & 1 \\ 0 & 0.3 \end{bmatrix}^{i-j-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} 2e^{-j}(1-e^{-1}) \in \Re_+^2, \quad \text{for} \quad i = 1, 2, \ldots \tag{50}$$

since $\Delta u(j) = u(j+1) - u(j) = 2e^{-j}(1-e^{-1}) > 0$ for $j \in Z_+$.

4. Fractional continuous-time linear systems

In this section, the following Caputo definition of the fractional derivative will be used [9]:

$$ {}_0D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad n-1 < \alpha < n \in N = \{1, 2, \ldots\}, \tag{51a}$$

where $\alpha \in \Re$ is the order of the derivative, plus:

$$f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n} \tag{51b}$$

and

$$\Gamma(x) = \int_0^\infty e^{-t} t^{s-1} dt \tag{51c}$$

and as such it is the Euler gamma function.

Consider the following fractional continuous-time linear system:

$$\frac{d^\alpha x}{dt^\alpha} = Ax + Bu, \qquad 0 < \alpha < 1, \tag{52a}$$

$$y = Cx + Du, \tag{52b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors, respectively, and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$, $D \in \Re^{p \times m}$.

Applying the Laplace transform to (52) and taking into account that:

$$\mathcal{L}\left[\frac{d^\alpha x}{dt^\alpha}\right] = s^\alpha X(s) - s^{\alpha-1}x(0), \qquad X(s) = \mathcal{L}[x(t)] = \int_0^\infty x(t)e^{-st}dt, \qquad 0 < \alpha < 1 \tag{53}$$

for zero initial conditions $x(0) = 0$, we obtain:

$$X(s) = [I_n s^\alpha - A]^{-1}BU(s), \quad U(s) = \mathcal{L}[u(t)]. \tag{54}$$

Taking account of [9] and:

$$[I_n s^\alpha - A]^{-1} = \sum_{k=0}^\infty A^k s^{-(k+1)\alpha} \tag{55}$$

we obtain:

$$X(s) = \sum_{k=0}^\infty A^k s^{-(k+1)\alpha} BU(s). \tag{56}$$

By applying the inverse Laplace transform and the convolution theorem to (56), we obtain [23]:

$$x(t) = \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \tag{57}$$

where

$$\Phi(t) = \sum_{k=0}^\infty \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \tag{58}$$

Substitution of (57) into (52b) yields:

$$y(t) = C\int_0^t \Phi(t-\tau)Bu(\tau)d\tau + Du(t). \tag{59}$$

Theorem 8. Equalities (57) and (59) imply, respectively:

$$\frac{d^\beta x(t)}{dt^\beta} = \int_0^t \Phi(t-\tau)B\frac{d^\beta u(\tau)}{d\tau^\beta}d\tau, \quad 0<\beta<1 \tag{60}$$

and

$$\frac{d^\beta y(t)}{dt^\beta} = C\int_0^t \Phi(t-\tau)B\frac{d^\beta u(\tau)}{d\tau^\beta}d\tau + D\frac{d^\beta u(t)}{dt^\beta}, \quad 0<\beta<1 \tag{61}$$

if and only if $u(0)=0$, $y(0)=0$, respectively.

Proof. By multiplying (56) by s^β, we obtain:

$$s^\beta X(s) - s^{\beta-1}x(0) = \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha}B[s^\beta U(s) - s^{\beta-1}u(0)] \tag{62}$$

since by assumption $x(0)=0$ and $u(0)=0$.

Applying the inverse Laplace transform to (62), we obtain (60) if and only if $u(0)=0$. The proof of (61) is similar.

Definition 6. [9] Fractional continuous-time linear system (52) is referred to as internally positive if $x(t)\in\Re_+^n$ and $y(t)\in\Re_+^p$, $t\in[0,+\infty)$ for all initial conditions $x(0)\in\Re_+^n$ and every $u(t)\in\Re_+^m$, $t\in[0,+\infty)$.

Theorem 9. [9] Fractional continuous-time linear system (52) is internally positive if and only if:

$$A\in M_n, \quad B\in\Re_+^{n\times m}, \quad C\in\Re_+^{p\times n}, \quad D\in\Re_+^{p\times m}. \tag{63}$$

Example 4. Consider fractional continuous-time system (4.2) with the following matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [\,1 \quad 0\,], \quad D = 0, \tag{64}$$

input $u(t) = 1 - e^{-t}$ and zero initial conditions.

The fractional system in (64) is positive since the conditions of (63) are satisfied.

Using (57)–(59) and (64), we obtain:

$$\Phi(t) = \sum_{k=0}^{\infty}\frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = I_2\frac{t^{\alpha-1}}{\Gamma(\alpha)} + A\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} = \begin{bmatrix} \frac{t^{\alpha-1}}{\Gamma(\alpha)} & \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} \\ 0 & \frac{t^{\alpha-1}}{\Gamma(\alpha)} \end{bmatrix} \tag{65}$$

and

$$x(t) = \int_0^t \Phi(t-\tau)Bu(\tau)d\tau = \int_0^t \frac{1}{\Gamma(2\alpha)}\begin{bmatrix} (t-\tau)^{2\alpha-1} \\ (t-\tau)^{\alpha-1} \end{bmatrix}(1-e^{-\tau})d\tau \in \Re_+^2, \quad t\geq 0,$$

$$y(t) = C\int_0^t \Phi(t-\tau)Bu(\tau)d\tau = \int_0^t \frac{(t-\tau)^{2\alpha-1}}{\Gamma(2\alpha)}(1-e^{-\tau})d\tau \in \Re_+^2, \quad t\geq 0. \tag{66}$$

Note that $u(0)=0$ while from (60), (61) and (66) we obtain:

$$\frac{d^\beta x(t)}{dt^\beta} = \int_0^t \Phi(t-\tau)B\frac{d^\beta u(\tau)}{d\tau^\beta}d\tau = \int_0^t \frac{1}{\Gamma(2\alpha)}\begin{bmatrix} (t-\tau)^{2\alpha-1} \\ (t-\tau)^{\alpha-1} \end{bmatrix}\frac{d^\beta}{d\tau^\beta}(1-e^{-\tau})d\tau, \quad 0<\beta<1 \tag{67}$$

and

$$\frac{d^\beta y(t)}{dt^\beta} = C\int_0^t \Phi(t-\tau)B\frac{d^\beta u(\tau)}{d\tau^\beta}d\tau = \int_0^t \frac{(t-\tau)^{2\alpha-1}}{\Gamma(2\alpha)}\frac{d^\beta}{d\tau^\beta}(1-e^{-\tau})d\tau \quad 0<\beta<1. \tag{68}$$

5. Fractional positive discrete-time linear systems

Consider the following fractional discrete-time linear system:

$$\Delta^\alpha x(i+1) = Ax(i) + Bu(i), \quad i\in Z_+, \tag{69a}$$

$$y(i) = Cx(i) + Du(i), \tag{69b}$$

where $x(i)\in\Re^n$, $u(i)\in\Re^m$, $y(i)\in\Re^p$, $i\in Z_+$ are the state, input and output vectors, respectively, while $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$, $D\in\Re^{p\times m}$ and the fractional difference of the order α is defined as follows:

$$\Delta^\alpha x(i) = \sum_{j=0}^{i}(-1)^j\begin{pmatrix}\alpha \\ j\end{pmatrix}x(i-j), \tag{69c}$$

$$\begin{pmatrix}\alpha \\ j\end{pmatrix} = \begin{cases} 1 & \text{for } j=0 \\ \frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for } j=1,2,\ldots. \end{cases} \tag{69d}$$

Substituting (69c) into (69a), we obtain:

$$x(i+1) = A_\alpha x(i) + \sum_{j=2}^{i+1}(-1)^{j+1}\begin{pmatrix}\alpha \\ j\end{pmatrix}x(i-j+1) + Bu(i), \tag{70a}$$

where

$$A_\alpha = A + I_n\alpha. \tag{70b}$$

The solution to equation (70a) has the following form [9]:

$$x(i) = \Phi(i)x(0) + \sum_{j=0}^{i-1}\Phi(i-j+1)Bu(j), \tag{71}$$

where

$$\Phi(j+1)=A_\alpha\Phi(j)+\sum_{k=2}^{j+1}(-1)^{k+1}\binom{\alpha}{k}\Phi(j-k+1), \quad \Phi(0) = I_n. \tag{72}$$

Substitution of (72) into (69b) yields the following:

$$y(i)=C\Phi(i)x(0)+\sum_{j=0}^{i-1}C\Phi(i-j+1)Bu(j)+Du(i). \tag{73}$$

Definition 7. [9] Fractional discrete-time system (69) is referred to as internally positive if $x(i) \in \Re_+^n$ and $y(i) \in \Re_+^p$, $i \in Z_+$ for all initial conditions $x(0) \in \Re_+^n$ and every $u(i) \in \Re_+^m$, $i \in Z_+$.

Theorem 10. [9] Fractional discrete-time system (69) is internally positive if and only if:

$$A \in \Re_+^{n\times n}, \qquad B \in \Re_+^{n\times m}, \qquad C \in \Re_+^{p\times n}, \qquad D \in \Re_+^{p\times m}. \tag{74}$$

Theorem 11. Equalities (71) and (73) for zero initial condition $x(0) = 0$ imply, respectively:

$$\Delta x(i) = \sum_{j=0}^{i-1}\Phi(i-j+1)B\Delta u(j), \quad i \in Z_+ \tag{75}$$

and

$$\Delta y(i)=\sum_{j=0}^{i-1}C\Phi(i-j+1)B\Delta u(j)+D\Delta u(i), \quad i \in Z_+ \tag{76}$$

if and only if $u(0) = 0$, $y(0) = 0$, respectively.

Proof. Using (71) for $x(0) = 0$, we obtain:

$$\begin{aligned}\Delta x(i) &= x(i+1) - x(i)\\ &= \sum_{j=0}^{i}\Phi(i-j)Bu(j) - \sum_{j=0}^{i-1}\Phi(i-j+1)Bu(j)\\ &= \sum_{j=0}^{i-1}\Phi(i-j+1)B\Delta u(j)\end{aligned} \tag{77}$$

if and only if $u(0) = 0$. The proof of (76) is similar.

These considerations can be easily extended to higher order difference.

6. Concluding remarks

The responses of positive standard and fractional continuous-time and discrete-time linear systems with derivatives of their inputs have been presented herein. It has been shown that the formulae for state vectors and outputs are also valid for their derivatives if the inputs and outputs and their derivatives of suitable order are zero for $t = 0$ (Theorems 1, 2). Similar results are also valid for discrete-time linear systems (Theorem 7) and fractional linear systems (Theorem 8 and Theorem 11). The considerations have been illustrated by examples of continuous-time and discrete-time linear systems. The considerations can be further extended to relate also to positive standard and fractional descriptor linear systems.

What remains an open problem is the extension of these considerations to positive standard and fractional linear systems with delays.

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