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BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES
Vol. 55, No. 1, 2007

On adaptive modelling and filtering in computer simulation and experimental mechanics

S.A. ŁUKASIEWICZ[1]*, M.H. HOJJATI[2], and R. QIAN[3]

[1]Department of Mechanical and Manufacturing Engineering, The University of Calgary, 2500 University Dr, N.W., Calgary, AB, Canada, T2N1N4
[2]Department of Mechanical Engineering, The University of Mazandaran, Bobol, Iran
[3]Pumpwell SolutionsLtd, 157-9th Ave SW, Alberta, Canada, T2G 5N4

Abstract. The paper presents the application of the newly developed method of the solution of nonlinear equations to the adaptive modelling and computer simulation. The approach is suitable when the system of equations can be arranged in such a way that it consists of a large number of linear equations and a smaller number of nonlinear equations. This situation occurs in the case of adaptive modelling of mechanical systems using finite elements or finite differences techniques. In this case the classical least square method becomes very effective. The paper presents several examples of the application of the method. A solution to the, so called, "black box" problem is also presented.

Key words: adaptive modelling, computer simulation, experimental mechanics.

1. Introduction

The most basic step in performing the computer simulation and filtering of the data obtained from physical experiments is the selection of a model itself. Very often the model is based on incomplete empirical data. Selection of an inadequate model and parameters, that characterize it, can be the most important cause of errors in the computer simulation. A systematic approach for selection of appropriate models from a well-defined class of models can be very beneficial for the process. Using the iterative approach it is possible to correct the model as well to eliminate the noise and systematic errors from the measurement data.

Any data obtained from the measurements as for example, displacements, strains or temperature carry some experimental errors due to inherent inaccuracies and deficiencies in the experimental techniques and measuring devices used. However, the quantities being measured must obey some laws of physics. In the cases involving Thermodynamics and Structural Analysis, these laws represent the equations of motion of thermo-elastic material and the equations of heat transfer. The quantities measured with errors do not satisfy the required model equations. However, this measured set of data can be enhanced substantially by determining a new set satisfying the model equations being at the same time as close as possible to the measured set. The transition from the measured set containing the experimental errors and noise, to the enhanced set is referred to as filtering. The proposed techniques and filters are based on the deterministic approach called Adaptive Matrix Filter (AMF). The algorithm can be achieved using the mathematical optimization technique in which the distance norm between the measured and calculated experimental data is selected as the objective function and then minimized subject to constrains of the state equations.

Recently developed photo-cameras for infrared photography make possible very precise detection of the temperature changes. It is also possible to measure the fields of the displacement using laser devices. The direct response of the system can be used as the source of information for defining the model. For example, the heat conduction equations, thermo-elasticity equations and equations of motion (elasto-dynamic equations) can be used as model equations if the thermal properties of the material are to be defined.. At present the method of neural networks is often used to solve this type of problems. However, the most important drawback of this approach are difficulties with implementation and easy utilization of the model of the analyzed system.

2. Numerical solution method

Application of the Adaptive Modelling in mechanical problems requires the solution of large systems of linear and nonlinear equations. Very often the solution that is based on the Finite Element Method is accompanied by a system of nonlinear equations. In this case the whole system becomes nonlinear and is solved using methods for the solution of non-linear equations. The method of Newton-Raphson [1–4] iterations is most commonly used. This method needs the calculation of first derivatives and the Jacobian of the matrix for the system. The solution is

*e-mail: lukasiew@ucalgary.ca

obtained by means of consecutive iterations. If the functions are differentiable with respect to all variables and behave well it is possible to find the solution in a reasonable number of iterations. However, this needs the initial approximation for all the variables taken sufficiently close to the simultaneous roots of the nonlinear system. This approach becomes not effective if the number of equations is large. There are problems with the convergence to correct solution and problems with the initial approximation for the variables [5–7].

This paper presents a new method [8] for the solution of a system of $\boldsymbol{m}+\boldsymbol{n}$ nonlinear equations when the system of equations can be presented as two groups of equations. The first group of $\boldsymbol{m}$ equations is linear with respect to the selected $\boldsymbol{m}$ variables; the second group of $\boldsymbol{n}$ equations is nonlinear. The solution for the first group does not require any iterative procedures and can be found by means of any method for the system of linear equations. The proposed method uses iterations only for the nonlinear part and needs fewer numbers of initial approximation as compared to those needed in Newton-Raphson method.

The general system of equations can be presented in the following form

$$\boldsymbol{f}(x,t)=0, \quad \boldsymbol{\phi}(x,t)=0,$$

where the system $\boldsymbol{f}_i$ is linear with respect to the variables $\boldsymbol{x}_i$ with the assumption that the values of the variables $\boldsymbol{t}_i$ are known. Equations φ_n are non-linear with respect to the variables $\boldsymbol{x}_i$ and $\boldsymbol{t}_i$. Suppose that the vector $\boldsymbol{t}$ is the initial guess solution to the nonlinear variables of the system of the equations. Similarly, the vector $\boldsymbol{x}$ is the vector of initial solution for the linear part of the system equations based on known $\boldsymbol{t}$. The vector $\boldsymbol{x}$ can be found by means of any method for the system of linear equations. Let $x+\Delta x$ and $t+\Delta t$ be a better approximate solution.

Representing the functions $\boldsymbol{f}$ and $\boldsymbol{\phi}$ by Taylor expansion in vector notation, we have

$$\boldsymbol{\phi}(x,t)+\frac{\partial \boldsymbol{\phi}}{\partial x}\Delta x+\frac{\partial \boldsymbol{\phi}}{\partial t}\Delta t=0, \tag{1}$$

$$\boldsymbol{f}(x,t)+\frac{\partial \boldsymbol{f}}{\partial x}\Delta x+\frac{\partial \boldsymbol{f}}{\partial t}\Delta t=0. \tag{2}$$

Solving Eqs. (1) and (2) with respect to Δx and Δt gives:

$$\Delta \boldsymbol{t}=-\left[I-(\frac{\partial \boldsymbol{\phi}}{\partial t})^{-1}(\frac{\partial \boldsymbol{\phi}}{\partial x})(\frac{\partial \boldsymbol{f}}{\partial x})^{-1}(\frac{\partial \boldsymbol{f}}{\partial t})\right]^{-1} \times\left[\frac{\partial \boldsymbol{\phi}}{\partial t}\right]^{-1}[\boldsymbol{\phi}(x,t)] \tag{3}$$

$$\Delta \boldsymbol{x}=-\left[\frac{\partial \boldsymbol{f}}{\partial x}\right]^{-1}\left[\boldsymbol{f}(x,t)+\frac{\partial \boldsymbol{f}}{\partial t}\Delta t\right].$$

The equation for $\Delta \boldsymbol{t}$ is reduced the above form because $\boldsymbol{f}(x,t)=0$ in each iteration. The method is based on consecutive iterations. Only the initial values of the variables $\boldsymbol{t}$ have to be assumed. The variables $\boldsymbol{x}$ can be found solving equations $\boldsymbol{f}(x,t)=0$. Then the new values of $\boldsymbol{t}$ are calculated from: $\boldsymbol{t}_{i+1}=\boldsymbol{t}_i+\Delta \boldsymbol{t}_i$.

Equation (4) is used to calculate the increments in nonlinear variables in two consecutive iterations. I is the unit diagonal matrix of the order of n. The procedure of the suggested method for the solution of a system of equations is as follows. Initial approximations for the nonlinear variables $\boldsymbol{t}_0$ are assumed. By substituting this set of initial approximation in Eq. (1), the corresponding initial guess for linear variables x_0 is found without any iteration by means of any method used for linear system of equations. Now, by using Eq. (3), Δt is calculated and the next iteration is repeated using $t_0+\Delta t$ for $\boldsymbol{t}$. This procedure proceeds until the following equation is satisfied:

$$\text{Max}\,|\Delta x_i|\leqslant\delta \quad \text{and} \quad \text{Max}\,|\Delta t_i|\leqslant\delta$$

where δ is a small number which is chosen according to the required accuracy.

Below the following simple example is presented to explain the method. Let us consider the following system of equations. The variables to be found are x,y,z and t.

$$\begin{aligned} tx+2y-z&=21,\\ 2x-y+5z&=26,\\ -2x+3y-z&=0,\\ tx^2-2xy+y^2-tz^2&=-213. \end{aligned} \tag{4}$$

The first and last equations are clearly nonlinear. However, following the procedure explained in the previous section, if we consider t as the variable to be found by iterations, then the first three equations will be linear in terms of x,y and z can be solved for any given $\boldsymbol{t}$. Now we have: the first three equations can be solved for x,y and z as follows for any given t:

$$x=\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} t & 2 & -1\\ 2 & -1 & 5\\ -2 & 3 & -1 \end{bmatrix}^{-1}\begin{bmatrix} 21\\ 26\\ 0 \end{bmatrix}. \tag{5}$$

The required derivatives of the equations can be calculated as follows:

$$\begin{aligned} \Delta t=-&\left[1-(\frac{1}{x^2-z^2})[2tx-2y-2x+2y-2tz]\right.\\ &\left.\times\begin{bmatrix} t & 2 & -1\\ 2 & -1 & 5\\ -2 & 3 & -1 \end{bmatrix}^{-1}\begin{bmatrix} x\\ 0\\ 0 \end{bmatrix}\right]^{-1}(\frac{1}{x^2-z^2})\\ &\times(tx^2-2xy+y^2-tz^2+213). \end{aligned} \tag{6}$$

The solution of the system of equations only needs the initial values of t. Table 1 presents the solutions obtained for x,y,z and t using Eq. (12). The results are obtained using $\delta=0.001$. Table 1 also records the number of iterations.

The method found three independent solutions for x,y,z, and t. It was proved that the method is very effective [8]. It needs less number of initial approximations and much less iterations as compared with needed in Netwton-Raphson method.

Table 1
Results

Initial approximation for t	x	y	z	t	No. of iterations
0.5	15.6300	10.7900	1.1036	0.0331	7
1	–10.6594	–4.2339	8.6170	–3.5729	8
1.5	2.0000	3.0000	5.0000	10.0000	6
2	2.0000	3.0000	5.0000	10.0000	5
5	2.0000	3.0000	5.0000	10.0000	4
9	2.0001	3.0001	5.0000	10.0000	3
10	2.0000	3.0000	5.0000	10.0000	1
100	2.0000	3.0000	5.0000	10.0000	4
1000	2.0001	3.0000	5.0000	10.0000	4
10000	2.0001	3.0000	5.0000	10.0000	4

3. Examples of adaptive modelling

In order to demonstrate the application of the proposed method in the field of adaptive modelling, let us consider the following simple examples.

Example 1. Modelling of a simple beam. A steel beam simply supported on two end bearings is under pure bending. The lateral deflection at 9 equally spaced nodes along the beam length has been measured. The Young's modulus of the beam is to be found using these measurements. The measured data are affected by the noise caused by inherent inaccuracies and deficiencies in the experimental techniques and measuring devices used.

Let us assume that $u_i^*(x_i)$ is a vector of measured lateral deflection of the beam at nine nodes that contain errors, i is the number of total measurements ($i = 9$). The model equations providing the additional information about the system are:

$$D \equiv EI\frac{d^2u}{dx^2} + M_0 = 0, \text{ with } u_1 = 0,\ u_9 = 0, \tag{7}$$

E, I and M_0 are Young's modulus, moment of inertia and applied bending moment respectively. Vector u_i represents the corrected values of u_i^*. u_1 and u_9 are deflections at left and right bearings respectively. By using the method of least square with Lagrange multipliers, the global error $\boldsymbol{R}$ in the interval of interest can be defined. $\boldsymbol{R}$ is calculated as the square of the differences between the measured data and data predicted by the model. The derivatives of $\boldsymbol{R}$ with respect to u_i and λ_j, where λ_j are the Lagrange multipliers, must be zero.. The finite difference representation of the differential operator is used for seven internal nodes. It can be shown that the set of equations obtained in this way can be presented in the matrix form:

$$\boldsymbol{IU} + \boldsymbol{N\lambda} - \boldsymbol{U}^* = 0, \tag{8}$$

$$\boldsymbol{N^T U} - \boldsymbol{F}^* = 0. \tag{9}$$

where U and U^* are the vectors of corrected and measured variables respectively. I is the unit, diagonal matrix with the order of 9. λ is the vector of Lagrange multipliers, $\boldsymbol{N}$ is system matrix and F^* represents the applied loads. $\boldsymbol{N}$ is a linear matrix while F^* is a nonlinear function of E, the Young's modulus of the beam. Equations (8) and (9) represent 18 linear equations in terms of 9 corrected node deflections, $u_i(i = 1-9)$ and 9 Lagrange multipliers $\lambda_j(j = 1-9)$. These equations, however, are nonlinear in terms of the unknown E, the Young's modulus of the beam.

In order to follow the same procedure as explained before, equations (8) and (9) can be written in the form that is more similar to previously explained notations:

$$f(u, \lambda, E) = \boldsymbol{Ax} - \boldsymbol{B} = 0, \tag{10}$$

where:

$$\boldsymbol{A} = \begin{bmatrix} I & N \\ N^T & 0 \end{bmatrix}, \quad \boldsymbol{x} = [u_1\ u_2 \ldots u_9\ \lambda_1\ \lambda_2 \ldots \lambda_9]^T$$

and

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{U}^* \\ \boldsymbol{F}^* \end{bmatrix}.$$

The vector $\boldsymbol{x}$ represents the linear part of variables; the nonlinear part of variables consists only of E, the unknown Young's modulus of the beam.

For any given value of E, the matrix representation provides the unique solution for $\boldsymbol{\lambda}$ and $\boldsymbol{U}$ as:

$$\boldsymbol{\lambda} = (\boldsymbol{N}^T\boldsymbol{N})^{-1}(\boldsymbol{N}^T\boldsymbol{U}^* - \boldsymbol{F}^*), \tag{11}$$

$$\boldsymbol{U} = (\boldsymbol{I} - \boldsymbol{N}(\boldsymbol{N}^T\boldsymbol{N})^{-1}\boldsymbol{N}^T)\boldsymbol{U}^* + \boldsymbol{N}(\boldsymbol{N}^T\boldsymbol{N})^{-1}\boldsymbol{F}^*, \tag{12}$$

The matrix

$$(\boldsymbol{I} - \boldsymbol{N}(\boldsymbol{N}^T\boldsymbol{N})^{-1}\boldsymbol{N}^T) \tag{13}$$

is referred to as the filter matrix [9].

Considering the fact that E is unknown, the derivative of R with respect to E should also be zero. This leads to a nonlinear equation in the following form:

$$\varphi_1(\boldsymbol{u},\boldsymbol{\lambda},E) = \frac{\partial \boldsymbol{R}}{\partial E} = (\boldsymbol{u} - \boldsymbol{u}^*)^T\frac{\partial \boldsymbol{u}}{\partial E} + \frac{\partial \boldsymbol{\lambda}^T}{\partial E}(\boldsymbol{N}^T\boldsymbol{u} - \boldsymbol{F}^*) + \boldsymbol{\lambda}^T(\boldsymbol{N}^T\frac{\partial \boldsymbol{u}}{\partial E} - \frac{\partial \boldsymbol{F}^*}{\partial E}) = 0. \tag{14}$$

The system of equations in this example consists of 19 equations totally with 19 unknown deflections at 9 nodes, 9 Lagrange multipliers and Young's modulus of the beam. The first 18 equations represented by Eq. (10) are linear in terms of $u_i(i = 1-9)$ and $\lambda_j(j = 1-9)$ if the Young's modulus is given a certain value. This set can be solved without any iteration. The only nonlinear equation is Eq. (14). In this case the vector of nonlinear variables consists only of one variable, E that is to be found by iteration. Solution of the system of equations with the suggested method only needs the initial approximation for E. Substituting the corresponding derivatives as follows in Eq. (3) gives the equations for E increment (for $t = E$).

$$\frac{\partial \boldsymbol{f}}{\partial x} = \boldsymbol{A}, \quad \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{t}} = \frac{\partial \boldsymbol{f}}{\partial E} = -\frac{\partial \boldsymbol{B}}{\partial E},$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \boldsymbol{\phi_1}}{\partial x}, \quad \frac{\partial \phi}{\partial t} = \frac{\partial \phi_1}{\partial E},$$

In this example a steel beam with a length of 600 mm, a width of 50 mm and a thickness of 5 mm has been considered. The bending moment was assumed as 10000 Nmm. The theoretical lateral deflections at 9 equally spaced nodes along the beam length were used as measured values using value of 200 GPa as Young Modulus. This data were randomly modified by introducing a measurements error of the order of 5%. The expected value for E was 200 GPa. Table 2 presents the calculated E for different initial approximations when a value of 0.00001 is used for δ. The calculated values for corrected deflections are almost identical with the theoretical values. Regardless of the initial approximation for E, the program converged to the value 201 GPa, the expected value was 200 GPa. The difference between expected and calculated values can be attributed to the errors intentionally introduced while simulating the measurements data. The number of iterations is small which indicates good convergence rate of the method. The range of initial approximation program is also very wide.

Table 2
Calculated Young Modulus for the steel beam under pure bending

Initial approximation (GPa)	Calculated E (GPa)	No. of iteration	Residue of ϕ_1 (10^{-7})
0.001	201.24	65	–9.5
1	201.23	34	–9.8
5	201.31	27	–8.3
10	201.35	24	–7.4
100	201.45	13	–5.5
200	201.26	5	–9.2
330	201.28	18	–8.4

Example 2. Modelling of the sucker rod in the oil well. Rod pumping is the oldest and still the most common method of artificial lift used extensively in the oil well industry. In this example an adaptive modelling method has been used to model the dynamic behaviour of the rod of the sucker string (Fig. 1), [10–14]. The main concept was to replace the solution of the exact complicated mathematical model of the pumping system by a simple matrix operation in which the bottom-hole values were obtained as the product of the vector of the data at the top of the well and by a certain matrix of the system. It means that we were looking for the equation in the form:

$$\{U\}_{bottom} = F \cdot \{U\}_{top} + \{C\} \tag{15}$$

where $\boldsymbol{F}$ is a unknown matrix and $\{C\}$ is an unknown vector but $\{U\}_{bottom}$ and $\{U\}_{top}$ are known data at the top and the bottom of the rod respectively. The motion of the sucker rod in the oil well is affected by many unknown and unpredictable factors, as for example: friction in the well, oil viscosity and temperature, pressure at the bottom of the well etc. Using adaptive modelling technique, the calculations of the bottom-hole values can be performed faster and in a simple way. In the example presented here we found that to create the system matrix for the well it was enough to replace the real rod system by a two-segment rod with appropriate dimensions. The simplified model was solved using D'Alembert's method [15]. The technique used the field dynamometer data measured at the top (polished) rod and the calculated force and displacement at the plunger end from the analysis of the actual model for the same data. Then, the parameters of the equivalent two-segment rod were found by minimizing the error calculated as a difference between the results of two models. Using the equivalent model the system matrix was created. The equivalent mathematical model was simple, however it could replace the multi-segment actual rod working in unpredictable environment. The Adaptive Transfer Matrix (ATM) $\boldsymbol{F}$ was created for this model using Eqs. (12), (13).

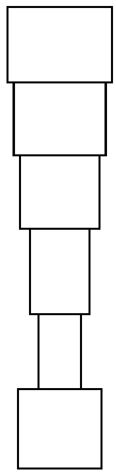

Fig. 1. A typical telescopic sucker rod with 6 segments

The data at the top of the sucker rod string were collected in a real oil well. The force and displacement at the top of the well (polished rod) were measured using a dynamometer. Then these data were used as the boundary conditions for the calculation of forces and displacements at the bottom of the rod string. The forces and displacements at the bottom of the well define the conditions of the pump, effectiveness of pumping, production rate, etc. The governing Eq. (1), for one dimensional motion of i-th segment of the rod, $u_i(x,t)$, is

$$a^2 \frac{\partial^2 u_i}{\partial x^2} = \frac{\partial^2 u_i}{\partial t^2} + b_i \frac{\partial u_i}{\partial t}, \text{ where } a^2 = \frac{E}{\rho},\ b_i = \frac{\eta}{EA_i}. \quad (16)$$

E and ρ are Young's Modulus and density of the rod material respectively. A_i is the cross section of i-th segment of the rod and η is the damping per unit length.

The D'Alembert's solution for the two-segment rod is given By setting $x = L$, the total length of the rod, displacement, u_2 and force, F_2 at the plunger end at any given time can be calculated.

$$u_2(x,t,b) = \frac{1}{4}\left\{\alpha_{12}\left[\Sigma u(t+\xi,b) - \frac{a}{EA_1}\hat{\Delta}Y(t,\xi,b)\right] + \bar{\alpha}_{12}\left[\Sigma u\left(t+\xi-2\lambda_1,b\right) + \frac{a}{EA_1}\hat{\Delta}Y(t+\xi-2\lambda_1,b)\right]\right\}, \quad (17)$$

$$F_2(x,t,b) = \frac{1}{4}\left\{\alpha_{12}\left[\frac{A_2}{A_1}\Sigma F(t+\xi,b) - \frac{EA_2}{a}\Delta V(t,\xi,b)\right] - \bar{\alpha}_{12}\left[\frac{A_2}{A_1}\Sigma F(t+\xi-2\lambda_1,b) + \frac{EA_2}{a}\Delta V(t+\xi-2\lambda_1,b)\right] - \frac{b}{2}\frac{EA_2}{a}\left[\alpha_{12}\Delta u(t+\xi,b) + \bar{\alpha}_{12}\Delta V(t+\xi-2\lambda_1,b)\right]\right\}$$

were the functions Δu, ΣF, ΔV are defined by Eq. (18)

In Eq. (17) V is the known velocity at the rod at the surface which is and other symbols are defined as followings:

$$\xi = \frac{x}{a},\ \alpha_{12} = 1 + \frac{A_1}{A_2},\ \bar{\alpha}_{12} = 1 - \frac{A_1}{A_2},\ \lambda_1 = \frac{l_1}{a},$$

where l_1 is the length of the first segment of the rod, a is given by Eq. (16)

$$\Sigma f(t+\theta,b) = f(t+\theta)e^{\frac{b}{2}\theta} + f(t-\theta)e^{-\frac{b}{2}\theta},$$
$$\Delta f(t+\theta,b) = f(t+\theta)e^{\frac{b}{2}\theta} - f(t-\theta)e^{-\frac{b}{2}\theta}, \quad (18)$$

where $f = u$, V and F and θ are a parameters that can have different values.

$$Y(t,\theta,b) = \int_0^\theta F(t+\tau)e^{\frac{b}{2}\tau}d\tau,$$

$$\hat{\Delta}Y(t,\theta,b) = Y(t,\theta,b) - Y(t,-\theta,b) = \int_{-\theta}^{\theta} F(t+\tau)e^{\frac{b}{2}\tau}d\tau, \quad (19)$$

where τ is the integration variable.

Equation (17) can be represented in a compact matrix form as:

$$\begin{Bmatrix} U \\ F \end{Bmatrix}_{bottom} = F \begin{Bmatrix} U \\ V \\ F \end{Bmatrix}_{top} + \begin{Bmatrix} C \\ 0 \end{Bmatrix} \quad (20)$$

where F is a matrix which allows to calculate the displacements and forces at the bottom of the well if the forces and displacements at the top of the well are known.

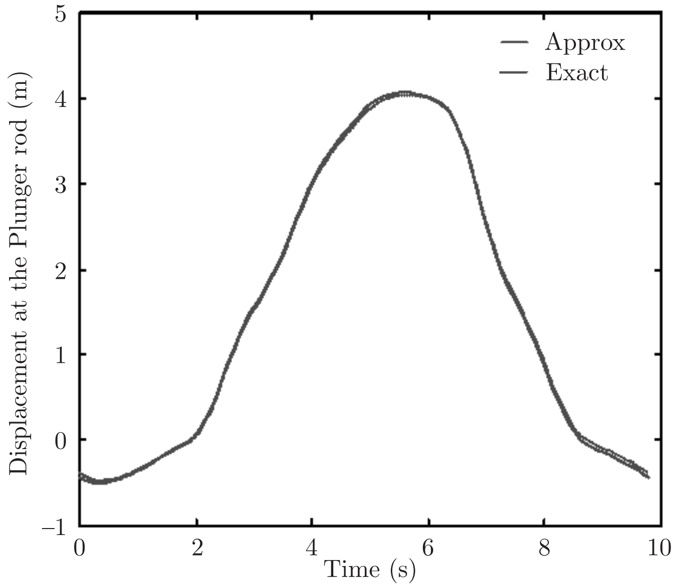

Fig. 2. Comparison between exact dynamic displacement at the plunger rod in the oil well of the 6 segment-rod and its 2-segments equivalent rod

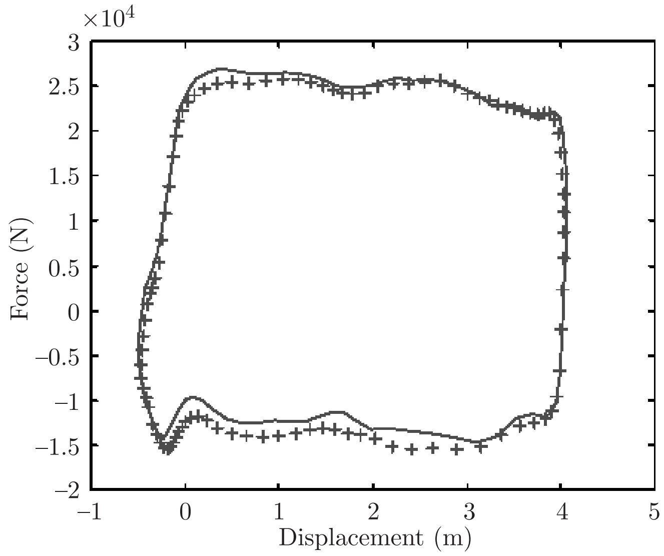

Fig. 3. Comparison between load-displacement curves at the plunger rod of the 6-segment rod (+) and its 2-segment equivalent model (Adaptive Transfer Matrix)

The matrix F is a matrix of coefficients of the right hand sides of the Eq. (17) over one full cycle of operation. We called it an Adaptive Transfer Matrix (ATM). This matrix depends only on the parameters of the system that were found through an optimization technique. Matrix F remains constant for a given system and was established only once. The vector C represents the constant terms of Eq. (17). The calculation of force and displacement at the plunger end has been reduced to the simple multiplication of matrix F and the vector of data from field dynamometer. The results of this approach were very successful. An ATM has been developed to predict the dynamic behaviour of a multi segment rod. The program successfully found the system matrix for equivalent 2-segment rod that gives the same displacement and force at the plunger end as those of the actual six-segment rod when the same input data at the top (polished) rod are applied to both systems. The equivalent model was used to develop the ATM (Eqs. 12, 13).and estimate the load and displacement at the plunger for any other dynamometer readings. The details of this work are presented in paper [16]. In order to demonstrate how the proposed approach reduces the amount of calculations, the equivalent model has been used with a different set of data at the top of the rod string. (Figs. 2, 3) show load-displacement curves at the polished rod and plunger obtained from exact calculations. This time, neither the exact data at the plunger nor the optimization program is used. The results were obtained from ATM for the equivalent model using the different dynamometer data. These results have been compared with the exact solution of the actual model in Fig. 4. The figure clearly shows that the equivalent model and adaptive transfer matrix is capable of estimating load and displacement at the plunger end for any other dynamometer data without loosing much accuracy. It was only necessary to multiply the data at the polished rod by the matrix F calculated for the analyzed system using the simplified model to obtain the results for the plunger. This operation was fast and simple.

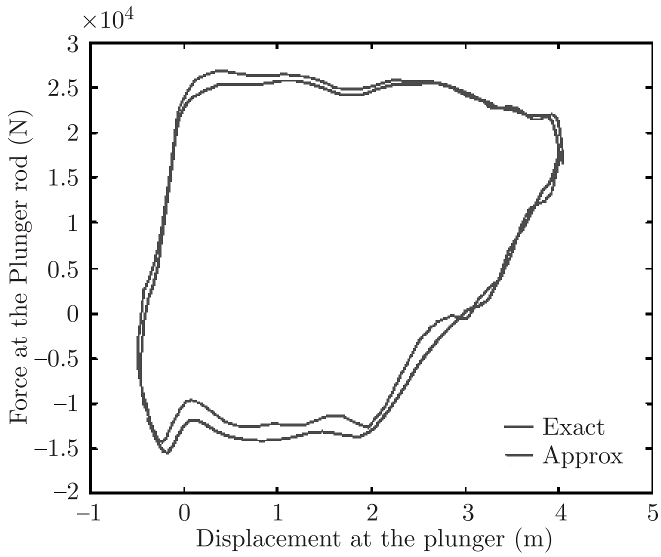

Fig. 4. Comparison between Force-Deflection curves at plunger rod from the actual model and the equivalent model (Adaptive Transfer Matrix)

The above described method could be used to find the ATM for any system for which we know the input and output data. This presents the solution for, the so called, "black box" problem.

Example 3. Modelling of thermal properties of a heated plate. The third example of adaptive modelling of a mechanical system presents the solution for a thermal problem. A layered plate heated at one surface is considered (Fig. 5). The purpose of the analysis is to define the material properties of the each layer using temperature distribution measured over the surface of the plate versus time is the source of information. Finite element method is used to solve the space-temperature equations of the theory of thermo-elasticity and finite difference method for the solution time-temperature equations. For a multilayered plate with simple support and isotropic material, the governing equations in a quasi-static problem are [17].

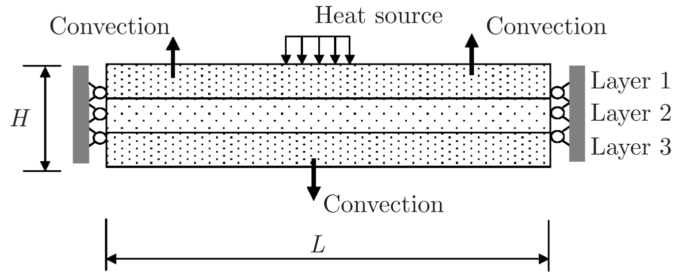

Fig. 5. 2D Model simplified from 3D model

$$k_I \nabla^2 T - (c\rho)_I \dot{T} + \dot{Q}_I = 0, \tag{21}$$

$$\sigma_{ij,i} = 0, \tag{22}$$

$$\sigma_{ij} = \lambda_I e_{\mu\mu} \delta_{ij} + 2G_I e_{ij} - \beta_I \delta_{ij} T, \tag{23}$$

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{24}$$

where $I = 1, 2, \ldots, N$, N is the total number of layers, T is the temperature, e_{ij} is the strain tensor and σ_{ij} is the stress tensor. The corresponding initial and boundary conditions for temperature T, heat flow q and displacements are given by:

$$T(\boldsymbol{x}, 0) = T^0(\boldsymbol{x}), \tag{25}$$

$$q(\boldsymbol{x}_0, t) = hA(T - T_\infty), \tag{26}$$

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}^0, \tag{27}$$

where the physical parameters are defined as

$$\beta_I = \frac{E_I a_I}{1 - 2\nu_I}, \quad G_I = \frac{E_I}{2(1 + \nu_I)}, \quad \lambda_I = \frac{2G_I \nu_I}{1 - 2\nu_I} \tag{28}$$

Here, for the i-th layer, ρ_I, c_I, k_I, $\dot{Q}_I$, a_I denote mass density, coefficient of the specific heat, heat conduction coefficient, volumetric heat generation rate, and coefficient of linear thermal expansion respectively, λ_I and G_I are Lame's constants, ν_I and E_I are Poisson's ratio and Young's modulus. The strain tensor must satisfy the compatibility condition:

$$e_{ij,kl} + e_{kl,ij} - e_{ik,jl} - e_{jl,ik} = 0. \tag{29}$$

The stress strain relations can be presented in the following matrix form

$$\boldsymbol{\sigma} = \boldsymbol{E}_I \boldsymbol{\varepsilon} + \boldsymbol{\beta}_I T, \tag{30}$$

Table 3
Material properties

Young's Modulus E (Pa)	Poisson's ratio ν	Heat conduction coefficient k (W/m K)	Coefficient of the specific heat c (W/kg K)	Density ρ (Kg/m^3)	Thermal expansion coefficient a(m/mK)
2.0e11	0.3	12	480	7800	2.65e-6
0.69e11	0.25	3	800	2330	1.5e-6

where

$$\boldsymbol{\sigma} = \left[\sigma_x\ \sigma_y\ \sigma_z\ \sigma_{xy}\ \sigma_{yz}\ \sigma_{xz}\right]^T,$$

$$\boldsymbol{\varepsilon} = \left[\varepsilon_x\ \varepsilon_y\ \varepsilon_z\ \varepsilon_{xy}\ \varepsilon_{yz}\ \varepsilon_{xz}\right]^T,$$

$$\boldsymbol{E}_I = \begin{bmatrix} \lambda_I + 2G_I & \lambda_I & \lambda_I & 0 & 0 & 0 \\ \lambda_I & \lambda_I + 2G_I & \lambda_I & 0 & 0 & 0 \\ \lambda_I & \lambda_I & \lambda_I + 2G_I & 0 & 0 & 0 \\ 0 & 0 & 0 & G_I & 0 & 0 \\ 0 & 0 & 0 & 0 & G_I & 0 \\ 0 & 0 & 0 & 0 & 0 & G_I \end{bmatrix},$$

$$\boldsymbol{\beta}_I = \left[-\beta_I\ -\beta_I\ -\beta_I\ 0\ 0\ 0\right]^T.$$

After applying FEM, the following governing equation for a whole body can be obtained in the following form:

$$\boldsymbol{KU} = \boldsymbol{F}.$$

By representing the derivatives of element temperature T^e and Q_I using the explicit finite difference scheme, difference equations are

$$k_I \nabla^2 \boldsymbol{T}^{em} - (c\rho)_I \frac{\boldsymbol{T}^{em+1} - \boldsymbol{T}^{em}}{\Delta t} + \frac{Q_I^{m+1} - Q_I^m}{\Delta t} = 0,$$
$$-k_I A \nabla^2 \boldsymbol{T}^{em}|_L = hA(\boldsymbol{T}^{em} - \boldsymbol{T}_\infty). \tag{31}$$

At time steps m and $m+1$, the governing equation and boundary condition of the whole body can be obtained as follows by assembling element governing equations and boundary conditions:

$$\boldsymbol{K}_1\boldsymbol{T} + \boldsymbol{CT} + \boldsymbol{Q} = 0, \tag{32}$$

$$\boldsymbol{K}_2\boldsymbol{T} + \boldsymbol{H}(\boldsymbol{T} - \boldsymbol{T}_\infty) = 0, \tag{33}$$

where

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{T}^m \\ \boldsymbol{T}^{m+1} \end{bmatrix},\ \boldsymbol{K}_1 = \left[k_I\nabla^2\ 0\right],\ \boldsymbol{K}_2 = \left[k_I\nabla^2\ 0\right]|_L,$$
$$\boldsymbol{C} = \left[\tfrac{(c\rho)_I}{\Delta t}\ -\tfrac{(c\rho)_I}{\Delta t}\right],\ \boldsymbol{H} = \left[h\ 0\right]. \tag{34}$$

with the initial condition $\boldsymbol{T} = \boldsymbol{T}^0$ and the boundary condition Eq. (32). Here $\boldsymbol{T}^0$ is the initial temperature distribution. $\boldsymbol{T}_\infty$ is the environment temperature that causes convection at the boundary. h is the coefficient of convection. A is the boundary area where convection occurs.

The global error functions for displacement u and temperature fields are

$$R^m = \frac{1}{2}[\boldsymbol{W}(\boldsymbol{B}_D\boldsymbol{U} - \boldsymbol{U}^*)]^T[\boldsymbol{W}(\boldsymbol{B}_D\boldsymbol{U} - \boldsymbol{U}^*)] + \boldsymbol{\lambda}(\boldsymbol{KU} - \boldsymbol{F})$$

$$R^T = \frac{1}{2}[\boldsymbol{W}(\boldsymbol{B}_{DT}\boldsymbol{T} - \boldsymbol{T}^*)]^T[\boldsymbol{W}(\boldsymbol{B}_{DT}\boldsymbol{T} - \boldsymbol{T}^*)] + \lambda^T(\boldsymbol{K}_1\boldsymbol{T} + \boldsymbol{CT} + \boldsymbol{Q}) + \eta^T(\boldsymbol{K}_2\boldsymbol{T} + \boldsymbol{H}(\boldsymbol{T} - \boldsymbol{T}_\infty)). \tag{35}$$

By minimizing R^m and R^T with respect to system material properties and responses, a set of linear-nonlinear algebraic equations for all identified parameters have been obtained. The above set of equations generates a correct solution for a given problem. The model problem represented in Fig. 5 has been used as an example to demonstrate the capability of the unified finite element-finite difference (FE-FD) approach. Figure presents the cross-section of 3D multi-layered rectangular plate with simple supports. The length of the plate was assumed infinite, therefore the problem is simplified to 2D model. It has been assumed that the material in layer 1 and layer 3 is steel and the material in layer 2 is ceramic, and their true material properties are shown in Table 3. The material data E_I, ρ_I, c_I, and k_I have been treated as temperature-independent parameters.

In the model presented in the Fig. 5 the heat source h = 10 W/m^2 °C is applied at the middle of the top surface and keeps constant temperature 80°C all the time. The fixed ends are at room temperature 250°C.

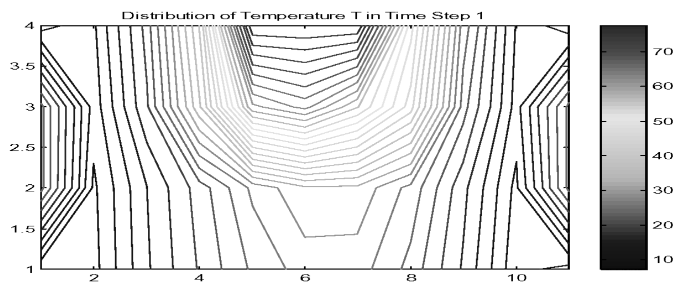

Fig. 6. Distribution of temperature in time step 1

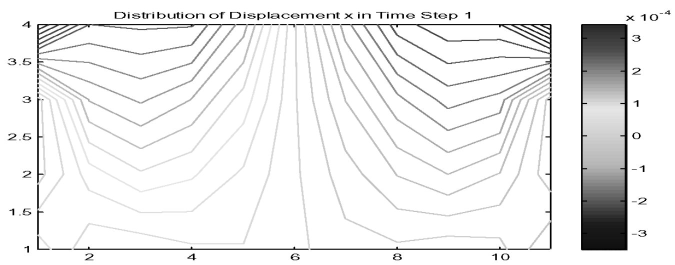

Fig. 7. Distribution of displacement x in time step 1

In order to find global minimum points, material property scale parameters have been introduced for every parameter as the ratios of expected value to the to the calculated values. The initial value of a material scale parameter is set to be 1. This guarantees that the minimum points obtained from Eq. (34) are the global minimum points. In this case the nonlinear set of equations was solved using Newton-Raphson method. The errors in the obtained result were less than 1%. The initial values of displacements and temperature distribution were calculated using computer software ANSYS (Figs. 6,7) without introducing any additional errors into the "measured data"

4. Conclusions

The described method of Adaptive Modelling [5] using the hybrid method of the solution of linear-nonlinear equations [8] proved to be effective tool in the modelling of the mechanical systems. In all examples the convergence was fast and the algorithms converged to the correct values. The method has also been successfully used in several other problems that are not presented here. This approach can be used to solve any, so called black box problems, where the input and output data are known but the system is unknown (sucker rod example). It has also been shown that the new method of the solution of linear – nonlinear equations makes this approach very effective. The solution needs smaller number of initial approximations as compared with those needed in Newton-Raphson method. The method is most useful for solving engineering problems in which a large number of linear equations are coupled with a limited number of nonlinear equations.

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