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Invited paper

Stability of positive linear discrete-time systems

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Abstract. The main focus of the paper is on the asymptotic behaviour of linear discrete-time positive systems. Emphasis is on highlighting the relationship between asymptotic stability and the structure of the system, and to expose the relationship between null-controllability and asymptotic stability. Results are presented for both time-invariant and time-variant systems.

Key words: positive systems, equilibrium points, stability.

1. Introduction

This paper is concerned with the positive linear discrete-time systems (PLDS) represented by the homogeneous system

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) \tag{1}$$

and the non-homogeneous system

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t) \tag{2}$$

where $\boldsymbol{x}(t) \in \Re_+^n$ is the state vector, $\boldsymbol{A} \in \Re_+^{n\times n}$ is the transition matrix, $u(t) \in \Re_+$ is the scalar input control for which $\boldsymbol{b} \in \Re_+^n$ is the corresponding control vector, and $t = 0, 1, 2, \ldots$ denotes the time index. In the case of single input–single output PDLS equation (2) is coupled with the output equation

$$y(t) = \boldsymbol{c}^T\boldsymbol{x}(t) \tag{3}$$

where $y(t) \in \Re_+$ is the output and $\boldsymbol{c} \in \Re_+^n$.

A number of models with positive linear system behaviour can be found in engineering, management science, economics, social sciences, compartmental analysis in biology and medicine, genetics and other areas (see [1–7] and references cited there) so (1) and (2) seem to represent important classes of systems. As indicated by Luenberger [5], the theory of positive systems is deep and elegant and simply the knowledge that a system is positive allows one to make some fairly strong statements about its behaviour; these statements being true no matter what values the parameters may happen to take. The most fundamental property of positive systems, resulting from the Perron-Frobenius theorems formulated at the beginning of the 20^{th} century (see, for example, [8]), relates to the existence of a dominant eigenvalue and its associated eigenvector (which determines the system's long term behaviour). Another important feature of positive systems is the relationship between stability and positivity and it is this that is the main focus of this paper. The two texts [9] and [10] provide a good introduction to the theory of positive systems.

The paper has been structured to be as self-contained as possible with emphasis on the bringing together of key results, with references to various sources for associated proofs. First some relevant background material is provided in Section 2. Section 3 deals with equilibrium points whilst Section 4 considers the asymptotic stability problem, with the relationship between null-controllability and asymptotic stability being developed. The important problem of localizing the value of the dominant eigenvalue is considered in Section 5, whilst Section 6 highlights the importance of the sub-dominant eigenvalue when considering system convergence. Section 7 extends the discussion to time-variant positive systems.

2. Preliminaries

In this section some of the relevant basic definitions and results associated with positive matrices and the PLDS given in (1) and (2) are presented.

Definition 1. A non-zero matrix $\boldsymbol{A}$ with real elements a_{ij} is called

(1) non-negative (notation $\boldsymbol{A} \geq 0$) if all the elements of $\boldsymbol{A}$ are non-negative ($a_{ij} \geq 0$ for all i, j);
(2) positive (notation $\boldsymbol{A} > 0$) if all the elements of $\boldsymbol{A}$ are positive ($a_{ij} > 0$ for all i, j).

Similar definitions and notations apply to vectors.

A common property of positive systems is that if the initial condition $\boldsymbol{x}(0)$ is positive (or at least non-negative) then the whole trajectory is entirely in the non-negative orthant $\Re_+^n$. Many of the structural properties of PDLS relate to the zero-nonzero pattern of the elements in the system matrix $\boldsymbol{A}$, and are not dependent on the actual values of its non-zero entries. Consequently in the study

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of such a system frequent use is made of the incidence matrix [11] (or associate matrix [10]).

Definition 2. The matrix $\tilde{\boldsymbol{A}} = (\tilde{a}_{ij})$ is called the incidence matrix of $\boldsymbol{A} = (a_{ij})$ if
$\tilde{a}_{ij} = 1$ if $a_{ij} > 0$ and $\tilde{a}_{ij} = 0$ if $a_{ij} = 0$, for $i, j = 1, 2, \ldots, n$.

Clearly $\tilde{\boldsymbol{A}}$ is a common incidence matrix for all non-negative matrices having the same zero-nonzero pattern as $\boldsymbol{A}$.

The structure of the system matrix $\boldsymbol{A} = [a_{ij}]_{n\times n}$ can be represented by a digraph (directed graph) $G(\boldsymbol{A}) = (N, \mathcal{A})$, where $N = \{1, 2, \ldots, n\}$ is the set of all vertices (or nodes) and $\mathcal{A}$ is the set of arcs; that is, ordered pairs (i, j) of elements taken from N. The vertices correspond to the rows and columns of $\boldsymbol{A}$, and there is an arc $(i, j) \in \mathcal{A}, i, j \in N$, if and only if $a_{ji} > 0$ (that is, arcs correspond to non-zero entries and indicate that the state variable $\boldsymbol{x}_i(t)$ influences the state variable $\boldsymbol{x}_j(t+1)$. Let $i_1,\ i_2, \ldots, i_s$ be a sequence of distinct vertices of a directed graph such that (i_k, i_{k+1}) is an arc, for each $k = 1, \ldots, s-1$. Then the sequence of distinct vertices and arcs $i_1,\ (i_1,\ i_2),\ i_2, \ldots, (i_{s-1}, i_s),\ i_s$ is called a directed path which leads from i_1 to i_s. A digraph is called strongly connected if for any two vertices i and $j, i, j \in N$, there exists a directed path that leads from i to j. For PDLS system (2) the positive elements of the vector $\boldsymbol{b}$ are identified with the corresponding vertices of $G(\boldsymbol{A})$ and an additional vertex labelled 0 is introduced to represent the scalar input. If the output equation (3) is also associated with (2) then a further vertex labelled $n+1$ is introduced. Note that, according to the definition given above, the digraphs of any two $n \times n$ non-negative matrices $\boldsymbol{A}$ and $\boldsymbol{A}_1$ which have the same zero-nonzero pattern are the same, i.e. $G(\boldsymbol{A}) = G(\boldsymbol{A}_1) = (N, \mathcal{A})$. (Note: This digraph representation of the system is sometimes referred to as an influence graph [10]).

Definition 3. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the matrix $\boldsymbol{A} \in \Re^{n\times n}$ than $\sigma(\boldsymbol{A}) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is called the spectrum of $\boldsymbol{A}$. If

$$\rho(\boldsymbol{A}) = \max_i\{|\lambda_i|, \lambda_i \in \sigma(\boldsymbol{A})\}$$

then $\rho(\boldsymbol{A})$ is called the spectral radius (dominant or maximal eigenvalue) of the matrix $\boldsymbol{A}$

Definition 4. A non-negative matrix $\boldsymbol{A}$ is called reducible if there exists a permutation matrix $\boldsymbol{S}$ such that

$$\boldsymbol{S}^T\boldsymbol{A}\boldsymbol{S} = \begin{bmatrix} \boldsymbol{A}_{11} & 0 \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix} \tag{4}$$

where $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ are square matrices and $\boldsymbol{0}$ is the zero matrix. Otherwise $\boldsymbol{A}$ is called irreducible. The corresponding systems (1) and (2) are called reducible and irreducible systems respectively.

It is possible that either or both of the matrices $\boldsymbol{A}_{11}$ and $\boldsymbol{A}_{22}$ in (4) are themselves reducible, so that further decomposition can be performed. Indeed, if $\boldsymbol{A}$ is a non-negative reducible matrix then there exists a permutation matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}^T\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} \boldsymbol{A}_{11} & 0 & \cdots & 0 \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ \boldsymbol{A}_{k1} & \boldsymbol{A}_{k2} & \cdots & \boldsymbol{A}_{kk} \end{bmatrix}, \boldsymbol{A}_{ii} \in \Re_+^{n_i\times n_i}, \sum_{i=1}^{k} \boldsymbol{n}_i = \boldsymbol{n}$$

where each block $\boldsymbol{A}_{ii}, i = 1, 2, \ldots, k$, is square and either irreducible or a 1×1 matrix. Consequently, when considering non-negative systems attention may be focused on irreducible systems, which enjoy interesting properties. Tests for establishing irreducibility are given in Theorem 1.

Theorem 1. The non-negative matrix $\boldsymbol{A}$, of order n, is irreducible if and only if

(a) its digraph $G(\boldsymbol{A})$ is strongly connected, or equivalently,
(b) the matrix $(\boldsymbol{I} + \boldsymbol{A})^{n-1}$ is positive; that is

$$(\boldsymbol{I} + \boldsymbol{A})^{n-1} = \boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^2 + \ldots + \boldsymbol{A}^{n-1} > 0$$

Theorem 1 also holds if the incidence matrix $\tilde{\boldsymbol{A}}$ replaces the matrix $\boldsymbol{A}$ in the expansion for the sum. It readily follows from the theorem that if $\boldsymbol{A} > 0$ then it is irreducible.

Irreducible non-negative matrices can be further subdivided into primitive or cyclic (called imprimitive by some authors). The following simple definitions of cyclic and primitive non-negative matrices, based on their spectra, can be found in [12].

Definition 5. Let $\boldsymbol{A} \geq 0$ be irreducible. The number h of eigenvalues of $\boldsymbol{A}$ of modulus $\rho(\boldsymbol{A})$ is called the index of cyclicity (imprimitivity index) of $\boldsymbol{A}$. If $h = 1$ then the matrix $\boldsymbol{A}$ is termed primitive. If $h > 1$ then the matrix $\boldsymbol{A}$ is said to be cyclic of index h.

Definition 5 is itself a test for classifying an irreducible non-negative matrix as primitive or cyclic. Other characterizations of primitivity are given in Theorem 2.

Theorem 2. An irreducible non-negative matrix $\boldsymbol{A} \in \Re_+^{n\times n}$ is primitive if and only if

(a) there exists a positive integer m such that $\boldsymbol{A}^m > 0$ or, equivalently,
(b) there exist paths of the same length m between each pair of vertices in the digraph $G(\boldsymbol{A})$.

It readily follows from Theorem 2 that the class of real positive matrices $\boldsymbol{A} > 0$ is primitive; that is $h = 1$ or, in other words, the moduli of all eigenvalues are strictly less than the dominant eigenvalue. The proof of Theorem 3 below can be found in [11].

Theorem 3. Let

$$\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = \lambda^n + a_1\lambda^{n_1} + a_2\lambda^{n_2} + \ldots + a_q\lambda^{n_q}$$

where $n > n_1 > n_2 > \ldots > n_q \geq 0$ and $a_j \neq 0$ for all j, be the characteristic polynomial of an irreducible matrix $\boldsymbol{A} \geq 0$ which is cyclic of index h. Then h is the greatest common divisor of the differences $n - n_j$, $j = 1, 2, \ldots, q$.

The result given in Theorem 4 is due to Wieland [13].

THEOREM 4. For any two nonnegative matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ such that $\boldsymbol{A} \geq \boldsymbol{B} \geq 0$ the corresponding dominant eigenvalues $\rho(\boldsymbol{A})$ and $\rho(\boldsymbol{B})$ satisfy the inequality $\rho(\boldsymbol{A}) \geq \rho(\boldsymbol{B})$. Moreover, if the matrix $\boldsymbol{A}$ is irreducible but $\boldsymbol{A} \neq \boldsymbol{B}$, this inequality holds as a strong inequality $\rho(\boldsymbol{A}) > \rho(\boldsymbol{B})$.

The Perron–Frobenius theorems [8] are key to the study of positive systems. The basic results associated with these theorems, relevant to this paper, are summarized in Theorem 5.

THEOREM 5. (1) If a matrix $\boldsymbol{A} \in \Re_+^{n\times n}$ is positive then:

- (a) there exists a real positive dominant eigenvalue $\rho(\boldsymbol{A})$ and a corresponding positive right eigenvector $\boldsymbol{f}_\rho$; that is, $\boldsymbol{A}\boldsymbol{f}_\rho = \rho(\boldsymbol{A}) f_\rho, \rho(\boldsymbol{A}) > 0, \boldsymbol{f}_\rho > 0$. ($\rho(\boldsymbol{A})$ and $\boldsymbol{f}_\rho$ are frequently referred to as the Frobenius eigenvalue and eigenvector repectively);
- (b) if $\lambda \neq \rho(\boldsymbol{A})$ is another eigenvalue of $\boldsymbol{A}$ then $|\lambda| < \rho(\boldsymbol{A})$; that is $\rho(\boldsymbol{A})$ is unique ;
- (c) $\rho(\boldsymbol{A})$ is an eigenvalue of algebraic and geometric multiplicity 1.

(2) A non-negative matrix $\boldsymbol{A} \in \Re_+^{n\times n}$ always has a real non-negative dominant eigenvalue $\rho(\boldsymbol{A}) \geq 0$ and a corresponding non-negative eigenvalue $\boldsymbol{f}_\rho \geq 0$ such that the moduli of all the eigenvalues of $\boldsymbol{A}$ are not greater than $\rho(\boldsymbol{A})$ and $\boldsymbol{f}_\rho \neq 0$.

(3) If the non-negative matrix $\boldsymbol{A}$ is also irreducible then

- (a) the dominant eigenvalue $\rho(\boldsymbol{A})$ and the corresponding eigenvector $\boldsymbol{f}_\rho$ are both positive, that is $\rho(\boldsymbol{A}) > 0$ and $\boldsymbol{f}_\rho > 0$;
- (b) if $\lambda \neq \rho(\boldsymbol{A})$ is another eigenvalue of $\boldsymbol{A}$ then $|\lambda| \leq \rho(\boldsymbol{A})$;
- (c) $\rho(\boldsymbol{A})$ is an eigenvalue of algebraic multiplicity 1.

(4) If the non-negative matrix $\boldsymbol{A}$ is irreducible and cyclic with cyclicity index $h > 1$, then there are h and only h, eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_h (h \geq 1)$ of $\boldsymbol{A}$ such that $|\lambda_j| = \rho(\boldsymbol{A})$, $j = 1, 2, \ldots, h$. In this case λ_j, $j = 1, 2, \ldots, h$, are the roots of the equation $\lambda^h - \rho^h = 0$ (that is, the eigenvalues are regularly distributed on the circle of radius $\rho(\boldsymbol{A})$).

(5) If the non-negative $\boldsymbol{A} \in \Re_+^{n\times n}$ is irreducible and primitive then $h = 1$ and the dominant eigenvalue $\rho(\boldsymbol{A})$ is unique.

DEFINITION 6. The PDLS (2), and the non-negative pair $(\boldsymbol{A}, \boldsymbol{b}) \geq 0$, is said to be null-controllable (or controllable to the origin) [14] if for any state $\boldsymbol{x}(t) \in \Re_+^n$ and some finite t there exists a non-negative control sequence $\{u(s), s = 0, 1, \ldots, t-1\}$ that transfers the system from the state $\boldsymbol{x} = \boldsymbol{x}(0)$ into the origin. Since the transition time is finite this is sometimes referred as finite time null-controllability.

Muratori and Rinaldi [15] introduce the classes of PDLS called excitable and transparent systems. These have specific structural properties, which will be exploited in Section 4 when considering the positivity of equilibrium points.

DEFINITION 7. The PLDS system (2) is said to be excitable if and only if each state variable can be made positive by applying an appropriate control $u^* \geq 0$ to the system initially at rest.

DEFINITION 8. The PDLS given in (2) with (3) is said to be transparent if and only if its free output y (that is, output with $u(t) = 0$) is positive for every $\boldsymbol{x}_0 > 0$.

Conditions for excitability and transparency are also developed in [15] and are summarized in Theorem 6. These are quite weak properties that are usually met in applications.

THEOREM 6. (a) The PDLS (2) is excitable if and only if there exists at least one path from the input node 0 to each node $i = 1, 2 \ldots n$ of the digraph $G(\boldsymbol{A})$, or equivalently, if and only if

$$\tilde{\boldsymbol{b}} + \tilde{\boldsymbol{A}}^T\tilde{\boldsymbol{b}} + (\tilde{\boldsymbol{A}}^T)^2\tilde{\boldsymbol{b}} + \ldots + (\tilde{\boldsymbol{A}}^T)^{n-1}\tilde{\boldsymbol{b}} > 0$$

(b) The PDLS given by (2) and (3) is transparent if and only if there exists at least one path from each node $i = 1, 2 \ldots n$ of the digraph $G(\boldsymbol{A})$ to the output node $n+1$ or equivalently, if and only if

$$\tilde{\boldsymbol{c}} + \tilde{\boldsymbol{A}}\tilde{\boldsymbol{c}} + \tilde{\boldsymbol{A}}^2\tilde{\boldsymbol{c}} + \ldots + \tilde{\boldsymbol{A}}^{n-1}\tilde{\boldsymbol{c}} > 0$$

DEFINITION 9. A square matrix $\boldsymbol{C} \in \Re^{n\times n}$ is an M-matrix if there exists a non-negative matrix $\boldsymbol{A} \in \Re_+^{n\times n}$ with dominant eigenvalue $\rho(\boldsymbol{A}) > 0$ such that $\boldsymbol{C} = c\boldsymbol{I} - \boldsymbol{A}$, where $c \geq \rho(\boldsymbol{A})$ and $\boldsymbol{I}$ is the $n \times n$ identity matrix.

Clearly an M-matrix is non-singular if $c > \rho(\boldsymbol{A})$ and singular if $c = \rho(\boldsymbol{A})$. Properties of non-singular M-matrices that are useful for considering the positivity of equilibrium points are contained in Theorem 7.

THEOREM 7. Let $\boldsymbol{C} = c\boldsymbol{I} - \boldsymbol{A}$ with $\boldsymbol{A} \geq 0$ and $c > 0$, then

- (a) $\boldsymbol{C}$ is non-singular and $\boldsymbol{C}^{-1} \geq 0$ if and only if $c > \rho(\boldsymbol{A})$
- (b) $\boldsymbol{C}$ is non-singular and $\boldsymbol{C}^{-1} > 0$ if and only if $c > \rho(\boldsymbol{A})$ and $\boldsymbol{A}$ is irreducible.

3. Equilibrium points

A point $\boldsymbol{x}_e$ is called an equilibrium point of the PDLS (2), subject to a constant input $u^* \geq 0$, if and only if it satisfies the condition

$$\boldsymbol{x}_e = \boldsymbol{A}\boldsymbol{x}_e + \boldsymbol{b}u^* \tag{5}$$

or equivalently

$$(\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x}_e = \boldsymbol{b}u^* \tag{6}$$

If unity is not an eigenvalue of $\boldsymbol{A}$ then $\boldsymbol{I} - \boldsymbol{A}$ is non-singular and there is a unique solution

$$\boldsymbol{x}_e = [\boldsymbol{I} - \boldsymbol{A}]^{-1}\boldsymbol{b}u^* \tag{7}$$

If unity is an eigenvalue of $\boldsymbol{A}$ then there may be no equilibrium point or an infinity of such points depending on whether or not (6) represent a consistent set of equations. In most cases of interest there is a unique equilibrium point given by (7). In the case of the homogeneous PLDS (1) $\boldsymbol{x}_e$ is an equilibrium point if and only if it satisfies the condition

$$\boldsymbol{x}_e = \boldsymbol{A}\boldsymbol{x}_e.$$

Clearly if unity is not an eigenvalue of $\boldsymbol{A}$ the origin ($\boldsymbol{x}_e = 0$) is the only equilibrium point of the system. If unity is an eigenvalue of $\boldsymbol{A}$, with algebraic multiplicity equal to geometric multiplicity, then any point on the ray generated by the corresponding eigenvector is an equilibrium point.

However, the equilibrium point $\boldsymbol{x}_e$ may not in itself be a non-negative vector, which is a desirable requirement in practice. It follows from Theorem 7 that if $\rho(\boldsymbol{A}) < 1$ then $(\boldsymbol{I} - \boldsymbol{A})$ is a non-singular M-matrix having a non-negative inverse so $(\boldsymbol{I} - \boldsymbol{A})^{-1} \geq 0$. Since $\boldsymbol{b} \geq 0$ and $u^* \geq 0$ it follows from (7) that in this case $\boldsymbol{x}_e \geq 0$. If $\rho(\boldsymbol{A}) = 1$ then $(\boldsymbol{I} - \boldsymbol{A})$ is a singular M-matrix which is generalized left inverse-positive; that is, there exists a non-negative matrix $\boldsymbol{S} \geq 0$ such that $\boldsymbol{S}(\boldsymbol{I} - \boldsymbol{A})^{k+1} = (\boldsymbol{I} - \boldsymbol{A})^k$ for some $k \geq I$ [12]. Moreover, it can be proved that every generalized inverse $\boldsymbol{S}$ of $\boldsymbol{C} = (\boldsymbol{I} - \boldsymbol{A})$ is non-negative on $V_C = \bigcap_{m=0}^{\infty} \Omega(\boldsymbol{C}^m)$, were $\Omega(\boldsymbol{T})$ is the range or column space of $\boldsymbol{T}$; that is $\boldsymbol{x} \geq 0$ and $\boldsymbol{x} \in V_C \rightarrow \boldsymbol{x} \geq 0$.

4. Asymptotic stability

The homogeneous PLDS (1) may be solved recursively, once an initial state $\boldsymbol{x}_0$ has been specified, to give

$$\boldsymbol{x}(t) = \boldsymbol{A}^t\boldsymbol{x}_0, \quad t = 0, 1, 2, \ldots \tag{8}$$

Definition 10. The homogeneous PLDS (1) is called asymptotically stable (a.s) if and only if the solution in equation (8) satisfies the condition

$$\lim_{t\to\infty} \boldsymbol{x}(t) = 0 \quad \text{for any} \quad \boldsymbol{x}_0 \in \Re^n_+$$

The system is said to be unstable if there exists an $\boldsymbol{x}_0$ such that $\underset{t\to\infty}{lim}\ \boldsymbol{A}^t\boldsymbol{x}_0$ is unbounded and marginally stable otherwise.

A condition for a.s is given in Theorem 8.

Theorem 8. The homogeneous PDLS (1) is a.s. if and only if the dominant eigenvalue $\rho(\boldsymbol{A}) < 1$. If $\rho(\boldsymbol{A}) > 1$ then the system is unstable and if $\rho(\boldsymbol{A}) = 1, (h = 1)$ then the system is said to be marginally stable.

If $\boldsymbol{x}_e$ as specified in (7) is the equilibrium point of the non-homogeneous PDLS (2), subject to a constant input u^* then denoting $\boldsymbol{z}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_e$ enables (2) to be reduced to the homogeneous form

$$\boldsymbol{z}(t+1) = \boldsymbol{A}\boldsymbol{z}(t) \tag{9}$$

The perturbed system (9) is not necessarily positive since $\boldsymbol{z}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_e$ can be greater, equal or less than zero. However the matrix $\boldsymbol{A}$ is non-negative since system (2) is positive and this simplifies the stability analysis.

Clearly the condition for $\lim_{t\to\infty} \boldsymbol{x}(t) = \boldsymbol{x}_e$ is identical to that for $\lim_{t\to\infty} \boldsymbol{z}(t) = 0$ in the homogeneous equation. Thus the conditions for the non-homogeneous PDLS (2) to be a.s are as given in Theorem 8. It is clear from Theorem 8 that the conditions for a.s of general linear discrete-time systems hold for PDLS. However, many of the necessary and/or sufficient conditions for a.s of general discrete-time linear systems become simpler in the case of PDLS. Examples are given in Theorem 9, and are proved in references [9,10].

Theorem 9. (1) The PLDS (1) is a.s if and only if either

(a) the coefficients of the characteristic polynomial

$$\Delta_{\boldsymbol{A}-\boldsymbol{I}}(\lambda) = \det[\boldsymbol{I}\lambda - \boldsymbol{A} + \boldsymbol{I}]$$

are positive, or

(b) all the leading principal minors of the matrix $(\boldsymbol{I} - \boldsymbol{A})$ are positive.

(2) The PDLS (1) is unstable if at least one diagonal entry of the matrix $\boldsymbol{A}$ is greater than 1, that is

$$a_{ii} > 1 \quad \text{for some} \quad i \in \{1, 2, \ldots, n\}$$

It is also of interest to expose the relation between null-controllability and a.s. It is shown in [14] that the PLDS is null-controllable (in finite time) if and only if the system matrix $\boldsymbol{A}$ is nil-potent (that is, $\boldsymbol{A}^s = 0$ for some integer $s \leq n$). A nil-potent matrix has all of its eigenvalues equal to zero so (finite time) null-controllability implies a.s. The converse is not true for PLDS, since a.s. does not imply finite time null-controllability. The concept of null-controllability can be extended to include infinite transition time to the origin [14]. Positive systems in which the trajectory of free motion converges to the origin as $t \rightarrow \infty$ are called weakly (asymptotically) null-controllable. Thus, it is clear that a.s. implies weak null-controllability.

Consider the open-loop PLDS (2) and the associated closed-loop system

$$\boldsymbol{x}(t+1) = \boldsymbol{A}_c\boldsymbol{x}(t) \tag{10}$$

where $\boldsymbol{A}_c = \boldsymbol{A} + \boldsymbol{b}\boldsymbol{k}$ is the closed-loop system matrix, $\boldsymbol{k} = (k_1, \ldots, k_n)$, is the constant feedback gain row-vector, and the linear state-feedback law is

$$u(t) = \boldsymbol{k}\boldsymbol{x}(t)$$

The closed-loop system (10) is positive if and only if $\boldsymbol{A}_c \geq 0$. The result contained in Theorem 10 is proved in [16].

THEOREM 10. The controls in the closed-loop PLDS (10) satisfy the constraints

$$u(t) \geq 0,\ t = 0, 1, 2, \dots \tag{11}$$

if and only if the feedback gains satisfy the condition

$$\boldsymbol{k} = (k_1, \dots, k_n) \geq 0 \tag{12}$$

independently of whether the open-loop system is positive or non-positive.

The negative but rather important result given in Theorem 11 is a direct consequence of Theorems 4 and 10.

THEOREM 11. An unstable PLDS cannot be stabilised by linear state-feedback if the restriction on nonnegativity of controls (11) in the closed loop is to be respected.

If the non-homogeneous PDLS (2) is a.s then $\rho(\boldsymbol{A}) < 1$ and it follows from Section 3 that the equilibrium point $\boldsymbol{x}_e$ determined by (7), is non-negative. The converse is also true for if there exists an $\boldsymbol{x}_e \geq 0$ satisfying (7) then the PLDS (2) is a.s.. No similar statement holds for general linear time-invariant systems. In many applications a question of interest is whether or not the state of the system tends towards a non-negative ($\boldsymbol{x}_e \geq 0$) or positive ($\boldsymbol{x}_e > 0$) equilibrium point $\boldsymbol{x}_e$, determined by (7), when a constant input $u^* > 0$ is applied. For a general PLDS the best response that can be given is that stated in Theorem 12.

THEOREM 12. The equilibrium point $\boldsymbol{x}_e$ of an a.s PLDS (2), with a constant input $u^* > 0$, is non-negative if $\boldsymbol{b} \geq 0$ and positive if $\boldsymbol{b} > 0$.

As shown in [9] the statement in Theorem 12 can be strengthened if the PDLS is excitable, as indicated in Theorem 13.

THEOREM 13. The equilibrium point $\boldsymbol{x}_e$ of an a.s and excitable PLDS (2), with a constant input $u^* > 0$, is positive.

In the case of excitable systems (as in Theorem 13) asymptotic stability is a necessary and sufficient condition for $\boldsymbol{x}_e > 0$.

We conclude this section by noting that the asymptotic behaviour of PLDS (1) and (2) can also be ascertained using Liapunov theory. First we recall that a general discrete-time linear system, having the structure of (1) but not necessary positive, is a.s if and only if there exists a positive definite matrix $\boldsymbol{P}$ such that $\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}$ is negative definite. That is, if and only if there exist positive definite matrices $\boldsymbol{P}$ and $\boldsymbol{Q}$ which satisfy the Liapunov equation

$$\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} - \boldsymbol{P} = -\boldsymbol{Q}$$

In the case of PDLS one can restrict attention to $\boldsymbol{P}$ being a diagonal matrix with all its diagonal elements positive (that is, $\boldsymbol{P}$ is a positive diagonal matrix) [9]. Then the condition for a.s is as given in Theorem 14.

THEOREM 14. A PDLS (1) or (2) is a.s if and only if there exists a positive diagonal matrix $\boldsymbol{P}$ such that the matrix $\boldsymbol{A}^T\boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}$ is negative definite.

5. Bounds on dominant eigenvalue

An obvious important use of the Perron-Frobenious results, summarized in Theorem 5, in stability analysis arises from the guarantee that the transition matrix $\boldsymbol{A}$ of a PDLS always has a non-negative dominant eigenvalue $\rho(\boldsymbol{A})$. Clearly, it follows from Theorem 8 that the PLDS (1) and (2) are a.s if and only if $\rho(\boldsymbol{A}) < 1$. Localizing the value of $\rho(\boldsymbol{A})$ is very important in practice, both for theoretical work and computation, where iterative processes require initial estimates of its value. Having tight theoretical bounds on the value of $\rho(\boldsymbol{A})$ can also be of value in the design of positive dynamical systems.

The most frequently used bounds for the dominant eigenvalue $\rho(\boldsymbol{A})$ of a non-negative matrix $\boldsymbol{A} = (a_{ij})$ are those due to Frobenius [8-10]. Denote by

$$r_i = \sum_{j=1}^{n} a_{ij} \quad \text{and} \quad c_j = \sum_{i=1}^{n} a_{ij}$$

the sum of the i^{th} row and j^{th} column, $i, j = 1, 2, \dots, n$, respectively of $\boldsymbol{A}$ Then the following evaluations for $\rho(\boldsymbol{A})$ hold

$$\min_i r_i \leq \rho(\boldsymbol{A}) \leq \max_i r_i \tag{13a}$$

$$\min_j c_j \leq \rho(\boldsymbol{A}) \leq \max_j c_j \tag{13b}$$

or, more compactly,

$$\max\{\min_i r_i, \min_j c_j\} \leq \rho(\boldsymbol{A}) \leq \min\{\max_i r_i, \max_j c_j\} \tag{13c}$$

If $\boldsymbol{A}$ is also irreducible then equality holds on both sides in (13a) if and only if all the row sums are equal and holds in (13b) if and only if all the column sums are equal. Sharper bounds on the dominant eigenvalue are provided in Theorem 15.

THEOREM 15. Let $\boldsymbol{A}$ be a non-negative matrix with non- zero row sums $r_i, (i = 1, 2, \dots, n)$, and dominant eigenvalue $\rho(\boldsymbol{A})$. Then

$$\min_i \left(\frac{1}{r_i} \sum_{j=1}^{n} a_{ij} r_j \right) \leq \rho(\boldsymbol{A}) \leq \max_i \left(\frac{1}{r_i} \sum_{j=1}^{n} a_{ij} r_j \right)$$

If $\boldsymbol{A} = (a_{ij})$ is a positive matrix with dominant eigenvalue $\rho(\boldsymbol{A})$ and maximum row sum R and minimum row sum r and if $r < R$ then, from (13a), $r < \rho(\boldsymbol{A}) < R$. Lederman proposed the problem of determining positive numbers p_1 and p_2 such that

$$r + p_1 \leq \rho(\boldsymbol{A}) \leq R - p_2$$

His result, together with later improvements by first Ostrowsky and then Brauer [8], is given below, with $\eta = \min_{i,j} a_{ij}$.

Lederman:

$$r + \eta\left(\frac{1}{\sqrt{\delta}} - 1\right) \le \rho(\boldsymbol{A}) \le R - \eta(1-\sqrt{\delta}),$$
$$\text{where} \quad \delta = \max_{r_i \prec r_j}(r_i/r_j).$$

Ostrowsky:

$$r + \eta\left(\frac{1}{\sigma} - 1\right) \le \rho(\boldsymbol{A}) \le R - \eta(1-\sigma),$$
$$\text{where} \quad \sigma = \sqrt{(r-\eta)/(R-\eta)}.$$

Brauer:

$$r + \eta(h-1) \le \rho(\boldsymbol{A}) \le R - \eta(1-1/g),$$

where

$$g = \frac{R - 2\eta + \sqrt{R^2 - 4\eta(R-r)}}{2(r-\eta)}$$
$$\text{and} \quad h = \frac{-r + 2\eta + \sqrt{\rho^2 + 4\eta(R-r)}}{2\eta}$$

6. Subdominant eigenvalues and convergence

It is clear from Section 4 that the asymptotic behaviour, as $t \to \infty$, of PLDS (1) and (2) is dependent on the asymptotic behaviour of $\boldsymbol{A}^t$ as $t \to \infty$. Using the spectral representation

$$\boldsymbol{A}^t = \sum_{i=1}^{n} \lambda_i^t \boldsymbol{f}_i \boldsymbol{g}_i^T \tag{14}$$

where $\lambda_i, i = 1,2,\ldots,n$ are the eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{f}_i$ and $\boldsymbol{g}_i$ are the corresponding right and left eigenvectors respectively, defined by $\boldsymbol{A}\boldsymbol{f}_i = \lambda_i \boldsymbol{f}_i$ and $\boldsymbol{g}_i^T\boldsymbol{A} = \lambda_i \boldsymbol{g}_i^T$, $i = 1,2,\ldots,n$. These sets of eigenvectors are bi-orthogonal and can be assumed to be mutually normalized, so that $\boldsymbol{g}_i^T \boldsymbol{f}_i = \delta_{ij}$, where δ_{ij} is the Kronecker delta, $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \ne j$. If $\boldsymbol{A} > 0$, or if $\boldsymbol{A} \ge 0$ is primitive, then from Theorem 5 $\rho(\boldsymbol{A}) > 0$ and is unique so, taking $|\lambda_1| \le |\lambda_2| \le \ldots \le |\lambda_{n-1}| < \rho(\boldsymbol{A})$, (14) can be written in the form

$$\left(\rho(\boldsymbol{A})^{-1}\boldsymbol{A}\right)^t = \boldsymbol{f}_\rho \boldsymbol{g}_\rho^T + \left[\left(\frac{\lambda_1}{\rho(\boldsymbol{A})}\right)^t \boldsymbol{f}_1\boldsymbol{g}_1^T + \ldots + \left(\frac{\lambda_{n-1}}{\rho(\boldsymbol{A})}\right)^t \boldsymbol{f}_{n-1}\boldsymbol{g}_{n-1}^T\right]. \tag{15}$$

Thus

$$\lim_{t\to\infty}\left(\rho(\boldsymbol{A})^{-1}\boldsymbol{A}\right)^t = \boldsymbol{f}_\rho \boldsymbol{g}_\rho^T,\ \boldsymbol{f}_\rho > 0,$$
$$\boldsymbol{g}_\rho > 0 \quad \text{and} \quad \boldsymbol{f}_\rho^T \boldsymbol{g}_\rho = 1.$$

Defining the sub-dominant eigenvalue of a matrix $\boldsymbol{A} \in \Re^{n\times n}$ by

$$\rho\_(\boldsymbol{A}) = \max_i\{|\lambda_i|/\lambda_i \in \sigma(\boldsymbol{A}),\ |\lambda_i| \ne \rho(\boldsymbol{A})\}$$

then in (15) $|\lambda_{n-1}| = \rho\_(\boldsymbol{A})$ but it may not be unique. However it is clear that the asymptotic behaviour of $\boldsymbol{A}^t$ is determined by the rate of convergence to zero of $\lim_{t\to\infty}[\rho\_(\boldsymbol{A})/\rho(\boldsymbol{A})]^t$. Even if $\rho(\boldsymbol{A})$ is known or easily estimated it may be difficult or impossible to compute an estimate of the sub-dominant eigenvalue $\rho\_(\boldsymbol{A})$ in order to get an useful bound on the ratio $\rho\_(\boldsymbol{A})/\rho(\boldsymbol{A})$. In such cases the following easily computed bound, due to Hopf [17] and holds for any positive matrix $\boldsymbol{A} = (a_{ij})$, may be used.

$$\frac{\rho\_(\boldsymbol{A})}{\rho(\boldsymbol{A})} \le \frac{\Lambda - \mu}{\Lambda + \mu} < 1$$

where $\Lambda = \max(a_{ij} : i,j = 1,2,\ldots,n)$ and $\mu = \min(a_{ij} : i,j = 1,2,\ldots,n)$.

A number of other bounds have been proposed in the literature and references [18–21] are indicative.

7. Time variant plds

Consider the time-variant PLDS

$$\boldsymbol{x}(t+1) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{b}(t)u(t), \quad t = 0,1,2,\ldots \tag{16}$$

and the associated homogeneous system

$$\boldsymbol{x}(t+1) = \boldsymbol{A}(t)\boldsymbol{x}(t) \tag{17}$$

with $\boldsymbol{A}(t) \in \Re_+^{n\times n}$, $\boldsymbol{b}(t) \in \Re_+^n$ and $u(t) \in \Re_+$ for all t and an initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \Re_+^n$.

It is not difficult to see that the trajectory of the system (16), respectively (17), is non-negative. As was shown in Section 4 the stability properties of time-invariant systems are determined by the stability properties of the associated homogeneous system. The trajectory of homogeneous system for any initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0 \ge 0$ is given by

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\boldsymbol{x}_0, \quad t = 0,1,2,\ldots$$

where

$$\boldsymbol{\Phi}(t) = \boldsymbol{A}(t-1)\boldsymbol{A}(t-2)\ldots\boldsymbol{A}(1)\boldsymbol{A}(0) = \prod_{k=0}^{t-1}\boldsymbol{A}(k),$$
$$\text{with} \quad \boldsymbol{\Phi}(0) = \boldsymbol{I}$$

is the fundamental (transition) matrix of the system (16). Clearly, $\boldsymbol{\Phi}(t)$ is transitive so for any two positive integers s and t such that $s < t$

$$\boldsymbol{\Phi}(t) = \boldsymbol{\Phi}(t-s)\boldsymbol{\Phi}(s).$$

In line with Definition 7 the positive system (16) (and the non-negative pair $(\boldsymbol{A}(t), \boldsymbol{b}(t)) \ge 0$) is said to be null-controllable if for any state $\boldsymbol{x} \in \Re_+^n$ and some finite s there exists a non-negative control sequence $\{u(t), t = 0,1,2,\ldots,s-1\}$ that transfers the system from the state $\boldsymbol{x} = \boldsymbol{x}(0)$ to the origin $\boldsymbol{x}(s) = 0$. Likewise, the system (16) is said to be weakly null-controllable if $\boldsymbol{x}(t) \to 0$ as $t \to \infty$. It is not difficult to see that since the system (16) is positive the control sequence does not contribute to reaching the origin [14]. Thus, the null-controllability property, respectively weak null-controllability property, depends entirely on the properties of the system matrix $\boldsymbol{A}(t)$ and, consequently, on the properties of fundamental matrix $\boldsymbol{\Phi}(t)$. The result given in Theorem 16 is proved in [22].

THEOREM 16. The positive system (16) is null-controllable if and only if there exists a finite time $s \geq 0$ such that the transition matrix $\boldsymbol{\Phi}(s) = 0$.

Since $\boldsymbol{A}(t) \geq 0$ for all $t \geq 0$ and $\boldsymbol{\Phi}(s) = \boldsymbol{A}(s-1)$ $\boldsymbol{A}(s-2)$ ... $\boldsymbol{A}(1)\boldsymbol{A}(0)$ Theorem 16 tells us that null-controllability of the system (16) is an entirely structural property, which depends only on the zero-nonzero pattern of the system matrix $\boldsymbol{A}(t)$ or its incidence matrix $\tilde{\boldsymbol{A}}(t)$. The time needed to reach the origin can be less, equal or greater than the dimension of the system [22]. This phenomenon has no equivalent in the case of time-invariant positive linear systems, where the time of reaching the origin is always less than or equal to the dimension of the system [10]. It readily follows from the transitivity property of $\boldsymbol{\Phi}(t)$ that if the system reaches the origin it can stay there forever, since $\boldsymbol{\Phi}(t) = \boldsymbol{\Phi}(t-s)\boldsymbol{\Phi}(s) = 0$ for any $t \geq s$ when $\boldsymbol{\Phi}(s) = 0$. That is, the time-variant PLDS is null-controllable. Thus, null-controllability implies asymptotic stability.

Let $D(\boldsymbol{A}) = (N, U)$ be a digraph of an $n \times n$ matrix $\boldsymbol{A} \geq 0$, where $N = \{1, 2, \ldots n\}$ is the set of all vertices and U is the set of all arcs. The digraph being constructed so that there is an arc $(i,j) \in U$, $(i,j) \in N$, if and only if $a_{ij} > 0$. To associate a matrix $\boldsymbol{A} \geq 0$ with this digraph the adjacency matrix $\bar{\boldsymbol{A}}$ is introduced [17]. The entries of $\bar{\boldsymbol{A}}$ being defined as

$$\tilde{a}_{ij} = \begin{cases} 1 \text{ if } (i,j) \in D(\boldsymbol{A}) \\ 0 \text{ if } (i,j) \notin D(\boldsymbol{A}) \end{cases}$$

The matrix $\bar{\boldsymbol{A}}$ is a binary matrix. It is clear that $\bar{\boldsymbol{A}} \geq 0$ and $D(\bar{\boldsymbol{A}}) = D(\boldsymbol{A})$, where $\boldsymbol{A}$ is any non-negative matrix having the same zero-nonzero pattern as $\bar{\boldsymbol{A}}$.(Note that the adjacency matrix of the digraph $D(\boldsymbol{A})$ has the same 0-1 pattern as the incidence matrix of $(\boldsymbol{A})$ given in Definition 2. Note also that the adjacency matrix of $G(\boldsymbol{A})$, introduced in Section 2, is the transpose of the adjacency matrix of $D(\boldsymbol{A})$ and vice-versa, so that $G(\boldsymbol{A})$ and $D(\boldsymbol{A})$ have different structures).

Let $D_1 = (N_1, U_1)$ and $D_2 = (N_2, U_2)$ be any two digraphs then the operation union $D_1 \cup D_2$ produces the digraph $(N_1 \cup N_2, U_1 \cup U_2)$. Thus, if the vertex sets are the same then the union of two digraphs is just the superposition of their arcs. Given m $n \times n$ nonnegative matrices $\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_m$, the joint digraph $D(\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_m)$ is defined as $D(\boldsymbol{A}_1) \cup D(\boldsymbol{A}_2) \cup \ldots \cup D(\boldsymbol{A}_m)$, in which each arc is labelled (coloured) with a subset of $\{1, 2, \ldots, m\}$ depending upon which of the digraphs $D(\boldsymbol{A}_1)$, $D(\boldsymbol{A}_2), \ldots, D(\boldsymbol{A}_m)$ includes that arc. A simple characterisation of null-controllability in terms of the joint digraph, and presented in Theorem 17, is developed in [22].

THEOREM 17. The system (16) is null-controllable if and only if for some finite $s \geq 0$ there exists no path of length s in the joint diagraph $D(\boldsymbol{A}(s-1), \ldots, \boldsymbol{A}(0))$ coloured with $1, 2, \ldots, s$ in the order of the matrices in the product from the left to the right.

For many reasons, it is important to determine classes of time-variant PDLS that are null-controllable and the time of reaching the origin is less than or at most equal to the dimension of the system. Theorems 18 and 19, proved in [22], define such classes of systems.

THEOREM 18. Let the adjacency matrix $\bar{\boldsymbol{A}}(t) = \bar{\boldsymbol{A}} \geq 0$ be a constant matrix for $0 < j \leq t < j + n$ and let $\bar{\boldsymbol{A}}$ be a nil-potent matrix. Then the positive system (16) is null-controllable and the time t of reaching the origin is $t < j + n$. In particular, if $j = 0$ the origin is reachable in $t \leq n$ steps.

THEOREM 19. Let $\bar{\boldsymbol{A}}(t) \geq 0$ be a nil-potent matrix for $k = 0, 1, 2, \ldots, n-1$ and let $\bar{\boldsymbol{A}}(t) \leq \bar{\boldsymbol{A}}(t+1)$ for at least $t = 0, 1, 2, \ldots, n-2$. Then the positive system is null-controllable and the origin can be reached in $s \leq n$ steps.

Since null-controllability implies asymptotic stability the characterisations of null-controllability provided in Theorems 18 – 19 can be used as tests for asymptotic stability of time invariant PLDS. Consider the (homogeneous) time-invariant PLDS

$$\boldsymbol{z}(t+1) = \boldsymbol{P}\boldsymbol{z}(t) \text{ with } z(0) = \boldsymbol{x}_0 \geq 0 \text{ and } t = 0, 1, 2, \ldots \tag{18}$$

having the following properties

(i) $\boldsymbol{P} \geq \boldsymbol{A}(t) \geq 0$ for $t = 0, 1, 2, \ldots$, and

(ii) $\rho(\boldsymbol{P}) < 1$ (that is, $\boldsymbol{P}$ is an asymptotically stable non-negative matrix).

As a consequence of property (ii), the trajectory of (18)

$$\boldsymbol{z}(t) = \boldsymbol{P}^t \boldsymbol{x}_0 \geq 0$$

is convergent to the origin as t approaches infinity, that is

$$\boldsymbol{z}(t) \to 0 \quad \text{as} \quad t \to \infty \tag{19}$$

so the time-invariant PLDS (16) corresponding to (18) is weakly null-controllable. On the other hand, as a consequence of property (i), the trajectory of the system (18) dominates the trajectory of the system (17); that is

$$0 \leq \boldsymbol{x}(t) \leq \boldsymbol{z}(t) \quad \text{for any} \quad t. \tag{20}$$

It readily follows from (19) and (20) that

$$\boldsymbol{x}(t) \to 0 \quad \text{as} \quad t \to \infty,$$

and, therefore, the system (17), respectively (16), is asymptotically stable.

8. Conclusions

The dominant eigenvalue $\rho(\boldsymbol{A})$ of the system matrix $\boldsymbol{A} \geq 0$ determines entirely the stability property of PLDS namely the system is asymptotically stable if $\rho(\boldsymbol{A}) < 1$, marginally stable if $\rho(\boldsymbol{A}) = 1$ and unstable when $\rho(\boldsymbol{A}) > 1$. Since there are a variety of estimates for the dominant eigenvalue, this eigenvalue being real and positive for positive matrices and real and nonnegative for nonnegative matrices can be easily evaluated.

Not any equilibrium point of a non-homogeneous PLDS is nonnegative but the nonnegative equilibrium points are of interest for the applications. As it is shown in Section 4 there is a remarkable connection between asymptotic stability and the non-negativity (positivity) of the equilibria; namely: the equilibrium points of an asymptotically stable PLDS are non-negative and if there exists a non-negative equilibrium point for the PLDS then it is asymptotically stable.

For PLDS the relationship between asymptotic stability and null-controllability is quite appealing. It is shown in the paper that (positive) null-controllability implies asymptotic stability whilst asymptotic stability implies weak null-controllability, in general. These results hold for time-invariant as well as for time-variant PLDS.

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