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Electric circuit analysis by means of optimization criteria Part II – complex circuits

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Abstract. In the paper the squared voltage-current functionals are minimized, which represent the global power losses in the network. In that way it is possible to find the voltage-current distributions on the net without the use of immitance operators and basing only on the Kirchhoff laws. Farther the individual branch parameters are defined in the syntheses process. Many optimal power analysis examples are also shown to illustrate the thesis included in the paper.

Key words: electrical networks, energy, analysis, synthesis, optimization.

1. Introduction

In the first part of the present article [1] the existence of certain equivocation was proved when analyzing power distribution in the one loop circuit. Such circuit is used in a power transmission process carried out e.g. by a single ideal overhead line. It was proved that there exists an infinity of possible current signals which assures constant voltage signal on a source and transmits the prescribed active power to the load. Using only some optimization criteria we get an unequivocal solution. This fact is of both theoretical and practical significance. It means that the transmission of power can meet various optimization criteria and satisfy various demands of sources and receivers. In the first part of this article some such criteria were considered, but its number is much bigger [1,2]. It then seems that the optimization approach lays the right direction to develop power quality and compensation theory [3–8].

The present study attempts to solve a complex problem of signals distribution in a transmission power net according to some optimization criteria. In comparison with one loop circuit, where the optimal signals are only one dimensional time depending ones, in the complex branched net the signals depend both on time and space dimension [2]. So far, the circuit theory has not dealt with such problems.

To make the problem clearer some examples will be presented at the beginning.

Example 1. Let us find the constant currents distribution in the box with prescribed constant values of currents in the inputs as to minimize the following functionals:

a) the sum of the module of inner branch currents (so called ‘taxi-norm’)

b) the sum of squared inner branch currents (the Euklides norm)

This task, in fact, consists of some norm minimization of the inner currents distribution. It is worth noting that there are no branch immittances used.

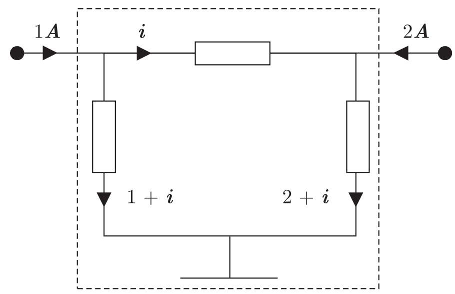

Fig. 1. The distribution of optimal inner currents in the box

The problem is obviously equivocal without the minimum condition. For the functional (a):

$$f(i) = |2 + i| + |i| + |1 - i| \to \min$$

where i is the free variable. In the intervals we get (Fig. 2)

$$f(i) = -3i - 1 \quad \text{for} \quad i < -2$$
$$f(i) = -i + 3 \quad \text{for} \quad -2 < i < 0$$
$$f(i) = i + 3 \quad \text{for} \quad 0 < i < 1$$
$$f(i) = 3i + 1 \quad \text{for} \quad i > 1$$

and the minimum point we reach for $i = 0$.

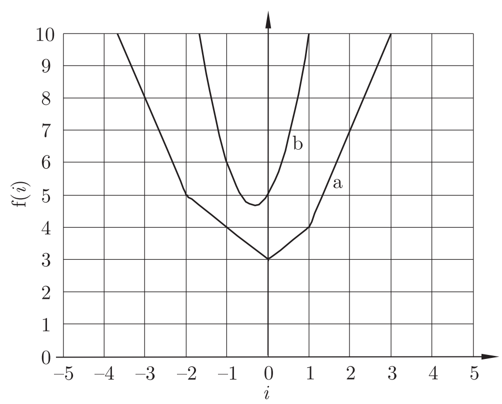

Fig. 2. The functionals (a) and (b)

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For the functional (b)

$$f(i) = (2+i)^2 + i^2 + (1-i)^2 \rightarrow \min$$

the minimum point is for $i = -1/3$. Thus the spatial distributions of inner currents are

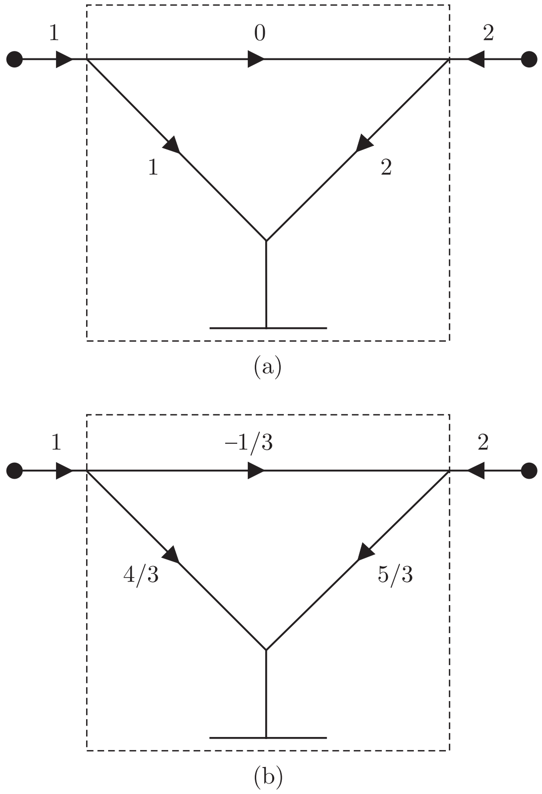


Fig. 3. The optimal spatial distributions of inner currents minimizing the functionals (a) and (b)

Example 2. Let us assume constant voltages (v_1, v_2) and currents (j_1, j_2) on the input and output of the box. We need to find the distribution of the inner currents which minimize the sum of the squared inner branch powers. Is there always a solution to this task?

The equivalent circuit is shown in Fig. 4.

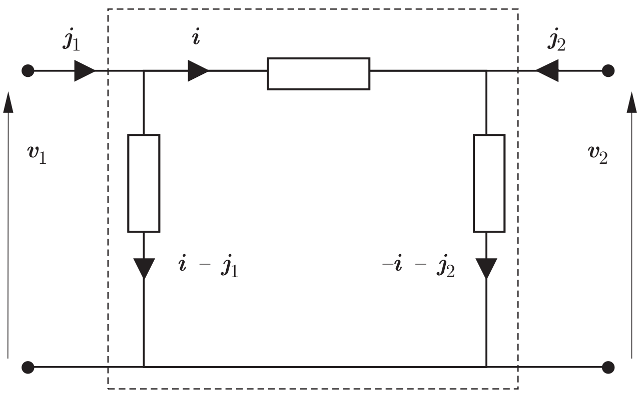


Fig. 4. The equivalent circuit of the box

This time the functional of the inner branch power is minimized, which is given by the formula

$$f(i) = v_1^2(i-j_1)^2 + (v_1 - v_2)^2 i^2 + v_2^2(i+j_2)^2 \rightarrow \min$$

where i is the free variable.

The minimum condition for the above task is

$$2(v_1^2 + v_2^2 - v_1 v_2)i = v_1^2 j_1 - v_2^2 j_2.$$

This gives the minimum point

$$i = \frac{v_1^2 j_1 - v_2^2 j_2}{v_1^2 + v_2^2 - v_1 v_2}.$$

The solution almost always exists because of the Schwarz inequality

$$v_1^2 + v_2^2 - |v_1 v_2| \geq 0.$$

The only exception occurs when bad conditioned task meets weekly the Schwarz inequality

$$v_1^2 + v_2^2 - |v_1 v_2| = 0.$$

Example 3. Let us assume constant voltages (v_1, v_2) and currents (j_1, j_2) on the input and output of the box. We need to find the distribution of the inner currents which minimize the sum of the squared inner branch powers.

The equivalent circuit is shown in Fig. 5.

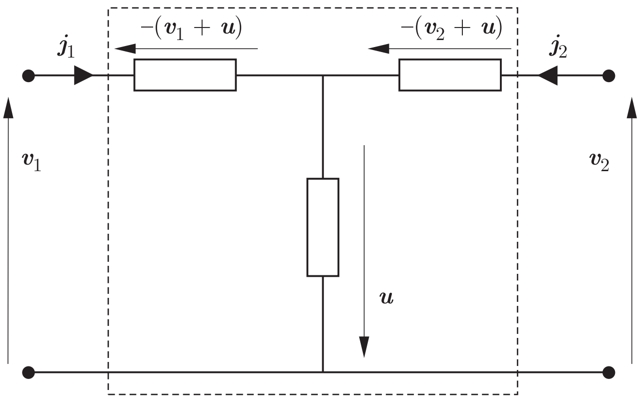


Fig. 5. The equivalent circuit of the box

This time the functional of the inner branch powers is minimized in the direction of free u variable

$$f(u) = j_1^2(u+v_1)^2 + (j_1+j_2)^2 u^2 + j_2^2(u+v_2)^2 \rightarrow \min.$$

Thus the minimum condition is

$$2(j_1^2 + j_2^2 + j_1 j_2)u = -j_1^2 v_1 - j_2^2 v_2.$$

Which gives the minimum point

$$u = -\frac{j_1^2 v_1 + j_2^2 v_2}{j_1^2 + j_2^2 + j_1 j_2}.$$

This solution also almost always exists if the Schwarz inequality is met

$$j_1^2 + j_2^2 - |j_1 j_2| \geq 0.$$

In the general situation the problem of finding the optimal inner signals concerns a multi terminal network which is depicted in Fig. 6.

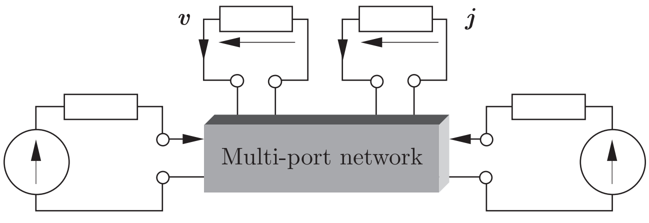


Fig. 6. The multi terminal network with prescribed input output signals

Let us consider the situation when the set of current j and voltage v signals on ports is given. It is possible to find the inner branch signal distribution according to the following theorem.

For the assumed inner structure of the network:

a) there exists an unequivocal voltage distribution on the inner branches for which the sum of squared the instantaneous voltage values is minimal
b) there exists an unequivocal current distribution on the inner branches for which the sum of the squared instantaneous current values is minimal.

In other words we can say: for the assumed inner structure of the network there exist unique vectors of the branch currents and branch voltages whose norms are minimal.

Proof. In order to determine the free voltages of the net we must use the topology theorems. We need to:

1) choose the tree starting from the port branches v
2) the sets subtraction gives:

TREE (of multi terminal network) $\setminus$ *PORT BRANCHES* (isomorphic to $\boldsymbol{v}$) = *FREE-BRANCHES* (isomorphic to $\boldsymbol{u}$) = *TREE-BRANCHES* $\setminus$ *OUTER TREE-BRANCHES* = *INNER TREE-BRANCHES*.

Thus using cut-set matrix $\boldsymbol{P}$ we can calculate the inner branch voltages $\boldsymbol{U}$ depending on port voltages $\boldsymbol{v}$ and free voltages $\boldsymbol{u}$.

$$\boldsymbol{U} = \boldsymbol{P}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{u}\end{bmatrix} \tag{1}$$

In order to calculate free currents we use links:

1) the prescribed branches (ports isomorphic to j) are the links. It is needed to add the rest of links as to make the complement be the tree,
2) then: *GIVEN LINKS* = *INNER FREE CURRENTS* i = *LINKS* $\setminus$ *OUTER LINKS*.

Thus using loop matrix $\boldsymbol{C}$ we can calculate the inner branch currents $\boldsymbol{I}$ depending on port currents j and free currents i.

$$\boldsymbol{I} = \boldsymbol{C}\begin{bmatrix}\boldsymbol{j}\\ \boldsymbol{i}\end{bmatrix} \tag{2}$$

The first minimization task is to choose the vector i to minimize norm

$$\boldsymbol{I}^T\boldsymbol{I} \rightarrow \text{min.} \tag{3}$$

Thus under (2) it results that

$$[\boldsymbol{j}^T\boldsymbol{i}^T]\boldsymbol{C}^T\boldsymbol{C}\begin{bmatrix}\boldsymbol{j}\\ \boldsymbol{i}\end{bmatrix} \rightarrow \text{min}$$

and after decomposition of $\boldsymbol{C}^T\ \boldsymbol{C}$ matrix on four submatrices

$$[\boldsymbol{j}^T\boldsymbol{i}^T]\begin{bmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12}\\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{j}\\ \boldsymbol{i}\end{bmatrix} \rightarrow \text{min}$$

we get

$$\boldsymbol{j}^T\boldsymbol{B}_{11}\boldsymbol{j} + \boldsymbol{j}^T\boldsymbol{B}_{12}\boldsymbol{i} + \boldsymbol{i}^T\boldsymbol{B}_{21}\boldsymbol{j} + \boldsymbol{i}^T\boldsymbol{B}_{22}i \rightarrow \text{min.} \tag{4}$$

Applying variation method to the (4) we get the following minimum condition

$$\boldsymbol{j}^T\boldsymbol{B}_{12}\delta\boldsymbol{i} + \delta\boldsymbol{i}^T\boldsymbol{B}_{21}\boldsymbol{j} + 2\delta\boldsymbol{i}^T\boldsymbol{B}_{22}\boldsymbol{i} + \delta\boldsymbol{i}^T\boldsymbol{B}_{22}\delta\boldsymbol{i} > 0 \tag{5}$$

for any $\delta\boldsymbol{i}$.

The condition (5) is met when

$$\delta\boldsymbol{i}^T\left[2\boldsymbol{B}_{22}\boldsymbol{i} + (\boldsymbol{B}_{21} + \boldsymbol{B}_{12}^T)\boldsymbol{j}\right] = 0$$

for any $\delta\boldsymbol{i}$.

Thus, because $\boldsymbol{B}_{21} = \boldsymbol{B}_{12}^T$, the $\boldsymbol{i}$ current meets the linear equation

$$2\boldsymbol{B}_{22}\boldsymbol{i} + 2\boldsymbol{B}_{21}\boldsymbol{j} = 0 \tag{6}$$

and since the matrix $\boldsymbol{B}_{22}$ is positively definite we get the unique solution

$$\boldsymbol{i} = -\boldsymbol{B}_{22}^{-1}\boldsymbol{B}_{21}\boldsymbol{j}. \tag{7}$$

Analogically we can choose voltage distribution $\boldsymbol{u}$ by minimizing

$$\boldsymbol{U}^T\boldsymbol{U} \rightarrow \text{min} \tag{8}$$

or using (1)

$$\boldsymbol{v}^T\boldsymbol{B}_{11}\boldsymbol{v} + \boldsymbol{v}^T\boldsymbol{B}_{12}\boldsymbol{u} + \boldsymbol{u}^T\boldsymbol{B}_{21}\boldsymbol{v} + \boldsymbol{u}^T\boldsymbol{B}_{22}\boldsymbol{u} \rightarrow \text{min} \tag{9}$$

the quested vector $\boldsymbol{u}$ meeting minimum condition (9) is calculated from

$$2\boldsymbol{B}_{22}u + 2\boldsymbol{B}_{21}\boldsymbol{v} = 0 \tag{10}$$

where $\boldsymbol{B}_{22}, \boldsymbol{B}_{21}$ are submatrices of $\boldsymbol{P}^T\boldsymbol{P}$ and $\boldsymbol{B}_{22}$ is positively definite matrix. Thus we get the unique solution $\boldsymbol{u}$

$$\boldsymbol{u} = -\boldsymbol{B}_{22}^{-1}\boldsymbol{B}_{21}\boldsymbol{v} \tag{11}$$

this completes the proof.

Example 4. Let us consider the three-port network depicted in Fig. 7.

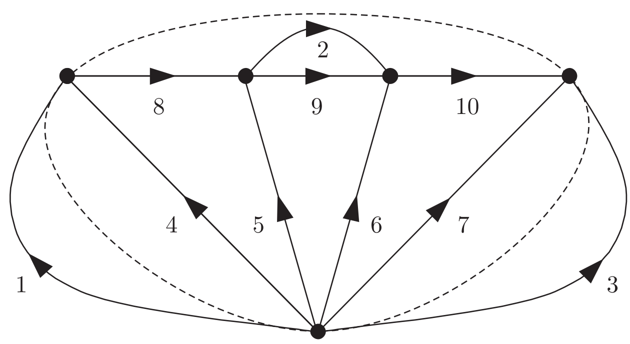


Fig. 7. The graph of three-port network

The set of ports {1, 2, 3} is isomorphic to prescribed vectors $\boldsymbol{v}, \boldsymbol{j}$; the set of the inner tree branches {5} is isomorphic to $\boldsymbol{u}$, as it is shown in Fig. 8.

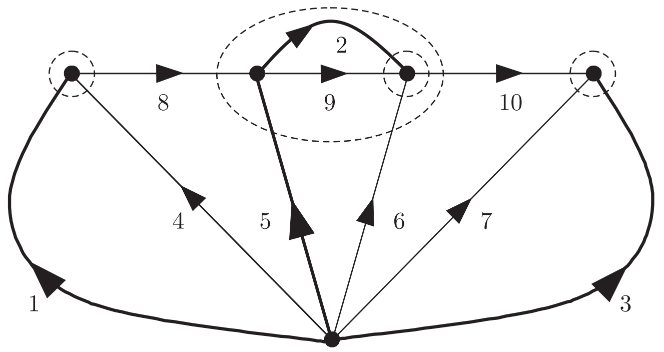


Fig. 8. The tree of the three-port network

The cut-set matrix $\boldsymbol{P}$ we find using the scheme shown in Fig. 8.

$$\boldsymbol{P} = \begin{array}{c|cccc|} & 1 & 2 & 3 & 5 \\ \hline 4 & 1 & & & \\ 5 & & & & 1 \\ 6 & & 1 & & 1 \\ 7 & & & 1 & \\ 8 & -1 & & & 1 \\ 9 & & 1 & & \\ 10 & & -1 & 1 & -1 \\ \hline \end{array}$$

Thus the block of $\boldsymbol{B}$ matrices have the following definition

$$\boldsymbol{P}^T\boldsymbol{P} = \begin{array}{c|ccc|c|} & 1 & 2 & 3 & 5 \\ \hline 1 & 2 & 0 & 0 & -1 \\ 2 & 0 & 3 & -1 & 2 \\ 3 & 0 & -1 & 2 & -1 \\ \hline 5 & -1 & 2 & -1 & 4 \\ \hline \end{array} = \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix}$$

Then the solution of set of equations, here reduced to the single equation, is:

$$u_5 = 0.25[1 - 2\ 1] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = 0.25(u_1 - 2u_2 + u_3).$$

Thus the optimal voltage distribution meeting $\sum\limits_{\alpha=4}^{10} (u_\alpha)^2 \to$ mincondition is found.

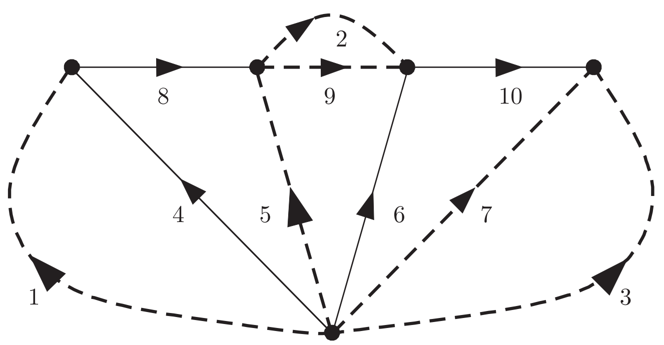


Fig. 9. The links (dotted lines) of the three port network

The choice of three port network links (from Fig. 7) is shown in Fig. 9. The loop matrix has form

$$\boldsymbol{C} = \begin{array}{c|cccccc|} & 1 & 2 & 3 & 5 & 7 & 9 \\ \hline 4 & -1 & 1 & & -1 & & 1 \\ 5 & & & & 1 & & \\ 6 & & -1 & -1 & & -1 & -1 \\ 7 & & & & & 1 & \\ 8 & & 1 & & -1 & & 1 \\ 9 & & & & & & 1 \\ 10 & & & -1 & & -1 & \\ \hline \end{array}$$

So the $\boldsymbol{B}$ submatrices are

$$\boldsymbol{C}^T\boldsymbol{C} = \begin{array}{c|ccc|ccc|} & 1 & 2 & 3 & 5 & 7 & 9 \\ \hline 1 & 1 & -1 & 0 & 1 & 0 & -1 \\ 2 & -1 & 3 & 1 & -2 & 1 & 3 \\ 3 & 0 & 1 & 2 & 0 & 2 & 1 \\ \hline 5 & 1 & -2 & 0 & 3 & 0 & -2 \\ 7 & 0 & 1 & 2 & 0 & 3 & 1 \\ 9 & -1 & 3 & 1 & -2 & 1 & 4 \\ \hline \end{array} = \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix}$$

Then the vector of inner currents is set by linear equations.

$$\begin{bmatrix} 3 & 0 & -2 \\ 0 & 3 & 1 \\ -2 & 1 & 4 \end{bmatrix} \begin{bmatrix} i_5 \\ i_7 \\ i_9 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -1 & -2 \\ 1 & -3 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}$$

Consequently the optimal current distribution meeting $\sum\limits_{\alpha=4}^{10} (i_\alpha)^2 \to$ mincondition is found.

2. Introduction to the timevarying spatial optimization

In the previous Section the optimal spatial distribution of the inner signals (on condition that port signals are given) was analyzed. Now we will consider timevarying spatial distributions. Besides the spatial coordinate (the branch number) the discrete time coordinate will be introduced.

We define the following symbols: $v_{n,\alpha}$, $i_{n,\alpha}$ – two dimensional discreet voltage and current signals, where: n – discrete time sample number, α – discrete space number (the branch number).

$$\boldsymbol{I} = \left[(i_0, i_1, ...)_\alpha\right]^T = \left[\mathrm{col}_\alpha\{\mathrm{col}_n\,[i_{n,\alpha}]\}\right]$$

$$= \begin{bmatrix} \boldsymbol{I}_1 \\ \vdots \\ \boldsymbol{I}_\alpha \\ \vdots \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} i_0 \\ \vdots \\ i_{N-1} \end{bmatrix}_1 \\ \vdots \\ \begin{bmatrix} i_0 \\ \vdots \\ i_{N-1} \end{bmatrix}_\alpha \\ \vdots \end{bmatrix}$$

where: $\alpha \in$ *INNER BRANCHES*, $n \in \{0, 1, .., N-1\}$,
$\boldsymbol{j} = \left[\mathrm{col}_\alpha\{\mathrm{col}_n\,[i_{n,\alpha}]\}\right], \alpha \in$ *OUTER LINKS*,
$n \in \{0, 1, ..., N-1\}$,

$\boldsymbol{i} = [\mathrm{col}_\alpha \{\mathrm{col}_n [i_{n,\alpha}]\}]$, $\alpha \in$ *INNER LINKS*,
$n \in \{0, 1, ..., N-1\}$,
and
$\boldsymbol{U} = [\mathrm{col}_\alpha \{\mathrm{col}_n [u_{n,\alpha}]\}]$, $\alpha \in$ *INNER BRANCHES*,
$n \in \{0, 1, ..., N-1\}$,
$\boldsymbol{v} = [\mathrm{col}_\alpha \{\mathrm{col}_n [u_{n,\alpha}]\}]$, $\alpha \in$ *OUTER TREE-BRANCHES*,
$n \in \{0, 1, ..., N-1\}$,
$\boldsymbol{u} = [\mathrm{col}_\alpha \{\mathrm{col}_n [u_{n,\alpha}]\}]$, $\alpha \in$ *INNER TREE-BRANCHES*,
$n \in \{0, 1, ..., N-1\}$.

The instant values of the branch signals for n sample will be also helpful:
$\boldsymbol{I}_n = \mathrm{col}_\alpha [i_{n,\alpha}]$ $\alpha \in$ *INNER BRANCHES*,
$\boldsymbol{j}_n = \mathrm{col}_\alpha [i_{n,\alpha}]$ $\alpha \in$ *OUTER LINKS*,
$\boldsymbol{i}_n = \mathrm{col}_\alpha [i_{n,\alpha}]$ $\alpha \in$ *INNER LINKS*,
$\boldsymbol{U}_n = \mathrm{col}_\alpha [u_{n,\alpha}]$ $\alpha \in$ *INNER BRANCHES*,
$\boldsymbol{v}_n = \mathrm{col}_\alpha [u_{n,\alpha}]$ $\alpha \in$ *OUTER TREE-BRANCHES*,
$\boldsymbol{u}_n = \mathrm{col}_\alpha [u_{n,\alpha}]$ $\alpha \in$ *INNER TREE-BRANCHES*.

Using the foregoing definitions, the connections (1) and (2) take the general form:

$$\boldsymbol{U} = [\boldsymbol{P}] \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix} \tag{12}$$

$$\boldsymbol{I} = [\boldsymbol{C}] \begin{bmatrix} \boldsymbol{j} \\ \boldsymbol{i} \end{bmatrix} \tag{13}$$

where $[\boldsymbol{P}]$, $[\boldsymbol{C}]$ stands for the multiplied loop and cut-set matrices where the scalars 0,1 are substituted by the $N \times N$ zeros and eye matrices.

Widening the problem (3), (8) on time coordinate we get the time depending solutions (see (7),(11))

$$\boldsymbol{i} = -[\boldsymbol{B}_{22}]^{-1} [\boldsymbol{B}_{21}] \boldsymbol{j} \tag{14}$$

$$\boldsymbol{u} = -[\boldsymbol{B}_{22}]^{-1} [\boldsymbol{B}_{21}] \boldsymbol{j} \tag{15}$$

where $[\boldsymbol{B}_{22}]$, $[\boldsymbol{B}_{21}]$ stands for the multiplied $\boldsymbol{B}_{22}$, $\boldsymbol{B}_{21}$ matrices.

For a sample $n \in \{0, 1, ..., N-1\}$, in a specific moment of time, the foregoing equations change to the form

$$\boldsymbol{i} = -[\boldsymbol{B}_{22}]^{-1} [\boldsymbol{B}_{21}] \boldsymbol{j}_n \tag{14}$$

$$\boldsymbol{u} = -[\boldsymbol{B}_{22}]^{-1} [\boldsymbol{B}_{21}] \boldsymbol{j}_n \tag{15}$$

which is the same as (7) and (11).

We can define the norms of the time-spatial signal distribution

$$\begin{aligned} \boldsymbol{I}^T\boldsymbol{I} &= \sum_n \sum_{\alpha \in \textit{INNER BRANCHES}} (i_n\alpha)^2 \\ &= \sum_{\alpha \in \textit{INNER BRANCHES}} \left(\sum_n (i_n\alpha)^2 \right) \\ &= \sum_{\alpha \in \textit{INNER BRANCHES}} ||\boldsymbol{I}_\alpha||^2 \end{aligned} \tag{16}$$

$$\begin{aligned} \boldsymbol{U}^T\boldsymbol{U} &= \sum_n \sum_{\alpha \in \textit{INNER BRANCHES}} (u_n\alpha)^2 \\ &= \sum_{\alpha \in \textit{INNER BRANCHES}} \left(\sum_n (u_n\alpha)^2 \right) \\ &= \sum_{\alpha \in \textit{INNER BRANCHES}} ||\boldsymbol{U}_\alpha||^2 \end{aligned} \tag{17}$$

where

$$\boldsymbol{I}_\alpha = \mathrm{col}_n [i_{n,\alpha}], \boldsymbol{U}_\alpha = \mathrm{col}_n [u_{n,\alpha}] \tag{18}$$

stands for current and voltage norms of α-branch. Thus we can formulate the theorem 1 in a different way.

THEOREM 1. For the assumed inner structure of the network there exist the unique voltage signal distribution assuring that the sum of squared rms voltage values of the inner branches is minimal, and there exist the unique current signal distribution assuring that the sum of squared rms current values of the inner branches is minimal.

The norms

$$||\boldsymbol{I}_\alpha||^2 = \boldsymbol{I}_n^T \boldsymbol{I}_n \tag{19}$$

$$||\boldsymbol{U}_\alpha||^2 = \boldsymbol{U}_n^T \boldsymbol{U}_n \tag{20}$$

can be called the spatial rms values.

Thus the Theorem 2 states that the following coefficients

$$S_n = \sqrt{\boldsymbol{I}_n^T \boldsymbol{I}_n} \quad \sqrt{\boldsymbol{U}_n^T \boldsymbol{U}_n} \rightarrow \min \tag{21}$$

(apparent instant power of the net) or

$$S = \sqrt{\boldsymbol{I}^T \boldsymbol{I}} \quad \sqrt{\boldsymbol{U}^T \boldsymbol{U}} \rightarrow \min \tag{22}$$

(apparent time-spatial power of the net) should be minimized.

3. The power functionals minimization

The minimization of the squared norm of instant power distribution in the net is the much more important issue. It is because the functional value represents the total instant power value of the net, which is a practical information, and it is the basis to some important generalizations. Using the vector notation from Section 2 we make the following new notation:

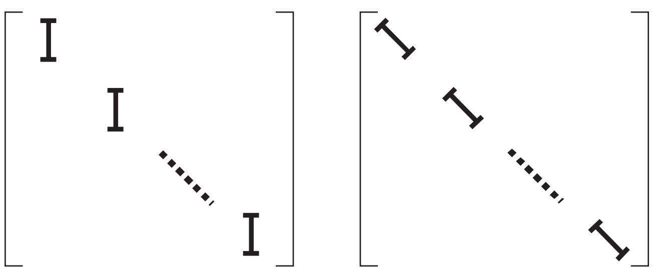
$$\mathrm{pdiag}(\boldsymbol{I}) \underset{\alpha}{\mathrm{diag}} [\boldsymbol{I}_\alpha] \tag{23}$$

is the multiplicated pseudo-diagonal matrix, where $\boldsymbol{I}_\alpha$ – column matrix of α – value spatial-time distribution, and

$$\mathrm{pdiag}(\boldsymbol{I}) \underset{\alpha}{\mathrm{diag}} (\mathrm{diag} \boldsymbol{I}_\alpha) \tag{23}$$

is the multiplicated diagonal matrix.

The pseudo- diagonal and diagonal matrices have the following structure:



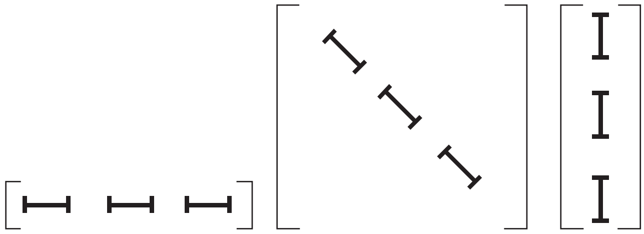
In this Section we minimize the following functionals:
– sum of the squared instantaneous branch powers inside the multi-port-network:

$$\boldsymbol{U}^T(\mathrm{diag}\boldsymbol{I})^2\boldsymbol{U} = \boldsymbol{I}^T(\mathrm{diag}\boldsymbol{U})^2\boldsymbol{I} \to \min \tag{25}$$

– sum of the squared apparent branch powers inside the multi-port-network:

$$\begin{aligned}\boldsymbol{U}^T(\mathrm{pdiag}\boldsymbol{U})(\mathrm{pdiag}\boldsymbol{I})^T\boldsymbol{I} = \boldsymbol{I}^T(\mathrm{pdiag}\boldsymbol{I})(\mathrm{pdiag}\boldsymbol{U})^T\\ \times \boldsymbol{U} \to \min\end{aligned} \tag{26}$$

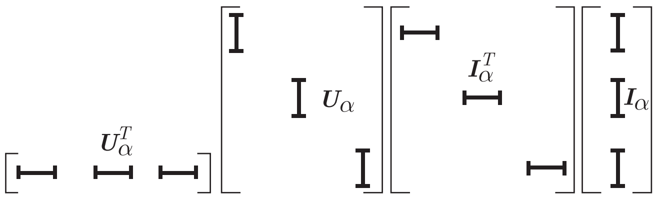
Condition (25) has the following matrix structure:



therefore it is equivalent to minimize instantaneous functional:

$$\boldsymbol{U}_n^T(\mathrm{diag}\boldsymbol{I}_n)^2\boldsymbol{U}_n = \boldsymbol{I}_n^T\mathrm{diag}(\boldsymbol{U}_n)^2\boldsymbol{I}_n \to \min \tag{27}$$

However condition (26) has a structure:



and it means the minimization of

$$\begin{aligned}\boldsymbol{U}^T\mathrm{col}[\boldsymbol{U}_\alpha\boldsymbol{I}_\alpha^T\boldsymbol{I}_\alpha] &= \boldsymbol{I}^T\mathrm{col}[\boldsymbol{I}_\alpha\boldsymbol{U}_\alpha^T\boldsymbol{U}_\alpha]\\ &= \boldsymbol{U}^T\underset{\alpha}{\mathrm{diag}}[||\boldsymbol{I}_\alpha||]^2\boldsymbol{U}\\ &= \boldsymbol{I}^T\underset{\alpha}{\mathrm{diag}}[||\boldsymbol{U}_\alpha||]^2\boldsymbol{I}\\ &= \sum_\alpha \boldsymbol{U}_\alpha^T\boldsymbol{U}_\alpha\boldsymbol{I}_\alpha^T\boldsymbol{I}_\alpha\\ &= \sum_\alpha \boldsymbol{I}_\alpha^T\boldsymbol{I}_\alpha\boldsymbol{U}_\alpha^T\boldsymbol{U}_\alpha\\ &= \sum_\alpha ||\boldsymbol{I}_\alpha||^2||\boldsymbol{U}_\alpha||^2.\end{aligned} \tag{28}$$

First we'll describe minimization of the functional (25) or (27). Minimizing this functional with fixed current vector we found the following problem (the discrete time index is omitted)

$$\underset{\boldsymbol{U}}{\mathrm{VAR}}\,\boldsymbol{U}^T\boldsymbol{Q}\boldsymbol{U} \to \min \tag{29}$$

where

$$\boldsymbol{Q} = (\mathrm{diag}\boldsymbol{I})^2 \tag{30}$$

is treated as symmetrical positively definite weight matrix, and

$$\boldsymbol{U} = \boldsymbol{P}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{u}\end{bmatrix}. \tag{31}$$

Differentiating vector (31) and taking into consideration that subvector $\boldsymbol{v}$ does not change, it yields:

$$d\boldsymbol{U} = \boldsymbol{P}\begin{bmatrix}\boldsymbol{0}\\ d\boldsymbol{u}\end{bmatrix}. \tag{32}$$

The differential of functional (29) has form:

$$\begin{aligned}(\boldsymbol{U}^T + d\boldsymbol{U}^T)\boldsymbol{Q}(\boldsymbol{U}+d\boldsymbol{U}) = \boldsymbol{U}^T\boldsymbol{Q}\boldsymbol{U} + \boldsymbol{U}^T\boldsymbol{Q}d\boldsymbol{U}\\ + d\boldsymbol{U}^T\boldsymbol{Q}\boldsymbol{U} + d\boldsymbol{U}^T\boldsymbol{Q}d\boldsymbol{U}.\end{aligned} \tag{33}$$

Taking into consideration the expressions (31) and (32) we get the minimum condition:

$$[\boldsymbol{v}^T\boldsymbol{u}^T]\boldsymbol{P}^T\,\boldsymbol{Q}\boldsymbol{P} + [0^T d\boldsymbol{u}^T]\boldsymbol{P}^T\boldsymbol{Q}\boldsymbol{P}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{u}\end{bmatrix} = 0$$

or

$$[\boldsymbol{v}^T\boldsymbol{u}^T]\begin{bmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12}\\ \boldsymbol{B} & \boldsymbol{A}\end{bmatrix}\begin{bmatrix}0\\ d\boldsymbol{u}\end{bmatrix} + [0^T d\boldsymbol{u}^T]\begin{bmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12}\\ \boldsymbol{B} & \boldsymbol{A}\end{bmatrix}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{u}\end{bmatrix} = 0 \tag{34}$$

where $\boldsymbol{B}_{11}, \boldsymbol{B}_{12}, \boldsymbol{B}, \boldsymbol{A}$ stand for matrices (blocks) resulted from division of matrix $\boldsymbol{P}^T\boldsymbol{Q}\boldsymbol{P}$ into four parts:

$$\boldsymbol{P}^T\boldsymbol{Q}\boldsymbol{P} = \begin{bmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12}\\ \boldsymbol{B} & \boldsymbol{A}\end{bmatrix}. \tag{35}$$

From (34) follows that

$$\boldsymbol{B}_{12}^T = \boldsymbol{B} \tag{36}$$

applying (35) and (36) in (34) yields the minimum condition:

$$[\boldsymbol{v}^T\boldsymbol{u}^T]\begin{bmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12}\\ \boldsymbol{B} & \boldsymbol{A}\end{bmatrix}\begin{bmatrix}0\\ d\boldsymbol{u}\end{bmatrix} + [d\boldsymbol{u}^T\boldsymbol{B}; d\boldsymbol{u}^T\boldsymbol{A}]\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{u}\end{bmatrix} = 0 \tag{37}$$

for any $d\boldsymbol{u}$.

Taking into consideration that components in (37) are mutually transposed, the minimum condition takes form:

$$\bigwedge_{\delta\boldsymbol{u}} d\boldsymbol{u}^T(\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{v}) = 0 \tag{38}$$

Expression (38) is met if and only if the sought vector u meets the linear equation:

$$\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{v} = \boldsymbol{0} \tag{39}$$

thus

$$\boldsymbol{u} = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{v} \tag{40}$$

As it was shown in [2] the elements of block matrices $\boldsymbol{B}, \boldsymbol{A}$ can be calculated by the following equation

$$\begin{aligned}\begin{bmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12}\\ \boldsymbol{B} & \boldsymbol{A}\end{bmatrix}_{\alpha\beta} &= [\boldsymbol{P}^T(\mathrm{diag}\boldsymbol{I})^2\boldsymbol{P}]_{\alpha\beta}\\ &= \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\}\\ TREE-BRANCHES}} \mathrm{sgn}(\gamma)(I_\gamma)^2\end{aligned} \tag{41}$$

The symbols $\{\alpha\}, \{\beta\}$ stand for cut-sets corresponding to the appropriate tree branches α, β The set $\{\alpha\}\cap\{\beta\}$ stands for logical product of two branch sets [2]. It follows that elements of matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ are defined by the following formulas

$$A_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\}\\ \mathit{TREE-BRANCHES}}} \operatorname{sgn}(\gamma)(I_\gamma)^2 \tag{42}$$

$$B_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\}\\ \mathit{INNER\cap OUTER}\\ \mathit{TREE-BRANCHES}}} \operatorname{sgn}(\gamma)(I_\gamma)^2 \tag{43}$$

Similarly the problem

$$\underset{\boldsymbol{I}}{\mathrm{VAR}}\, \boldsymbol{I}^T(\mathrm{diag}\boldsymbol{U})^2\boldsymbol{I} \to \min \tag{44}$$

has solution

$$\boldsymbol{Ai} + \boldsymbol{Bj} = \boldsymbol{0} \tag{45}$$

thus

$$\boldsymbol{i} = -\boldsymbol{A}^{-1}\boldsymbol{Bj} \tag{46}$$

where

$$\begin{aligned} \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B} & \boldsymbol{A} \end{bmatrix}_{\alpha\beta} &= [\boldsymbol{C}^T(\mathrm{diag}\boldsymbol{U})^2\boldsymbol{C}]_{\alpha\beta} \\ &= \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\}\\ \mathit{LINKS}}} \operatorname{sgn}(\gamma)(U_\gamma)^2 \end{aligned} \tag{47}$$

then

$$A_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\}\\ \mathit{INNER\ LINKS}}} \operatorname{sgn}(\gamma)(U_\gamma)^2 \tag{48}$$

$$B_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\}\\ \mathit{INNER\cap OUTER}\\ \mathit{LINKS}}} \operatorname{sgn}(\gamma)(U_\gamma)^2. \tag{49}$$

Further functional (25) will be marked by the symbol:

$$f(\boldsymbol{U},\boldsymbol{I}) = \boldsymbol{U}^T(\mathrm{diag}\boldsymbol{I})^2\boldsymbol{U} = \boldsymbol{I}^T(\mathrm{diag}\boldsymbol{U})^2\boldsymbol{I} \tag{50}$$

Expression (29), (40), (31), (44), (46) define functions:

$$\begin{aligned} &\boldsymbol{I} \to \boldsymbol{u} = -[\boldsymbol{A}_P(\boldsymbol{I})]^{-1}\boldsymbol{B}_P(\boldsymbol{I})\boldsymbol{v} \to \\ &\boldsymbol{U} = \boldsymbol{P}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix} = \boldsymbol{\Phi}^U(\boldsymbol{I}) \end{aligned} \tag{51}$$

and

$$\begin{aligned} &\boldsymbol{U} \to \boldsymbol{i} = -[\boldsymbol{A}_C(\boldsymbol{U})]^{-1}\boldsymbol{B}_C(\boldsymbol{U})\boldsymbol{j} \to \\ &\boldsymbol{I} = \boldsymbol{C}\begin{bmatrix} \boldsymbol{j} \\ \boldsymbol{i} \end{bmatrix} = \boldsymbol{\Phi}^I(\boldsymbol{U}) \end{aligned} \tag{52}$$

Thus for any $\boldsymbol{I}_0$ functional $f(\boldsymbol{U},\boldsymbol{I}_0)$ of $\boldsymbol{U}$ variable has the unique minimum: $\min\limits_{\boldsymbol{U}=\boldsymbol{U}_1} f(\boldsymbol{U},\boldsymbol{I}_0)$ in point $\boldsymbol{U} = \boldsymbol{U}_1$ as well for any $\boldsymbol{U}_0$ functional $f(\boldsymbol{U}_0,\boldsymbol{I})$ of $\boldsymbol{I}$ variable has the only minimum: $\min\limits_{\boldsymbol{I}=\boldsymbol{I}_1} f(\boldsymbol{U}_0,\boldsymbol{I})$ in point $\boldsymbol{I} = \boldsymbol{I}_1$.

Thus the following sequence of inequality is met:

$$\begin{aligned} &\min_{\boldsymbol{I}=\boldsymbol{I}_1} f(\boldsymbol{U}_0,\boldsymbol{I}) > \min_{\boldsymbol{U}=\boldsymbol{U}_1} f(\boldsymbol{U},\boldsymbol{I}_1) > \min_{\boldsymbol{I}=\boldsymbol{I}_2} f(\boldsymbol{U}_1,\boldsymbol{I}) \\ &\quad > \min_{\boldsymbol{U}=\boldsymbol{U}_2} f(\boldsymbol{U},\boldsymbol{I}_2)\ldots \\ &\quad > \min_{\boldsymbol{U}=\boldsymbol{U}_+} f(\boldsymbol{U},\boldsymbol{I}_+) = \min_{\boldsymbol{I}=\boldsymbol{I}_+} f(\boldsymbol{U}_+,\boldsymbol{I}) \\ &\quad = \min_{\boldsymbol{U}=\boldsymbol{U}_+}\ \min_{\boldsymbol{I}=\boldsymbol{I}_+} f(\boldsymbol{U},\boldsymbol{I}). \end{aligned} \tag{53}$$

It means that the point $\boldsymbol{I}_+, \boldsymbol{U}_+$ is the only minimum point of functional $f(\boldsymbol{U},\boldsymbol{I})$ and also the only limit point of sequence (53) of functional. Thus minimum point $\boldsymbol{U}_+, \boldsymbol{I}_+$ could be possibly received as a limit of sequence:

$$\boldsymbol{I}_0; \boldsymbol{U}_1, \boldsymbol{I}_1; \boldsymbol{U}_2, \boldsymbol{I}_2; \ldots \boldsymbol{U}_n, \boldsymbol{I}_n; \ldots \tag{54}$$

definite as follows:

$$\begin{aligned} \boldsymbol{U}_1 &= \boldsymbol{\Phi}^U(\boldsymbol{I}_0), \quad \boldsymbol{I}_1 = \boldsymbol{\Phi}^I(\boldsymbol{U}_1) \\ \boldsymbol{U}_2 &= \boldsymbol{\Phi}^U(\boldsymbol{I}_1), \quad \boldsymbol{I}_2 = \boldsymbol{\Phi}^I(\boldsymbol{U}_2) \\ &\cdots \\ \boldsymbol{U}_n &= \boldsymbol{\Phi}^U(\boldsymbol{I}_{n-1}), \quad \boldsymbol{I}_n = \boldsymbol{\Phi}^I(\boldsymbol{U}_n) \\ \boldsymbol{U}_{n+1} &= \boldsymbol{\Phi}^U(\boldsymbol{I}_n), \quad \boldsymbol{I}_{n+1} = \boldsymbol{\Phi}^I(\boldsymbol{U}_{n+1}) \end{aligned} \tag{55}$$

It results from (55) that voltage and current sequences can be determined separately with help of composite functions

$$\boldsymbol{U}_{n+1} = \boldsymbol{\Phi}^U[\boldsymbol{\Phi}^I(\boldsymbol{U}_n)] = \boldsymbol{\Phi}^U \circ \boldsymbol{\Phi}^I(\boldsymbol{U}_n) \tag{56}$$

$$\boldsymbol{I}_{n+1} = \boldsymbol{\Phi}^I[\boldsymbol{\Phi}^U(\boldsymbol{I}_n)] = \boldsymbol{\Phi}^I \circ \boldsymbol{\Phi}^U(\boldsymbol{I}_n) \tag{57}$$

In Fig. 10 iterative functions $\boldsymbol{\Phi}^U(\boldsymbol{I})$ and $\boldsymbol{\Phi}^I(\boldsymbol{U})$ were illustrated in block diagrams.

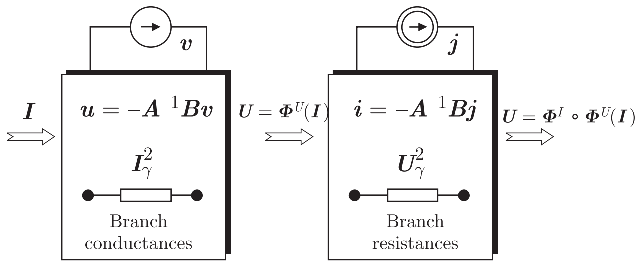


Fig. 10. Interpretation of iterative functions $\boldsymbol{\Phi}^U(\boldsymbol{I})$ and $\boldsymbol{\Phi}^I(\boldsymbol{U})$ illustrated in block diagrams

The first block symbolizes voltage analysis of the circuit fed by voltage sources $\boldsymbol{v}$ with the branch conductance equal to the squared branch currents. The elements $\boldsymbol{A}, \boldsymbol{B}$ are

$$A_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\}\\ \mathit{INNER\ TREE-BRANCHES}}} \operatorname{sgn}(\gamma)(I_\gamma)^2 \tag{58}$$

$$B_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\}\\ \mathit{INNER\cap OUTER}\\ \mathit{TREE-BRANCHES}}} \operatorname{sgn}(\gamma)(I_\gamma)^2 \tag{59}$$

The second block performs current analysis of circuit fed by the current sources from ports j with branch 'resistances' equal to the squared branch voltage. The elements $\boldsymbol{A}, \boldsymbol{B}$ are

$$A_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\}\\ \mathit{INNER\ LINKS}}} \operatorname{sgn}(\gamma)(U_\gamma)^2 \tag{60}$$

$$B_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\}\\ \mathit{INNER\cap OUTER}\\ \mathit{LINKS}}} \operatorname{sgn}(\gamma)(U_\gamma)^2 \tag{61}$$

Example 5. The two port network with prescribed DC currents and voltages on ports is shown in Fig. 11.

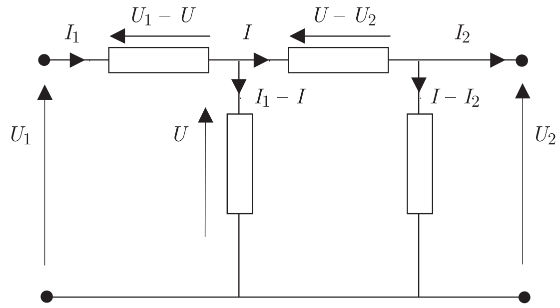


Fig. 11. The two port network with prescribed port signals

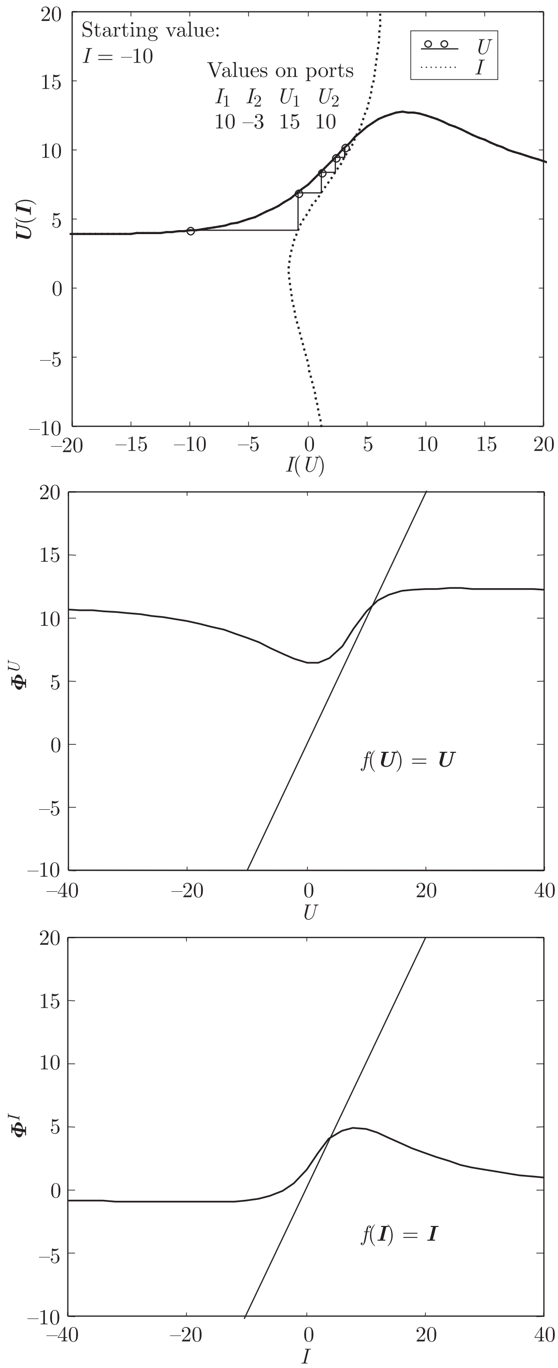


Fig. 12. Function Φ^U, Φ^I and the iteration process while minimizing the power functional

In the foregoing figure the current i and voltage u signals are marked. The power functional being minimized is

$$f(u,i) = i_1^2(u_1-u)^2 + u^2(i_1-i)^2 + i^2(u-u_2)^2 + u_2^2(i-i_2)^2 \tag{62}$$

The functions $\Phi^U(i)$ and $\Phi^I(u)$ minimizing the functional (62) with fixed $\boldsymbol{u}, \boldsymbol{i}$ signals have form:

$$\Phi^U(i) = 1/2\frac{i_1^2 u_1 + i^2 u_2}{i_1^2 + i_1 i + i^2} =: u^{\text{opt}}(i) \tag{63}$$

$$\Phi^I(U) = 1/2\frac{i_1 u^2 + i_2 u_2^2}{u^2 + u u_2 + u_2^2} =: i^{\text{opt}}(u) \tag{64}$$

The process of appropriate iterations (55) or (56) and (57) is shown in Fig. 12.

Example 6. Figure 13 shows graph of two-port network with 1, 2 ports with chosen set of links 1, 2, 3, 6 and tree branches 1, 2, 5.

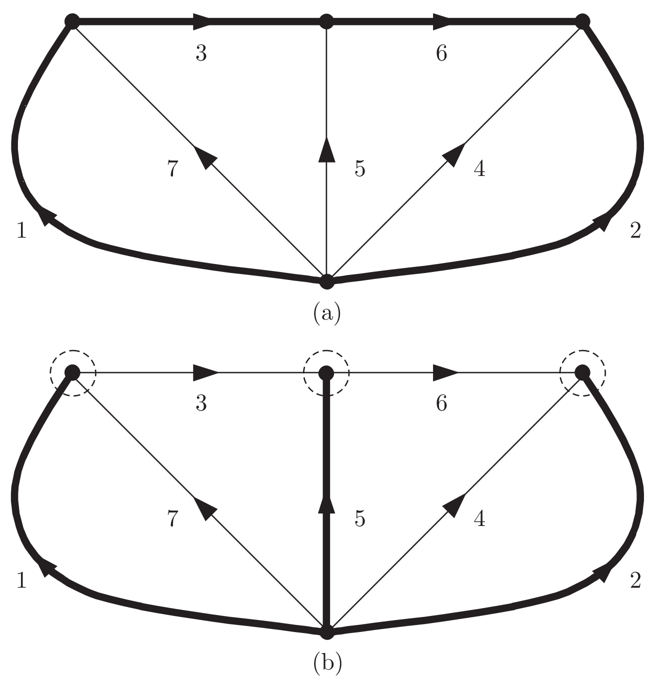


Fig. 13. The chosen set of links 1, 2, 3, 6 (a) and tree branches 1, 2, 5 (b)

The loop and cut-set matrices with external and internal links and tree branches marked, as well as current and voltage coordinates have form:

$\boldsymbol{C}$ =

	j_1 (OUTER LINKS)	j_2 (OUTER LINKS)	i_3 (INNER LINKS)	i_6 (INNER LINKS)
	1	2	3	6
3			1	
4		−1		−1
5			−1	1
6				1
7	−1		1	

$\boldsymbol{P}$=

	v_1 (OUTER TREE-BRANCHES)	v_2 (OUTER TREE-BRANCHES)	u_5 (INNER TREE-BRANCHES)
	1	2	5
3	−1		1
4		1	
5			1
6		1	−1
7	1		

It yields cut matrices of links and tree branches (compare Eqs. 60,61,58,59):

A:

	3	6
3	{3,5,7}	{–5}
6	{–5}	{4,5,6}

	5
5	{3,5,6}

B:

	1	2
3	{–7}	
6		{4}

	1	2
5	{–3}	{–6}

On the basis of cut matrices it is noted the set of linear equations determining the iterative functions (38) and (37):

$$\boldsymbol{U} \Rightarrow \begin{bmatrix} u_3^2+u_5^2+u_7^2 & -u_5^2 \\ -u_5^2 & u_4^2+u_5^2+u_6^2 \end{bmatrix} \begin{bmatrix} i_3 \\ i_6 \end{bmatrix} = \begin{bmatrix} u_7^2 & \\ & u_4^2 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} \Rightarrow \boldsymbol{i} = \begin{bmatrix} i_3 \\ i_6 \end{bmatrix}$$

$$\boldsymbol{I} = \boldsymbol{C}\begin{bmatrix} \boldsymbol{j} \\ \boldsymbol{i} \end{bmatrix} = \boldsymbol{\Phi}^I(\boldsymbol{U})$$

$$\boldsymbol{I} \Rightarrow \begin{bmatrix} I_3^2+I_5^2+I_6^2 \end{bmatrix}\begin{bmatrix} u_5 \end{bmatrix} = \begin{bmatrix} I_3^2 & I_6^2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_1 \end{bmatrix} \Rightarrow \boldsymbol{u} = \begin{bmatrix} u_5 \end{bmatrix}$$

$$\boldsymbol{P}\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix} = \boldsymbol{\Phi}^U(\boldsymbol{I}) = \boldsymbol{\Phi}^U \circ \boldsymbol{\Phi}^I(\boldsymbol{U}).$$

4. Minimization of the power differential functional

Introducing inner standard vectors for U0 voltage and I0 currents we minimize differential power functional:

$$(\boldsymbol{U}-\boldsymbol{U}_0)^T[\text{diag}(\boldsymbol{I}-\boldsymbol{I}_0)]^2(\boldsymbol{U}-\boldsymbol{U}_0) = (\boldsymbol{I}-\boldsymbol{I}_0)^T \times [\text{diag}(\boldsymbol{U}-\boldsymbol{U}_0)]^2(\boldsymbol{I}-\boldsymbol{I}_0) \to \text{min}. \tag{65}$$

Minimization of a partial-voltage functional

$$(\boldsymbol{U}-\boldsymbol{U}_0)^T\boldsymbol{Q}(\boldsymbol{U}-\boldsymbol{U}_0) \underset{\boldsymbol{u}\ \text{var}}{\to} \text{min}, \tag{66}$$

where

$$\boldsymbol{Q} = [\text{diag}(\boldsymbol{I}-\boldsymbol{I}_0)]^2_{\text{const}}. \tag{67}$$

Thus the necessary and sufficient minimum condition is formulated as:

$$\boldsymbol{A}\boldsymbol{u} = -\boldsymbol{B}\boldsymbol{v} + \boldsymbol{a}^0, \tag{68}$$

(68) where $\boldsymbol{A}, \boldsymbol{B}$ matrices and $\boldsymbol{a}$ vector are defined as:

$$A_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \text{INNER TREE–BRANCHES}}} \text{sgn}(\gamma)(I_\gamma - I_{0,\gamma})^2 \tag{69}$$

$$B_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \text{INNER}\cap\text{OUTER} \\ \text{TREE – BRANCHES}}} \text{sgn}(\gamma)(I_\gamma - I_{0,\gamma})^2 \tag{70}$$

$$a^0_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \text{INNER TREE–BRANCHES}}} \text{sgn}(\gamma)(I_\gamma - I_{0,\gamma})^2 \tag{71}$$

Similarly minimization of partial -current functional:

$$(\boldsymbol{I}-\boldsymbol{I}_0)^T\boldsymbol{Q}(\boldsymbol{I}-\boldsymbol{I}_0) \underset{\boldsymbol{i}\ \text{var}}{\to} \text{min}, \tag{72}$$

where

$$\boldsymbol{Q} = [\text{diag}(\boldsymbol{U}-\boldsymbol{U}_0)]^2_{\text{const}}. \tag{73}$$

has solution in form of a set of linear equations:

$$\boldsymbol{A}\boldsymbol{i} = -\boldsymbol{B}\boldsymbol{j} + \boldsymbol{a}^0 \tag{74}$$

where

$$A_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \text{INNER LINKS}}} \text{sgn}(\gamma)(U_\gamma - U_{0,\gamma})^2 \tag{75}$$

$$B_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \text{INNER}\cap\text{OUTER} \\ \text{LINKS}}} \text{sgn}(\gamma)(U_\gamma - U_{0,\gamma})^2 \tag{76}$$

$$a^0_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \text{INNER LINKS}}} \text{sgn}(\gamma)(U_\gamma - U_{0,\gamma})^2 I_{0,\gamma} \tag{77}$$

Warning: both in (65) as in remaining expressions the time index was omitted.

5. Minimization of complex variable functionals

If U_α, I_α stand for spatial distribution of voltages and currents in the net, then the functional :

$$\sum_\alpha U^*_\alpha I_\alpha I^*_\alpha U_\alpha = \boldsymbol{U}^*\text{diag}|I_\alpha|^2\boldsymbol{U} = \boldsymbol{I}^*\text{diag}|U_\alpha|^2\boldsymbol{I} \tag{78}$$

is a sum of squared apparent powers of net elements.

However if $U_{n,\alpha}, I_{n,\alpha}$ stand for spatial frequency distribution of a complex variable (n – discreet frequency coordinate, α – number of branch), then the functional:

$$\sum_n\sum_\alpha U^*_{n,\alpha}I_{n,\alpha}I^*_{n,\alpha}U_{n,\alpha} = \sum_n \boldsymbol{U}^*_n \underset{\alpha}{\text{diag}}\,|I_{n,\alpha}|^2\boldsymbol{U}_n = \sum_n \boldsymbol{I}^*_n \underset{\alpha}{\text{diag}}\,|U_{n,\alpha}|^2\boldsymbol{I}_n \tag{79}$$

is an average (along frequencies) sum of squared apparent powers of circuit.

Minimization of functional (79) is equivalent to minimizing functional:

$$\boldsymbol{U}^*_n \underset{\alpha}{\text{diag}}\,|I_{n,\alpha}|^2\boldsymbol{U}_n = \boldsymbol{I}^*_n \underset{\alpha}{\text{diag}}\,|U_{n,\alpha}|^2\boldsymbol{I}_n \to \text{min} \tag{80}$$

for every n separately. Thereby the problems of functionals minimization (78) and (79) agree. Building method of iterative function for minimizing this functional is shown in Fig. 14 with the help of block diagrams.

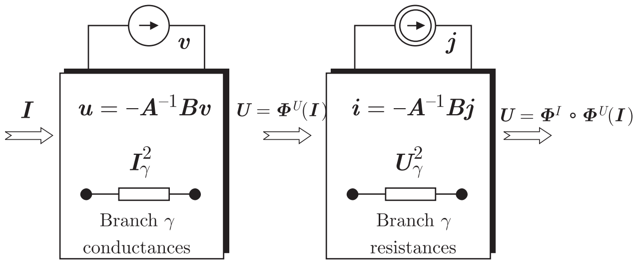


Fig. 14. Block diagram for determining iterative function

The elements $\boldsymbol{A}, \boldsymbol{B}$ marked on the foregoing block diagram are defined as:

$$A_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \textit{INNER TREE – BRANCHES}}} \operatorname{sgn}(\gamma)|I_\gamma|^2 \tag{81}$$

$$B_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \textit{INNER}\cap\textit{OUTER} \\ \textit{TREE – BRANCHES}}} \operatorname{sgn}(\gamma)|I_\gamma|^2 \tag{82}$$

for the first block, and:

$$A_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \textit{INNER LINKS}}} \operatorname{sgn}(\gamma)|U_\gamma|^2 \tag{83}$$

$$B_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \textit{INNER}\cap\textit{OUTER} \\ \textit{LINKS}}} \operatorname{sgn}(\gamma)|U_\gamma|^2 \tag{84}$$

When Laplace's transform of signals is used, then the minimization functional becomes:

$$\sum_\alpha \oint_{\mathbb{C}} U_\alpha(-s)I_\alpha(s)I_\alpha(-s)U_\alpha(s)ds \to \min \tag{85}$$

where circulation is taken on contour consisting of imaginary axis and semicircle in left (or right) complex plain with the ray approaching infinity $r \to \propto$. It is easily shown that minimizing of functional (85) is reduced to minimizing:

$$\begin{aligned} &[\boldsymbol{U}(-s)]^T \operatorname*{diag}_\alpha[I_\alpha(s)I_\alpha(-s)]\boldsymbol{U}(s) = [\boldsymbol{I}(-s)]^T \\ &\times \operatorname*{diag}_\alpha[U_\alpha(s)U_\alpha(-s)]\boldsymbol{I}(s) \to \min \end{aligned} \tag{86}$$

for every s separately. Thus the set of equations meeting this optimization task take form:

$$\boldsymbol{A}(s)\boldsymbol{u}(s) = -\boldsymbol{B}(s)\boldsymbol{v}(s) \tag{87}$$

where

$$A_{\alpha\beta}(s) = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \textit{INNER TREE – BRANCHES}}} \operatorname{sgn}(\gamma)I_\gamma(s)I_\gamma(-s) \tag{88}$$

$$B_{\alpha\beta}(s) = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \textit{INNER}\cap\textit{OUTER} \\ \textit{TREE – BRANCHES}}} \operatorname{sgn}(\gamma)I_\gamma(s)I_\gamma(-s) \tag{89}$$

and

$$\boldsymbol{A}(s)\boldsymbol{i}(s) = -\boldsymbol{B}(s)\boldsymbol{j}(s) \tag{90}$$

where

$$A_{\alpha\beta}(s) = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \textit{INNER LINKS}}} \operatorname{sgn}(\gamma)U_\gamma(s)U_\gamma(-s) \tag{91}$$

$$B_{\alpha\beta}(s) = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \textit{INNER}\cap\textit{OUTER} \\ \textit{LINKS}}} \operatorname{sgn}(\gamma)U_\gamma(s)U_\gamma(-s) \tag{92}$$

Minimization of functional (26) which is the sum of squared apparent powers of branches inside the multi port network proceeds as follows:

From comparing structures (25) and (28) follows that functional (28) can be minimized according to analogous iterative algorithm:

$$\boldsymbol{I} \to \operatorname*{VAR}_{\boldsymbol{U}} \boldsymbol{U}^T \operatorname*{diag}_\alpha[||\boldsymbol{I}||^2]\boldsymbol{U} \to \min$$

$$A_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \textit{INNER LINKS}}} \operatorname{sgn}(\gamma)||\boldsymbol{I}_\gamma||^2$$

$$B_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \textit{INNER}\cap\textit{OUTER} \\ \textit{LINKS}}} \operatorname{sgn}(\gamma)||\boldsymbol{I}_\gamma||^2$$

$$\boldsymbol{u} = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{v}$$

$$\boldsymbol{U} = \boldsymbol{P}\begin{bmatrix}\boldsymbol{v}\\ \boldsymbol{u}\end{bmatrix} = \boldsymbol{\Phi}^U(\boldsymbol{I})$$

$$\operatorname*{VAR}_{\boldsymbol{I}} \boldsymbol{I}^T \operatorname*{diag}_\alpha[||\boldsymbol{U}||^2]\boldsymbol{I} \to \min$$

$$A_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \textit{INNER LINKS}}} \operatorname{sgn}(\gamma)||\boldsymbol{U}_\gamma||^2$$

$$B_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \textit{INNER}\cap\textit{OUTER} \\ \textit{LINKS}}} \operatorname{sgn}(\gamma)||\boldsymbol{U}_\gamma||^2$$

$$\boldsymbol{i} = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{j} \to \boldsymbol{I} = \boldsymbol{C}\begin{bmatrix}\boldsymbol{j}\\ \boldsymbol{i}\end{bmatrix} = \boldsymbol{\Phi}^I \circ \boldsymbol{\Phi}^U(\boldsymbol{I}) \tag{93}$$

Indeed, it has analogous structure as functional (50), that is:

$$f(\boldsymbol{U}, \boldsymbol{I}) = \sum_\alpha \boldsymbol{U}_\alpha^T \boldsymbol{U}_\alpha \boldsymbol{I}_\alpha^T \boldsymbol{I}_\alpha \tag{94}$$

It means that for any fixed $\boldsymbol{I}_0$ functional $f(\boldsymbol{U}, \boldsymbol{I}_0)$ of $\boldsymbol{U}$ variable reaches the unique minimum, and for any $\boldsymbol{U}_0$ functional $f(\boldsymbol{U}_0, \boldsymbol{I})$ of $\boldsymbol{I}$ variable also reaches the unique minimum. Thus the sequence of inequality (53) is valid, then it follows that functional (94) has the unique minimum point attainable with the help of iterative function (93).

Norms in the formulas (93) are defined as follows:

$$||\boldsymbol{U}_\alpha||^2 = \sum_{m=0}^{N-1}(U_{m,\alpha})^2 \tag{95}$$

in discrete time domain and

$$||\boldsymbol{U}_\alpha||^2 = \int_0^T [U_\alpha(t)]^2 dt \tag{96}$$

in continuous time domain.

Procedure (93) is valid in frequency domain, too. If by n we mark discrete index of frequency, then applying the Parcevall theorem we can write

$$||\boldsymbol{U}_\alpha||^2 = \sum_{m=0}^{N-1} |U_{m,\alpha}|^2 \tag{97}$$

However in continuous frequencies domain:

$$||\boldsymbol{U}_\alpha||^2 = \frac{1}{2\pi j} \oint\limits_{\mathbb{C}} U_\alpha(-s) U_\alpha(s) ds \tag{98}$$

Example 7. Figure 15 shows graph of two – port network of Π type, with a chosen set of links and tree branches.

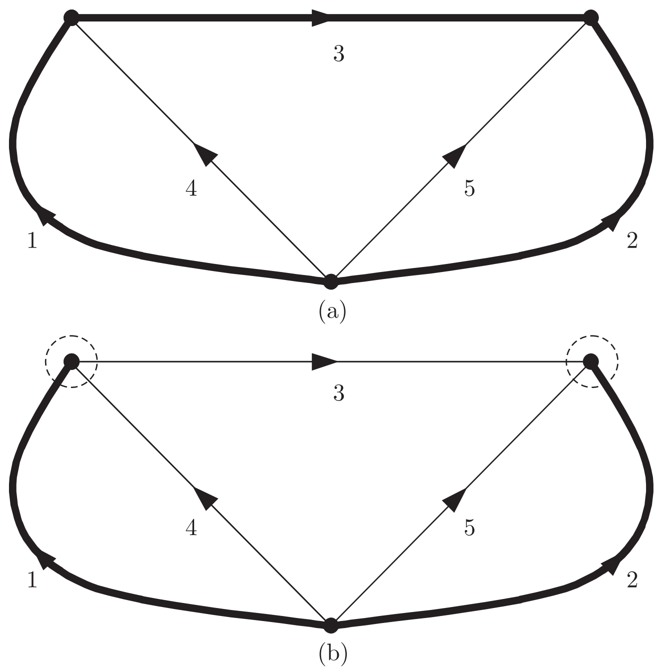


Fig. 15. The chosen set of links and tree branches (a) of two – port network of Π type (b)

Matrices of: links (loops) and tree branches (cut-sets) with marked indices have form:

$\boldsymbol{C}$ =

	j_1	j_2	i_3
	OUTER LINKS		INNER LINKS
	1	2	3
3			1
4	–1		1
5		–1	–1

$\boldsymbol{P}$=

	v_1	v_2	
	OUTER TREE-BRANCHES		INNER TREE-BRANCHES
	1	2	
3	–1	1	$v_2 - v_1$
4	1		v_1
5		1	v_2

The set of internal tree branches is empty. There is no voltage optimizing equations. Distribution of voltage signals is assumed – there are no independent voltages. Only current distribution is given by a single equation:

$\boldsymbol{A}$:

	3
3	{3,4,5}

$\boldsymbol{B}$:

	1	2
3	{–4}	{5}

$$\left(||\boldsymbol{U}_4||^2 + ||\boldsymbol{U}_5||^2 + ||\boldsymbol{U}_3||^2\right) \boldsymbol{I}_3 = \left[||\boldsymbol{U}_4||^2 ||\boldsymbol{U}_5||^2\right] \begin{bmatrix} \boldsymbol{j}_1 \\ \boldsymbol{j}_2 \end{bmatrix}$$

or

$$\left(||\boldsymbol{v}_1||^2 + ||\boldsymbol{v}_2||^2 + ||\boldsymbol{v}_2 - \boldsymbol{v}_1||^2\right) \boldsymbol{I}_3 = ||\boldsymbol{v}_1||^2 \boldsymbol{j}_1 - ||\boldsymbol{v}_2||^2 \boldsymbol{j}_2$$

which yields

$$\boldsymbol{I}_3 = \frac{||\boldsymbol{v}_1||^2 \boldsymbol{j}_1 - ||\boldsymbol{v}_2||^2 \boldsymbol{j}_2}{||\boldsymbol{v}_1||^2 + ||\boldsymbol{v}_1||^2 + ||\boldsymbol{v}_2 - \boldsymbol{v}_1||^2}$$

(see example 2)

Example 8. Figure 16 shows the two-port network of T type, in the graph the set of links and tree branches is chosen.

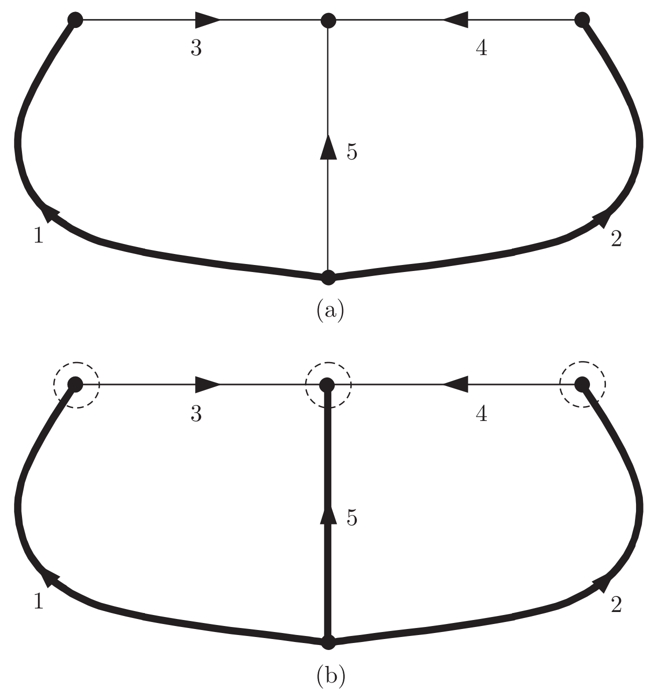


Fig. 16. The chosen set of links and tree branches (a) of two-port network of T type (b)

Loop and cut-set matrices with marked indices are following:

$\boldsymbol{C}$ =

	j_1	j_2	i_3
	OUTER LINKS		INNER LINKS
	1	2	
3	1		j_1
4		1	j_2
5	–1	–1	$j_2 - j_1$

$\boldsymbol{P}$=

	v_1	v_2	
	OUTER TREE-BRANCHES		INNER TREE-BRANCHES
	1	2	5
3	–1		1
4		–1	1
5			1

This time the set of internal links is empty. There are no current optimizing equations. The only independent coordinate – voltage signal $\boldsymbol{u}5$ – is given instantly from single equations:

$\boldsymbol{A}$:

	5
5	{3,4,5}

$\boldsymbol{B}$:

	1	2
5	{–3}	{–4}

where

$$\left(||\boldsymbol{I}_3||^2 + ||\boldsymbol{I}_4||^2 + ||\boldsymbol{I}_5||^2\right) \boldsymbol{U}_5 = \left[||\boldsymbol{I}_3||^2 ||\boldsymbol{I}_4||^2\right] \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix}$$

$$U_5 = \frac{||j_1||^2 v_1 + ||j_2||^2 v_2}{||j_1||^2 + ||j_1||^2 + ||j_2 + j_1||^2}$$

6. Minimization of conditional power functionals

Often besides the global minimum condition the additional constrains are necessary, e.g. the active power of inner branches constrain.

Let us consider again the multi- port- network shown in Fig. 6. The functional made as sum of the squared apparent powers of inner branches will be minimized (see (26) or 28) with the prescribed active power value of any individual inner branch). At the same time the active power balance condition of the whole circuit is met:

$$\sum_{\alpha \in INNER\ BRANCHES} \boldsymbol{I}_\alpha^T \boldsymbol{U}_\alpha + \sum_{\alpha \in OUTER\ (PORT)\ BRANCHES} \boldsymbol{I}_\alpha^T \boldsymbol{U}_\alpha = 0$$

which follows that the sum of active power of inner branches has assumed value:

$$\sum_{\alpha \in INNER\ BRANCHES} \boldsymbol{I}_\alpha^T \boldsymbol{U}_\alpha = p$$

where: p – assumed value.

It is then possible to set independent values of active power to the branches belonging to a certain branch set Ω.

Thus the minimum task has the form of a conditional optimization problem:

$$\boldsymbol{U}^T \operatorname*{diag}_{\alpha} \left[||\boldsymbol{I}_\alpha||^2\right] \boldsymbol{U} \to \min \tag{99}$$

$$\boldsymbol{W}_\Omega (\text{pdiag}\boldsymbol{I})^T \boldsymbol{U} = \boldsymbol{p} \tag{100}$$

or

$$\boldsymbol{I}^T \operatorname*{diag}_{\alpha} \left[||\boldsymbol{U}_\alpha||^2\right] \boldsymbol{I} \to \min \tag{101}$$

$$\boldsymbol{W}_\Omega (\text{pdiag}\boldsymbol{U})^T \boldsymbol{I} = \boldsymbol{p} \tag{102}$$

where: $\boldsymbol{U}, \boldsymbol{I}$ so far stand for multiplicated vector of voltage and current signals inside the multi- port-network, $\boldsymbol{p} = \operatorname*{col}_{\alpha\in\Omega}[p_\alpha]$ stands for assumed active power vector for chosen inner branches, $\boldsymbol{W}_\Omega$ – the choice matrix of branches .

Obviously the optimization problems (99), (100) and (101), (102), are equivalent. Let us set currents distribution $\boldsymbol{I}$, then let us seek voltage distribution $\boldsymbol{U}$ meeting problem (99), (100), that is:

$$\boldsymbol{U}^T \boldsymbol{Q} \boldsymbol{U} + \boldsymbol{U}^T (\text{pdiag}\boldsymbol{I}) \boldsymbol{W}_\Omega^T \boldsymbol{x} \to \min \tag{103}$$

where:
$\boldsymbol{x} = \operatorname*{col}_{\alpha\in\Omega}[x_\alpha]$ – stands for the vector of indeterminate Lagrange factors,
$\boldsymbol{Q} = \operatorname*{diag}_{\alpha}\left[||\boldsymbol{I}_\alpha||^2\right]$

Problem (103) will be solved by means of a differential method. As it results from the considerations in Section 3 the necessary and sufficient minimum condition for the functional (103) is (at the fixed $\boldsymbol{x}$):

$$2d\boldsymbol{U}^T \boldsymbol{Q}\boldsymbol{U} + d\boldsymbol{U}^T (\text{pdiag}\boldsymbol{I}) \boldsymbol{W}_\Omega^b x = 0$$

or under (31,32,34)

$$2\left[\boldsymbol{0}^T d\boldsymbol{u}^T\right] \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B} & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix} + \left[\boldsymbol{0}^T d\boldsymbol{u}^T\right] \begin{bmatrix} \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{bmatrix} = \boldsymbol{x} = 0 \tag{104}$$

where:

$$\begin{bmatrix} \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{P}^T (\text{pdiag}\boldsymbol{I}) \boldsymbol{W}_\Omega^T \tag{105}$$

and where (104) has to be met for any change of $d\boldsymbol{u}$. Thus:

$$\bigwedge_{\delta \boldsymbol{u}} d\boldsymbol{u}^T (\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{v} + 0.5\boldsymbol{B}_1 \boldsymbol{x}) = 0 \tag{106}$$

Condition (106) is met by the following set of equations:

$$\boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{v} + 0.5\boldsymbol{B}_1\boldsymbol{x} = \boldsymbol{0} \tag{107}$$

which yields

$$\boldsymbol{u} = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{v} - 0.5\boldsymbol{A}^{-1}\boldsymbol{B}_1\boldsymbol{x} \tag{108}$$

Power condition (100) can be recorded in form:

$$\boldsymbol{W}_\Omega (\text{pdiag}\boldsymbol{I})^T \boldsymbol{P} \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix} = \boldsymbol{p}$$

or under (105):

$$\left[\boldsymbol{B}_2^T \boldsymbol{B}_1^T\right] \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix} = \boldsymbol{p}$$

thus

$$\boldsymbol{B}_2^T \boldsymbol{v} + \boldsymbol{B}_1^T \boldsymbol{u} = \boldsymbol{p} \tag{109}$$

Substituting (91) to (92) yields a set of linear equations for the Lagrange factor vector:

$$0.5\boldsymbol{B}_1^T \boldsymbol{A}^{-1} \boldsymbol{B}_1 \boldsymbol{x} = (\boldsymbol{B}_2^T - \boldsymbol{B}_1^T \boldsymbol{A}^{-1}\boldsymbol{B})\boldsymbol{v} - \boldsymbol{p} \tag{110}$$

Analogous proceedings with dual problem (101), (102) give:

$$\boldsymbol{U} \to \boldsymbol{Q} = \operatorname*{diag}_{\alpha}\left[||\boldsymbol{U}_\alpha||^2\right] \xrightarrow{\boldsymbol{C}} \boldsymbol{B}, \boldsymbol{A}$$

$$\begin{bmatrix} \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{bmatrix} \boldsymbol{x} = \boldsymbol{C}^T (\text{pdiag}\boldsymbol{U}) \boldsymbol{W}_\Omega^T \tag{111}$$

$$\boldsymbol{i} = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{j} - 0.5\boldsymbol{A}^{-1}\boldsymbol{B}_1\boldsymbol{x} \tag{112}$$

$$0.5\boldsymbol{B}_1^T \boldsymbol{A}^{-1} \boldsymbol{B}_1 \boldsymbol{x} = (\boldsymbol{B}_2^T - \boldsymbol{B}_1^T \boldsymbol{A}^{-1}\boldsymbol{B})\boldsymbol{j} - \boldsymbol{p} \tag{113}$$

With independent set of chosen branches Ω matrix $\boldsymbol{B}_1^T \boldsymbol{A}^{-1} \boldsymbol{B}_1$ is positively definite, thus the set of Eqs. (110) or (113) has a unique solution $\boldsymbol{x}$. Thereby, for the fixed $\boldsymbol{I}$ problem (99), (100) have one minimum point , similarly for the fixed $\boldsymbol{U}$ problem (101) and (102) also have the unique minimum point. Thus the following sequence of inequality (164) is met, which means that minimum point of problems (99,100), (or 101,102) can be received by means of an iterative process (55) (or 56,57).

The individual iterative functions for this process can be calculated as follows: $\boldsymbol{I} \rightarrow$

$$A_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \text{INNER TREE – BRANCHES}}} \operatorname{sgn}(\gamma)||\boldsymbol{I}_\gamma||^2,$$

$$B_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \text{INNER}\cap\text{OUTER} \\ \text{TREE – BRANCHES}}} \operatorname{sgn}(\gamma)||\boldsymbol{I}_\gamma||^2$$

$$\begin{bmatrix} \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{bmatrix} = \boldsymbol{P}^T(\operatorname{pdiag}\boldsymbol{I})\boldsymbol{W}_\Omega^T \qquad (\Omega = \Omega(\boldsymbol{I}))$$

$$\begin{aligned} 0.5\boldsymbol{B}_1^T\boldsymbol{A}^{-1}\boldsymbol{B}_1\boldsymbol{x} &= \left(\boldsymbol{B}_2^T - \boldsymbol{B}_1^T\boldsymbol{A}^{-1}\boldsymbol{B}\right) \\ &\quad \times \boldsymbol{v} - \boldsymbol{p} \rightarrow \boldsymbol{x} \rightarrow \boldsymbol{u} \\ &= -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{v} - 0.5\boldsymbol{A}^{-1}\boldsymbol{B}_1\boldsymbol{x} \\ &\rightarrow \boldsymbol{U} = [\boldsymbol{P}]\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{u} \end{bmatrix} = \boldsymbol{\Phi}^U(\boldsymbol{I}) \rightarrow \end{aligned} \tag{114}$$

$$A_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \text{INNER LINKS}}} \operatorname{sgn}(\gamma)||\boldsymbol{U}_\gamma||^2,$$

$$B_{\alpha\beta} = \sum_{\substack{\gamma\in\{\alpha\}\cap\{\beta\} \\ \text{INNER}\cap\text{OUTER} \\ \text{LINKS}}} \operatorname{sgn}(\gamma)||\boldsymbol{U}_\gamma||^2$$

$$\begin{bmatrix} \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{bmatrix} = \boldsymbol{C}^T(\operatorname{pdiag}\boldsymbol{U})\boldsymbol{W}_\Omega^T \qquad (\Omega = \Omega(\boldsymbol{U}))$$

$$\begin{aligned} 0.5\boldsymbol{B}_1^T\boldsymbol{A}^{-1}\boldsymbol{B}_1\boldsymbol{x} &= \left(\boldsymbol{B}_2^T - \boldsymbol{B}_1^T\boldsymbol{A}^{-1}\boldsymbol{B}\right) \\ &\quad \times \boldsymbol{j} - \boldsymbol{p} \rightarrow \boldsymbol{x} \rightarrow \boldsymbol{i} \\ &= -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{j} - 0.5\boldsymbol{A}^{-1}\boldsymbol{B}_1\boldsymbol{x} \\ &\rightarrow [\boldsymbol{C}]\begin{bmatrix} \boldsymbol{j} \\ \boldsymbol{i} \end{bmatrix} = \boldsymbol{\Phi}^I \circ \boldsymbol{\Phi}^U(\boldsymbol{I}) \end{aligned} \tag{115}$$

Example 9. The graph of two-port network of T type will be considered again, which is characterized by the lack of independent current co-ordinates see Fig. 17.

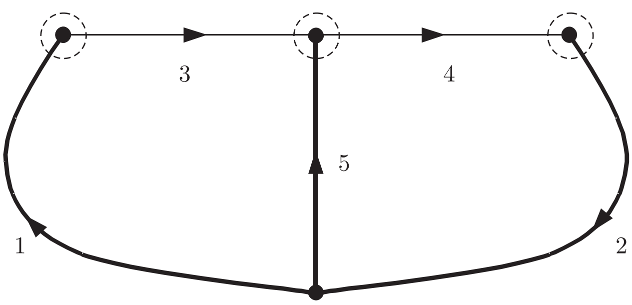


Fig. 17. Graph of two-port network of T type with the chosen tree

The conditional functional of voltage variable will be minimized:

$$\boldsymbol{U}^T\boldsymbol{U} \rightarrow \min$$

$$\boldsymbol{W}_\Omega(\operatorname{pdiag}\boldsymbol{I})T\boldsymbol{U} = \begin{bmatrix} p_3 \\ p_4 \end{bmatrix} \qquad \Omega = \{3, 4\}$$

where: p_3, p_4 – fixed active powers in branches 3, 4.

The individual matrices taking part in optimization process have form:

$$\boldsymbol{C} = \begin{array}{c|cc|c} & 1 & 2 & \\ \hline 3 & 1 & & j_1 \\ 4 & & 1 & j_2 \\ 5 & -1 & 1 & j_2 - j_1 \end{array} \qquad \boldsymbol{P} = \begin{array}{c|cc|c} & 1 & 2 & 5 \\ \hline 3 & -1 & & 1 \\ 4 & & -1 & -1 \\ 5 & & & 1 \end{array}$$

$$\boldsymbol{W} = \begin{array}{c|ccc} & 3 & 4 & 5 \\ \hline 3 & 1 & & \\ 4 & & 1 & \end{array}$$

this yields following matrices:

$$\boldsymbol{A} = [3], \quad \boldsymbol{B} = [-1,\ 1]$$

$$\begin{bmatrix} \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_3 & & \\ & \boldsymbol{I}_4 & \\ & & \boldsymbol{I}_5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{array}{[cc]l} -\boldsymbol{I}_3 & & \\ & -\boldsymbol{I}_4 & \boldsymbol{B}_2 \\ \boldsymbol{I}_3 & -\boldsymbol{I}_4 & \boldsymbol{B}_1 \end{array}$$

Thus Eqs. (110) take form:

$$\begin{aligned} \frac{1}{6}\begin{bmatrix} \boldsymbol{I}_3^T \\ -\boldsymbol{I}_3^T \end{bmatrix}[\boldsymbol{I}_3, \boldsymbol{I}_4]\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} &= \left(\begin{bmatrix} \boldsymbol{I}_3^T & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I}_4^T \end{bmatrix}\right. \\ &\quad \left. - \begin{bmatrix} \boldsymbol{I}_3^T \\ -\boldsymbol{I}_4^T \end{bmatrix}\frac{1}{3}[-1,\ 1]\right)\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} - \begin{bmatrix} p_3 \\ p_4 \end{bmatrix} \end{aligned}$$

or

$$\begin{bmatrix} ||\boldsymbol{I}_3||^2 & -\boldsymbol{I}_3^T\boldsymbol{I}_4 \\ -\boldsymbol{I}_4^T\boldsymbol{I}_3 & ||\boldsymbol{I}_4||^2 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4\boldsymbol{I}_3^T & -2\boldsymbol{I}_3^T \\ -2\boldsymbol{I}_4^T & -4\boldsymbol{I}_4^T \end{bmatrix}\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} - \begin{bmatrix} 6p_3 \\ 6p_4 \end{bmatrix}$$

therefore we get following set of linear equations for the Lagrange factors:

$$\begin{bmatrix} ||\boldsymbol{j}_1||^2 & -(\boldsymbol{j}_1, \boldsymbol{j}_2) \\ -(\boldsymbol{j}_1, \boldsymbol{j}_2) & ||\boldsymbol{j}_2||^2 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4(\boldsymbol{j}_1, \boldsymbol{v}_1) - 2(\boldsymbol{j}_1, \boldsymbol{v}_2) - 6p_3 \\ -2(\boldsymbol{j}_2, \boldsymbol{v}_1) - 4(\boldsymbol{j}_2, \boldsymbol{v}_2) - 6p_4 \end{bmatrix}$$

Determinant of this set of equations meet the Schwartz inequality:

$$||\boldsymbol{j}_1||^2||\boldsymbol{j}_2||^2 - (\boldsymbol{j}_1, \boldsymbol{j}_2)^2 \geq 0$$

So, the problem can have no solution when inequality becomes equality. The unique independent voltage is defined by the formula:

$$\begin{aligned} \boldsymbol{u}_5 &= -\frac{1}{3}[-1,\ 1]\begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} - \frac{1}{6}[\boldsymbol{I}_3,\ -\boldsymbol{I}_4]\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \\ &= \frac{1}{3}\boldsymbol{v}_1 - \frac{1}{3}\boldsymbol{v}_2 - \frac{1}{6}x_3\boldsymbol{j}_1 + \frac{1}{6}x_4\boldsymbol{j}_2 \end{aligned}$$

Example 10. Graph of two – port network of type Π with chosen set of links is drawn again in Fig. 18. This time, because of the lack of independent voltages, the following optimization problem will be solved:

$$\boldsymbol{I}^T\boldsymbol{I} \rightarrow \min$$

$$\boldsymbol{W}_\Omega(\operatorname{pdiag}\boldsymbol{U})^T\boldsymbol{I} = \begin{bmatrix} p_4 \\ p_5 \end{bmatrix} \qquad \Omega = \{4, 5\},$$

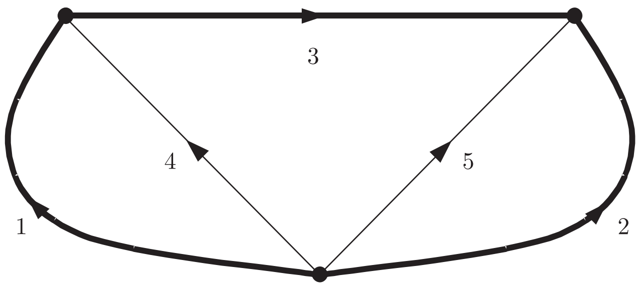


Fig. 18. The graph of two-port network of type Π with chosen set of links

where p_4 and p_5 stand for independent active powers in branches 4 and 5.

With the help of graph we can determine:

$\boldsymbol{P}$=

	v_1	v_2	
	1	2	
3	–1	1	$v_2 - v_1$
4	–1		v_1
5		–1	v_2

$\boldsymbol{C}$=

	j_1	j_2	I_3
	1	2	3
3			1
4	–1		1
5		–1	–1

$\boldsymbol{W}$=

	3	4	5
4		1	
5			1

Hence follow further matrices:

$$\boldsymbol{A} = [3], \quad \boldsymbol{B} = [-1,\ 1]$$

$$\begin{bmatrix} \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{bmatrix} = \begin{bmatrix} & -1 & \\ & & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_3 & & \\ & \boldsymbol{U}_4 & \\ & & \boldsymbol{U}_5 \end{bmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \begin{array}{c} \begin{bmatrix} -\boldsymbol{U}_4 & \\ & -\boldsymbol{U}_5 \\ \boldsymbol{U}_4 & -\boldsymbol{U}_5 \end{bmatrix} \end{array} \begin{array}{c} \\ \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{array}$$

Set of Eqs. (96) has form:

$$\frac{1}{6}\begin{bmatrix} \boldsymbol{U}_4^T \\ -\boldsymbol{U}_5^T \end{bmatrix} [\boldsymbol{U}_3, \boldsymbol{U}_4] \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \left(\begin{bmatrix} \boldsymbol{U}_4^T & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{U}_5^T \end{bmatrix} - \begin{bmatrix} \boldsymbol{U}_4^T \\ -\boldsymbol{U}_5^T \end{bmatrix} \frac{1}{3}[-1,\ 1] \right) \begin{bmatrix} \boldsymbol{j}_1 \\ \boldsymbol{j}_2 \end{bmatrix} - \begin{bmatrix} p_4 \\ p_5 \end{bmatrix}$$

thus

$$\begin{bmatrix} ||\boldsymbol{v}_1||^2 & -\boldsymbol{v}_1^T\boldsymbol{v}_2 \\ -\boldsymbol{v}_1^T\boldsymbol{v}_2 & ||\boldsymbol{v}_2||^2 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4\boldsymbol{v}_1^T & -2\boldsymbol{v}_1^T \\ -2\boldsymbol{v}_2^T & -4\boldsymbol{v}_2^T \end{bmatrix} \begin{bmatrix} \boldsymbol{j}_1 \\ \boldsymbol{j}_2 \end{bmatrix} - \begin{bmatrix} 6p_4 \\ 6p_5 \end{bmatrix}$$

or

$$\begin{bmatrix} ||\boldsymbol{v}_1||^2 & -(\boldsymbol{v}_1\boldsymbol{v}_2) \\ -(\boldsymbol{v}_1, \boldsymbol{v}_2) & ||\boldsymbol{v}_2||^2 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -4(\boldsymbol{j}_1, \boldsymbol{v}_1) - 2(\boldsymbol{j}_2, \boldsymbol{v}_1) - 6p_4 \\ -2(\boldsymbol{j}_1, \boldsymbol{v}_2) - 4(\boldsymbol{j}_2, \boldsymbol{v}_2) - 6p_5 \end{bmatrix}$$

Also this time the Schwartz inequality in determinant:

$$||\boldsymbol{v}_1||^2||\boldsymbol{v}_2||^2 - (\boldsymbol{v}_1, \boldsymbol{v}_2)^2 \geq 0$$

is met , thus the set of equations for x_4, x_5 can be badly conditioned when the difference between $||v_1||||v_2||$ and scalar product $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ approaches zero.

The unique independent current co- ordinate can be received now from expression (112):

$$\boldsymbol{I}_5 = -\frac{1}{3}[-1,\ 1]\begin{bmatrix} \boldsymbol{j}_1 \\ \boldsymbol{j}_2 \end{bmatrix} - \frac{1}{6}[\boldsymbol{U}_4,\ -\boldsymbol{U}_5]\begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \frac{1}{3}\boldsymbol{j}_1 - \frac{1}{3}\boldsymbol{j}_2 - \frac{1}{6}x_4\boldsymbol{v}_1 + \frac{1}{6}x_5\boldsymbol{v}_2$$

Example 11. It is given to illustrate iterative process of minimization (114,115), and to find the choice set of branches Ω.

The graph of two-port network with port {1,2} is shown in Fig. 19, where *TREE BRANCHES* {1, 2; 6} and *LINKS* {1,2;6} are marked with thick line

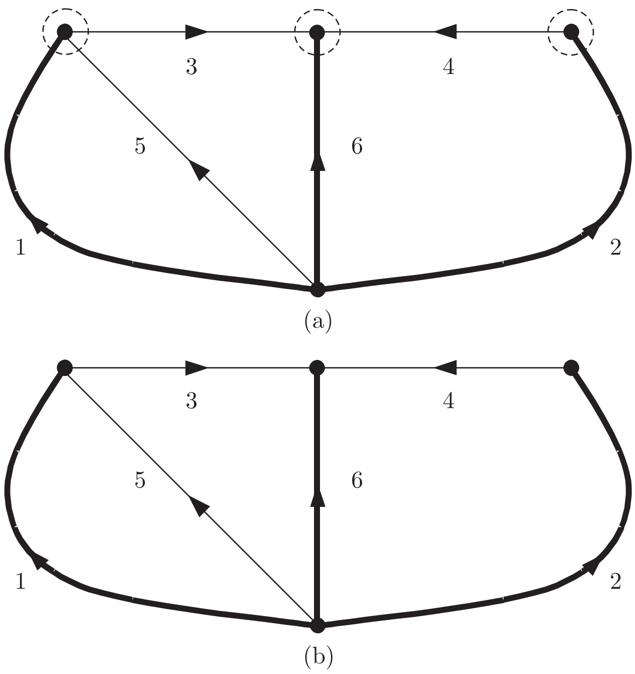


Fig. 19. Tree branches and links (a) of two port network (b)

Determining the iterative function $\boldsymbol{I} \rightarrow \boldsymbol{\Phi}^U(\boldsymbol{I})$ proceeds as follows:

$\boldsymbol{P}$=

		v_1	v_2	U_6	
		1	2	6	
U_3	3	–1		1	$-v_1 + U_6$
U_4	4		–1	1	$-v_2 + U_6$
U_5	5	1			v_1
U_6	6			1	U_6

$\boldsymbol{W}_\Omega$=

	3	4	5	6		
3	1				p_3	←The vector
4		1			p_4	of prescribed powers
6				1	p_6	of independent branches
Ω						

Warning: with the fixed vector $\boldsymbol{I}$ the active power p_5 in branch 5 is fixed too. This branch should be removed from *INNER BRANCH* set by creating set of choice Ω . The further indispensable matrices are: the cut-set matrix:

$\boldsymbol{A}$:

	6
6	{3,4,6}

$\boldsymbol{B}$:

	1	2
6	{–3}	{–4}

thus

$$\boldsymbol{A} = [|\boldsymbol{I}|^2]$$

$$\boldsymbol{B} = [-|\boldsymbol{I}_3|^2], \ -||\boldsymbol{I}_4||^2]$$

where:

$$|\boldsymbol{I}|^2 = |\boldsymbol{I}_3|^2 + ||\boldsymbol{I}_4||^2 + ||\boldsymbol{I}_6||^2$$

and

$$\begin{bmatrix} \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{bmatrix} = \boldsymbol{P}^T(\text{pdiag}\, \boldsymbol{I})\boldsymbol{W}_\Omega^T = \begin{array}{|ccc|l} \hline -\boldsymbol{I}_3 & & & \\ & -\boldsymbol{I}_4 & & \boldsymbol{B}_2 \\ \hline \boldsymbol{I}_3 & \boldsymbol{I}_4 & \boldsymbol{I}_6 & \boldsymbol{B}_1 \\ \hline \end{array}$$

which yields set of linear Eqs. (110):

$$\frac{1}{2|\boldsymbol{I}|^2} \begin{bmatrix} \boldsymbol{I}_3^T \\ \boldsymbol{I}_4^T \\ \boldsymbol{I}_6^T \end{bmatrix} [\boldsymbol{I}_3, \boldsymbol{I}_4, \boldsymbol{I}_6] \begin{bmatrix} x_3 \\ x_4 \\ x_6 \end{bmatrix} = \left(\begin{bmatrix} -\boldsymbol{I}_3^T & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I}_4^T \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \right.$$
$$\left. - \begin{bmatrix} \boldsymbol{I}_3^T \\ \boldsymbol{I}_4^T \\ \boldsymbol{I}_6^T \end{bmatrix} \frac{1}{|\boldsymbol{I}|^2} [-||\boldsymbol{I}_3||^2, -||\boldsymbol{I}_4||^2] \right) \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} - \begin{bmatrix} p_3 \\ p_4 \\ p_6 \end{bmatrix}$$

or

$$\frac{1}{2|\boldsymbol{I}|^2} \begin{bmatrix} \boldsymbol{I}_3^T\boldsymbol{I}_3 & \boldsymbol{I}_3^T\boldsymbol{I}_4 & \boldsymbol{I}_3^T\boldsymbol{I}_6 \\ \boldsymbol{I}_4^T\boldsymbol{I}_3 & \boldsymbol{I}_4^T\boldsymbol{I}_4 & \boldsymbol{I}_4^T\boldsymbol{I}_6 \\ \boldsymbol{I}_6^T\boldsymbol{I}_3 & \boldsymbol{I}_6^T\boldsymbol{I}_4 & \boldsymbol{I}_6^T\boldsymbol{I}_6 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_6 \end{bmatrix}$$

$$= \begin{array}{|c|c|} \hline \left(\frac{||\boldsymbol{I}_3||^2}{|\boldsymbol{I}|^2} - 1\right)\boldsymbol{I}_3^T & \frac{||\boldsymbol{I}_4||^2}{|\boldsymbol{I}|^2}\boldsymbol{I}_3^T \\ \hline \frac{||\boldsymbol{I}_3||^2}{|\boldsymbol{I}|^2}\boldsymbol{I}_4^T & \left(\frac{||\boldsymbol{I}_4||^2}{|\boldsymbol{I}|^2} - 1\right)\boldsymbol{I}_4^T \\ \hline \frac{||\boldsymbol{I}_3||^2}{|\boldsymbol{I}|^2}\boldsymbol{I}_6^T & \frac{||\boldsymbol{I}_4||^2}{|\boldsymbol{I}|^2}\boldsymbol{I}_6^T \\ \hline \end{array} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} - \begin{array}{|c|} \hline p_3 \\ \hline p_4 \\ \hline p_6 \\ \hline \end{array}$$

We can see that equation matrix is the Gram matrix (matrix of scalar products). Thus it is positively definite , and the set of linear equations has a solution. In the worst case the set of equations can be bad conditioned, when a determinant of the Gram matrix will approach zero.

To determine the unique independent voltage signal the equation (108) is designed for:

$$\boldsymbol{u}_6 = -\frac{1}{|\boldsymbol{I}|^2}[-||\boldsymbol{I}_3||^2, -||\boldsymbol{I}_4||^2] \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_1 \end{bmatrix} - \frac{1}{2||\boldsymbol{I}||^2}[\boldsymbol{I}_3, \boldsymbol{I}_4, \boldsymbol{I}_6]$$
$$\times \begin{bmatrix} x_3 \\ x_4 \\ x_6 \end{bmatrix} = \frac{||\boldsymbol{I}_3||^2}{|\boldsymbol{I}|^2}\boldsymbol{v}_1 + \frac{||\boldsymbol{I}_4||^2}{|\boldsymbol{I}|^2}\boldsymbol{v}_2$$
$$- \frac{1}{2|\boldsymbol{I}|^2}[x_3\boldsymbol{I}_3 + x_4\boldsymbol{I}_4 + x_6\boldsymbol{I}_6]$$

and

$$\boldsymbol{U} = \boldsymbol{P} \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \boldsymbol{u}_6 \end{bmatrix} = \boldsymbol{\Phi}^U(\boldsymbol{I})$$

Determining of iterative function $\boldsymbol{U} \rightarrow \boldsymbol{\Phi}^I(\boldsymbol{U})$ proceeds according to diagram:

$$\boldsymbol{C} = \begin{array}{cc|cc|c|l} & & j_1 & j_2 & I_6 & \\ & & 1 & 2 & 6 & \\ \hline I_3 & 3 & & -1 & 1 & -j_1 - I_6 \\ I_4 & 4 & & 1 & 1 & j_2 \\ I_5 & 5 & -1 & -1 & -1 & -j_1 - j_2 - I_6 \\ I_6 & 6 & & & 1 & I_6 \\ \hline \end{array}$$

$$\boldsymbol{W}_\Omega = \begin{array}{c|cccc|} & 3 & 4 & 5 & 6 \\ \hline 3 & 1 & & & \\ 5 & & & 1 & \\ 6 & & & & 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline p_3 \\ p_5 \\ p_6 \\ \hline \end{array}$$

The vector
of assumed powers
of independent branches

$$\boldsymbol{A}: \quad \begin{array}{c|c|} & 6 \\ \hline 6 & \{3,4,6\} \\ \hline \end{array} \qquad \boldsymbol{B}: \quad \begin{array}{c|c|c|} & 1 & 2 \\ \hline 6 & \{5\} & \{3, 5\ \} \\ \hline \end{array}$$

thus

$$\boldsymbol{A} = [|\boldsymbol{U}|^2],$$

$$\boldsymbol{B} = [||\boldsymbol{U}_5||^2, ||\boldsymbol{U}_3||^2 + ||\boldsymbol{U}_5||^2]$$

where:

$$|\boldsymbol{U}|^2 = ||\boldsymbol{U}_3||^2 + ||\boldsymbol{U}_5||^2 + ||\boldsymbol{U}_6||^2$$

and

$$\begin{bmatrix} \boldsymbol{B}_2 \\ \boldsymbol{B}_1 \end{bmatrix} = \boldsymbol{C}^T(\text{pdiag}\, \boldsymbol{U})\boldsymbol{W}_\Omega^T = \begin{array}{|ccc|l} \hline -\boldsymbol{U}_3 & -\boldsymbol{U}_5 & & \boldsymbol{B}_2 \\ \hline -\boldsymbol{U}_3 & -\boldsymbol{U}_5 & \boldsymbol{U}_6 & \boldsymbol{B}_1 \\ \hline \end{array}$$

which yields the set of linear Eqs. (113):

$$\frac{1}{2|\boldsymbol{U}|^2} \begin{bmatrix} -\boldsymbol{U}_3^T \\ -\boldsymbol{U}_5^T \\ \boldsymbol{U}_6^T \end{bmatrix} [-\boldsymbol{U}_3, -\boldsymbol{U}_5, \boldsymbol{U}_6] \begin{bmatrix} x_3 \\ x_5 \\ x_6 \end{bmatrix} = \left(\begin{bmatrix} \boldsymbol{0} & -\boldsymbol{U}_3^T \\ \boldsymbol{0} & -\boldsymbol{U}_5^T \\ \boldsymbol{0} & -\boldsymbol{0} \end{bmatrix} \right.$$
$$\left. - \begin{bmatrix} -\boldsymbol{U}_3^T \\ -\boldsymbol{U}_5^T \\ -\boldsymbol{U}_6^T \end{bmatrix} \frac{1}{|\boldsymbol{U}|^2} [||\boldsymbol{U}_5||^2, ||\boldsymbol{U}_3||^2 + ||\boldsymbol{U}_5||^2] \right) \begin{bmatrix} \boldsymbol{j}_1 \\ \boldsymbol{j}_2 \end{bmatrix}$$
$$- \begin{bmatrix} p_3 \\ p_5 \\ p_6 \end{bmatrix}$$

or

$$\frac{1}{2|\boldsymbol{I}|^2} \begin{bmatrix} \boldsymbol{U}_3^T\boldsymbol{U}_3 & \boldsymbol{U}_3^T\boldsymbol{U}_5 & -\boldsymbol{U}_3^T\boldsymbol{U}_6 \\ \boldsymbol{U}_5^T\boldsymbol{U}_3 & \boldsymbol{U}_5^T\boldsymbol{U}_5 & -\boldsymbol{U}_5^T\boldsymbol{U}_6 \\ -\boldsymbol{U}_6^T\boldsymbol{U}_3 & -\boldsymbol{U}_6^T\boldsymbol{U}_5 & \boldsymbol{U}_6^T\boldsymbol{U}_6 \end{bmatrix} \begin{bmatrix} x_3 \\ x_5 \\ x_6 \end{bmatrix}$$

$$= \begin{array}{|c|c|} \hline \frac{||\boldsymbol{U}_5||^2}{|\boldsymbol{U}|^2}\boldsymbol{U}_3^T & \left(\frac{||\boldsymbol{U}_3||^2 + ||\boldsymbol{U}_5||^2}{|\boldsymbol{U}|^2} - 1\right)\boldsymbol{U}_3^T \\ \hline \frac{||\boldsymbol{U}_5||^2}{|\boldsymbol{U}|^2}\boldsymbol{U}_5^T & \left(\frac{||\boldsymbol{U}_3||^2 + ||\boldsymbol{U}_5||^2}{|\boldsymbol{U}|^2} - 1\right)\boldsymbol{U}_5^T \\ \hline \frac{||\boldsymbol{U}_5||^2}{|\boldsymbol{U}|^2}\boldsymbol{U}_6^T & \frac{||\boldsymbol{U}_3||^2 + ||\boldsymbol{U}_5||^2}{|\boldsymbol{U}|^2}\boldsymbol{U}_6^T \\ \hline \end{array} \begin{bmatrix} æv_1 \\ æv_2 \end{bmatrix} - \begin{array}{|c|} \hline p_3 \\ \hline p_5 \\ \hline p_6 \\ \hline \end{array}$$

This time the equation matrix also turns out to be the Gram matrix . But the size of a linear equation set can be reduced by one. From observation of matrix $\boldsymbol{C}$ follows that with fixed vector $\boldsymbol{U}$ active power p_4 in branch 4 is also fixed (the current in this branch $\boldsymbol{I}_4 = \boldsymbol{j}_2$ is forced) . Thus from the balance of power it follows that: $p_3 + p_4 + p_5 + p_6 = $ const it yields $p_3 + p_5 + p_6 = $ const it means that only two values of the sum out of three (p_3, p_5, p_6) are independent. Thus the set of branch choice Ω can be assumed as $\Omega = \{5, 6\}$ or $\{3, 5\}$ or $\{3, 6\}$.

Suitable matrices of choice $\boldsymbol{W}_\Omega$ take form:

$$\begin{array}{c|cccc|} & 3 & 4 & 5 & 6 \\ \hline 5 & & & 1 & \\ 6 & & & & 1 \\ \hline \end{array} \qquad \begin{array}{c|cccc|} & 3 & 4 & 5 & 6 \\ \hline 3 & 1 & & & \\ 5 & & & 1 & \\ \hline \end{array} \qquad \begin{array}{c|cccc|} & 3 & 4 & 5 & 6 \\ \hline 3 & 1 & & & \\ 6 & & & & 1 \\ \hline \end{array}$$

Then block matrices $\begin{bmatrix}\boldsymbol{B}_2\\ \boldsymbol{B}_2\end{bmatrix}$ are respectively:

$-\boldsymbol{U}_5$		
$-\boldsymbol{U}_5$		$\boldsymbol{B}_2$
$-\boldsymbol{U}_5$	$\boldsymbol{U}_6$	$\boldsymbol{B}_1$

	$-\boldsymbol{U}_5$
$-\boldsymbol{U}_3$	$-\boldsymbol{U}_5$
$-\boldsymbol{U}_3$	$-\boldsymbol{U}_5$

$-\boldsymbol{U}_3$	
$-\boldsymbol{U}_3$	$\boldsymbol{U}_6$

which yields following set of linear equations

$$\frac{1}{2|\boldsymbol{U}|^2}\begin{bmatrix} \boldsymbol{U}_5^T\boldsymbol{U}_5 & -\boldsymbol{U}_5^T\boldsymbol{U}_6 \\ -\boldsymbol{U}_6^T\boldsymbol{U}_5 & \boldsymbol{U}_6^T\boldsymbol{U}_6 \end{bmatrix}\begin{bmatrix} x_5 \\ x_6 \end{bmatrix} = \left(\begin{bmatrix} -\boldsymbol{U}_5^T & -\boldsymbol{U}_5^T \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} - \frac{1}{|\boldsymbol{U}|^2}\begin{bmatrix} -\boldsymbol{U}_5^T \\ \boldsymbol{U}_6^T \end{bmatrix}\left[||\boldsymbol{U}_5||^2, ||\boldsymbol{U}_3||^2 + ||\boldsymbol{U}_5||^2\right]\right) \times \begin{bmatrix} \boldsymbol{j}_1 \\ \boldsymbol{j}_2 \end{bmatrix} - \begin{bmatrix} p_5 \\ p_6 \end{bmatrix}$$

or

$$\frac{1}{2|\boldsymbol{U}|^2}\begin{bmatrix} \boldsymbol{U}_3^T\boldsymbol{U}_3 & -\boldsymbol{U}_3^T\boldsymbol{U}_5 \\ -\boldsymbol{U}_5^T\boldsymbol{U}_3 & \boldsymbol{U}_5^T\boldsymbol{U}_5 \end{bmatrix}\begin{bmatrix} x_3 \\ x_5 \end{bmatrix} = \left(\begin{bmatrix} \boldsymbol{0} & -\boldsymbol{U}_3^T \\ -\boldsymbol{U}_5^T & -\boldsymbol{U}_5^T \end{bmatrix} - \frac{1}{|\boldsymbol{U}|^2}\begin{bmatrix} -\boldsymbol{U}_3^T \\ \boldsymbol{U}_5^T \end{bmatrix}\left[||\boldsymbol{U}_5||^2, ||\boldsymbol{U}_3||^2 + ||\boldsymbol{U}_5||^2\right]\right) \times \begin{bmatrix} \boldsymbol{j}_1 \\ \boldsymbol{j}_2 \end{bmatrix} - \begin{bmatrix} p_3 \\ p_5 \end{bmatrix}$$

or

$$\frac{1}{2|\boldsymbol{U}|^2}\begin{bmatrix} \boldsymbol{U}_3^T\boldsymbol{U}_3 & -\boldsymbol{U}_3^T\boldsymbol{U}_6 \\ -\boldsymbol{U}_6^T\boldsymbol{U}_3 & \boldsymbol{U}_6^T\boldsymbol{U}_6 \end{bmatrix}\begin{bmatrix} x_3 \\ x_6 \end{bmatrix} = \left(\begin{bmatrix} \boldsymbol{0} & -\boldsymbol{U}_3^T \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} - \frac{1}{|\boldsymbol{U}|^2}\begin{bmatrix} -\boldsymbol{U}_3^T \\ \boldsymbol{U}_6^T \end{bmatrix}\left[||\boldsymbol{U}_5||^2, ||\boldsymbol{U}_3||^2 + ||\boldsymbol{U}_5||^2\right]\right) \times \begin{bmatrix} \boldsymbol{j}_1 \\ \boldsymbol{j}_2 \end{bmatrix} - \begin{bmatrix} p_3 \\ p_6 \end{bmatrix}$$

Similarly while determining previous function $\boldsymbol{\Phi}^U(\boldsymbol{I})$, from the observation of matrix $\boldsymbol{P}$ it results that with the assumed distribution of currents power p5 in branch 5 is also prescribed. Thus out of three remaining powers two can be chosen independently i.e. $\Omega = \{3, 4\}$ or $\{3, 6\}$ or $\{4, 6\}$. It yields then the following matrices $\boldsymbol{W}_\Omega$:

$$\begin{array}{c|cccc|} & 3 & 4 & 5 & 6 \\ \hline 3 & 1 & & & \\ 4 & & 1 & & \\ \hline \end{array} \qquad \begin{array}{c|cccc|} & 3 & 4 & 5 & 6 \\ \hline 3 & 1 & & & \\ 6 & & & & 1 \\ \hline \end{array} \qquad \begin{array}{c|cccc|} & 3 & 4 & 5 & 6 \\ \hline 4 & & 1 & & \\ 6 & & & & 1 \\ \hline \end{array}$$

and block matrices $\boldsymbol{B}_2, \boldsymbol{B}_1$:

$-\boldsymbol{I}_3$		
	$-\boldsymbol{I}_4$	$\boldsymbol{B}_2$
$\boldsymbol{I}_3$	$\boldsymbol{I}_4$	$\boldsymbol{B}_1$

$-\boldsymbol{I}_3$	
$\boldsymbol{I}_3$	$\boldsymbol{I}_6$

$-\boldsymbol{I}_4$	
$\boldsymbol{I}_4$	$\boldsymbol{I}_6$

then the set of linear Eqs. (110)takes form:

$$\frac{1}{2|\boldsymbol{I}|^2}\begin{bmatrix} \boldsymbol{I}_3^T\boldsymbol{I}_3 & \boldsymbol{I}_3^T\boldsymbol{I}_4 \\ \boldsymbol{I}_4^T\boldsymbol{I}_3 & \boldsymbol{I}_4^T\boldsymbol{I}_4 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \left(\begin{bmatrix} -\boldsymbol{I}_3^T & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I}_4^T \end{bmatrix} - \frac{1}{|\boldsymbol{I}|^2}\begin{bmatrix} \boldsymbol{I}_3^T \\ \boldsymbol{I}_4^T \end{bmatrix}\left[-||\boldsymbol{I}_3||^2, -||\boldsymbol{I}_4||^2\right]\right) \times \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} - \begin{bmatrix} p_3 \\ p_4 \end{bmatrix}$$

or

$$\frac{1}{2|\boldsymbol{I}|^2}\begin{bmatrix} \boldsymbol{I}_3^T\boldsymbol{I}_3 & \boldsymbol{I}_3^T\boldsymbol{I}_6 \\ \boldsymbol{I}_6^T\boldsymbol{I}_3 & \boldsymbol{I}_6^T\boldsymbol{I}_6 \end{bmatrix}\begin{bmatrix} x_3 \\ x_6 \end{bmatrix} = \left(\begin{bmatrix} -\boldsymbol{I}_3^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} - \frac{1}{|\boldsymbol{I}|^2}\begin{bmatrix} \boldsymbol{I}_3^T \\ \boldsymbol{I}_6^T \end{bmatrix}\left[-||\boldsymbol{I}_3||^2, -||\boldsymbol{I}_4||^2\right]\right) \times \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} - \begin{bmatrix} p_3 \\ p_6 \end{bmatrix}$$

or

$$\frac{1}{2|\boldsymbol{I}|^2}\begin{bmatrix} \boldsymbol{I}_4^T\boldsymbol{I}_4 & \boldsymbol{I}_4^T\boldsymbol{I}_6 \\ \boldsymbol{I}_6^T\boldsymbol{I}_4 & \boldsymbol{I}_6^T\boldsymbol{I}_6 \end{bmatrix}\begin{bmatrix} x_4 \\ x_6 \end{bmatrix} = \left(\begin{bmatrix} \boldsymbol{0} & -\boldsymbol{I}_4^T \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} - \frac{1}{|\boldsymbol{I}|^2}\begin{bmatrix} \boldsymbol{I}_4^T \\ \boldsymbol{I}_6^T \end{bmatrix}\left[-||\boldsymbol{I}_3||^2, -||\boldsymbol{I}_4||^2\right]\right) \times \begin{bmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \end{bmatrix} - \begin{bmatrix} p_4 \\ p_6 \end{bmatrix}$$

7. Conclusion

As a matter of fact the problems which were considered in this article lie between the synthesis and the analysis of branched circuits. Just in the preface the thesis was proved that there exists an unequivocal voltage distribution on the inner branches whose sum of squared instantaneous voltage values is minimal. At the same time, there exists an unequivocal current distribution on the inner branches whose sum of squared instantaneous current values is minimal. By means of minimization of the positively defined functionals it is possible to find the voltage-current distributions on the net without the use of the immitance operators and basing only on the Kirchhoff laws. Consequently it is possible to find the individual branch parameters in the syntheses process.

This issue is next developed in Sections 3,4,5 where the power functionals are minimized. These are the squared voltage-current functionals, which represent the global power losses in the network. The generalization of that issue is presented in Section 6, where the synthesis of control of power net was done in such a way as to minimize the global power functional with some branch active power constrains. The practical application of that method is to synthesize the globally passive compensators of two port networks structure connected between the source and the receiver [6–8,2]. In Figs. 20,21 the two-port compensators of T and Π shape are shown. They have only one free signal x (voltage and current respectively). These signals can be determined uniquely when using some minimum criteria.

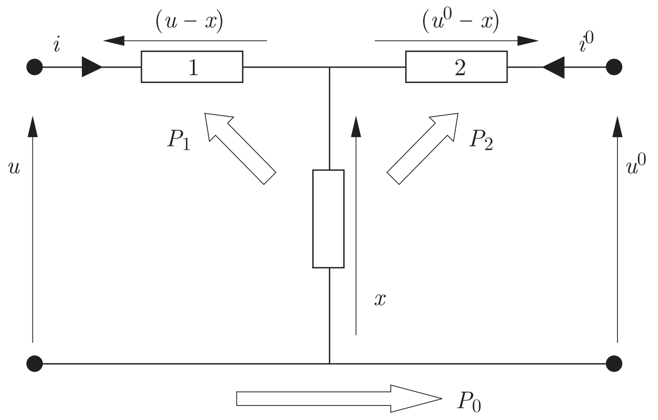


Fig. 20. Four terminal network with 'global minimal voltage'

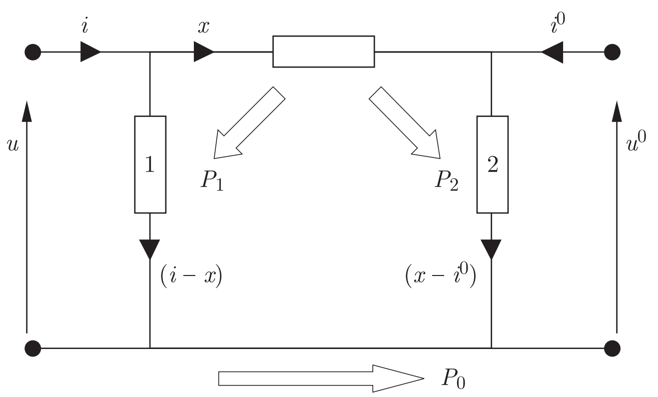


Fig. 21. Four terminal network with 'global minimal current'

Both four terminal networks shown in Figs. 20,21 are globally passive, i.e.

$$(u,i) = (u^0, i^0)$$

The power conditions of the four terminal network shown in Fig. 20 can be recorded as:

$$\begin{aligned}(u-x,i) &= P_1\\ (x-u^0,i^0) &= P_2\end{aligned} \tag{116}$$

or

$$\begin{aligned}(x,i) &= P_0 - P_1\\ (x,i^0) &= P_0 + P_2\end{aligned} \tag{117}$$

where P_0 is the prescribed Power flux transmitted by the four terminal network. In order to find unequivocal x the 'global minimal voltage condition' must be put.

$$(x,x) + (u-x,u-x) + (x-u^0,x-u^0) \to \min \tag{118}$$

or

$$0.5(x,x) - 1/3(u+u^0,x) \to \min \tag{119}$$

The suitable functional for tasks (117)-(119) has form:

$$f_{\lambda,\mu}(\boldsymbol{i}) = 0.5(x,x) - 1/3(u+u^0,x) - \lambda(x,i) - \mu(x,i^0) \to \min \tag{120}$$

For the four terminal network from Fig. 21 the analogous power condition can be put

$$\begin{aligned}(x,i) &= P_0 - P_1\\ (x,i^0) &= P_0 + P_2\end{aligned} \tag{121}$$

and the 'global minimal current condition'

$$(x,x) + (i-x,i-x) + (x-i^0,x-i^0) \to \min \tag{122}$$

or

$$0.5(x,x) - 1/3(i+i^0,x) \to \min \tag{123}$$

The suitable functional for tasks (121)-(123) has form:

$$f_{\lambda,\mu}(x) = 0.5(x,x) - 1/3(i+i^0,x) - \lambda(x,u) - \mu(x,u^0) \to \min \tag{124}$$

The minimization tasks (120) and (124) are solved by the following equations:

$$x = 1/3(u+u^0) + \lambda i + \mu u^0 \tag{125}$$

$$\begin{bmatrix}(i,i) & (i^0,i)\\ (i^0,i) & (i^0,i^0)\end{bmatrix}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} = \begin{bmatrix}\frac{2}{3}P_0 - P_1 - \frac{1}{3}(u^0,i)\\ \frac{2}{3}P_0 + P_2 - \frac{1}{3}(u,i^0)\end{bmatrix} \tag{126}$$

$$x = 1/3(i+i^0) + \lambda i + \mu i^0 \tag{127}$$

and

$$\begin{bmatrix}(u,u) & (u^0,u)\\ (u^0,u) & (u^0,u^0)\end{bmatrix}\begin{bmatrix}\lambda\\ \mu\end{bmatrix} = \begin{bmatrix}\frac{2}{3}P_0 - P_1 - \frac{1}{3}(u,i^0)\\ \frac{2}{3}P_0 + P_2 - \frac{1}{3}(u^0,i)\end{bmatrix} \tag{128}$$

We can see the similarity between the results (125–128) and those from the examples 9 and 10 in Section 6.

It is important to notice that the above described four terminal networks are globally passive. Thanks to that it is possible to supply its branches from the constant energy accumulators or capacitors by means of switching system resulting from the synthesis of procedures (125–128) [2,6,8].

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