BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES
Vol. 52, No. 1, 2004

Reachability index of the positive 2D general models

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Abstract. It is shown that $2(n+1)$ is the upper bound for the reachability index of the n-order positive 2D general models.

Keywords: reachability index, positive 2D general model, upper bound.

1. Introduction

In recent years a growing interest in positive two-dimensional (2D) systems has been observed [1–9]. An overview of some recent results in positive systems has been given in the monographs [1,10] and papers [5–9] and on the controllability of 1D and 2D systems in [11]. The asymptotic behaviour of positive 2D systems and their internal stability have been investigated in [8,9]. The local reachability of positive 2D systems described by the second Fornasini-Marchesini models [2–4] has been analyzed in [5]. It was shown that the reachability index of the n-order positive 2D systems is not bounded by n.

In this note it will be shown that $2(n+1)$ is the upper bound for the reachability index of the n-order positive 2D systems described by the general model.

2. Problem formulation

Let $R^{n\times m}$ be the set of $n\times m$ real matrices and $R^n = R^{n\times 1}$.

Consider the 2D general model

$$x_{i+1,j+1} = A_0x_{ij} + A_1x_{i+1,j} + A_2x_{i,j+1} + B_0u_{ij} + B_1u_{i+1,j} + B_2u_{i,j+1} \quad (1a)$$

$i,j \in Z_+$ (the set of nonnegative integers)

$$y_{ij} = Cx_{ij} + Du_{ij} \quad (1b)$$

where $x_{ij}\in R^n$ is the local state vector at the point (i,j), $u_{ij}\in R^m$ and $y_{ij}\in R^p$ are the input and output vectors and $A_k\in R^{n\times n}$, $B_k\in R^{n\times m}$, $k=0,1,2$, $C\in R^{p\times n}$, $D\in R^{p\times m}$.

Boundary conditions for (1a) are given by

$$x_{i0},\ i\in Z_+ \quad \text{and} \quad x_{0j},\ j\in Z_+ \quad (2)$$

Let R^n_+ be the set of n-dimensional vectors with nonnegative components.

Definition 1. The model (1) is called the positive 2D general model (P2DGM) if for all boundary conditions

$$x_{i0}\in R^n_+,\ i\in Z_+,\ x_{0j}\in R^n_+,\ j\in Z_+ \quad (3)$$

and every sequence of inputs $u_{ij}\in R^m_+$, $i,j\in Z_+$ we have $x_{ij}\in R^n_+$ and $y_{ij}\in R^p_+$ for $i,j\in Z_+$.

Theorem 1 [10]. The model (1) is a P2DGM if and only if

$$A_k\in R^{n\times n}_+,\ B_k\in R^{n\times m}_+,\ k=0,1,2,\ C\in R^{p\times n}_+,\ D\in R^{p\times m}_+ \quad (4)$$

where $R^{p\times q}_+$ is the set of $p\times q$ real matrices with nonnegative entries.

The transition matrix T_{ij} of the model (1) is defined by

$$T_{ij} = \begin{cases} I_n & \text{(identity matrix) for } i=j=0 \\ A_0T_{i-1,j-1} + A_1T_{i,j-1} + A_2T_{i-1,j} & \text{for } i,j>0\ (i+j>0) \\ 0 & \text{(zero matrix) for } i<0 \text{ or/and } j<0 \end{cases} \quad (5)$$

From (5) it follows that for P2DGM (1) $T_{ij}\in R^{n\times n}_+$ for $i,j\in Z_+$.

Definition 2. The P2DGM (1) is called reachable at the point $(h,k)\in Z_+\times Z_+$ if for zero boundary conditions (ZBC) (2) and every vector $x_f\in R^n_+$ there exists a sequence of inputs $u_{ij}\in R^m_+$ for $(i,j)\in D_{hk}$ such that $x_{hk}=x_f$, where

$$D_{hk} = \{(i,j)\in Z_+\times Z_+ : 0\leqslant i\leqslant h,\ 0\leqslant j\leqslant k, i+j\neq h+k\}. \quad (6)$$

Definition 3. The P2DGM (1) is called reachable for ZBC if it is reachable at any point $(h,k)\in Z_+\times Z_+$. If $x_f\in R^n_+$ is reachable at the point (h,k) then it will be said that the state x_f is reached in $h+k$ steps. The number $h+k$ steps is called the reachability index of (1) and it will be denoted by I_R, i.e. $I_R=h+k$.

Theorem 2 [10]. The P2DGM (1) is reachable for ZBC if and only if the reachability matrix

$$R_{hk} := \left[M_0, M_i^1, 1\leqslant i\leqslant h; M_j^2, 1\leqslant j\leqslant k; M_{ij}, 1\leqslant i\leqslant h; 1\leqslant j\leqslant k; i+j\neq h+k\right] \quad (7)$$

$$M_0 = T_{h-1,k-1}B_0,\ M_i^1 = T_{h-i,k-1}B_1 + T_{h-i-1,k-1}B_0,\quad i=1,...,h$$

$$M_j^2 = T_{h-1,k-j}B_2 + T_{h-1,k-j-1}B_0, j=1,...,k$$

$$M_{ij} = T_{h-i-1,k-j-1}B_0 + T_{h-i,k-j-1}B_1 + T_{h-i-1,k-j}B_2,\ i=1,...,h;\ j=1,...,k,\ i+j\neq h+k$$

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contains an $n \times n$ monomial matrix (in each of its rows and in each of its columns only one entry is positive and the remaining entries are zero).

For standard 1D n-order linear systems the reachability index is equal to n.

It is also known [5] that for standard (i.e. not necessarily positive) 2D general models the reachability index is equal to n ($I_R = n$) i.e. any local state of (1) starting from ZBC can be reached in $h+k$ steps for $h+k \leqslant n$.

For P2DGM (1) the set X^+_{h+k} of all local states that can be reached in $h+k$ steps starting from ZBC by means of an input sequence $u_{ij} \in R^m_+$ coincides with the set of all nonnegative combinations of the columns of the matrix (7), i.e. $X^+_{h+k} = coneR_{hk}$.

It is known [5] that the reachability index I_R of a positive 2D linear systems is not bounded by n.

In [5] it was shown that the reachability index of the system (1) with $A_0 = 0$, $B_0 = B_1 = 0$ and

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

is equal to $I_R = 13$ (for $n = 7$). In [5] the conjecture was also given that $n^2/4$ represents an upper bound for the reachability index of every 2D positive system.

In this paper it will be shown that $2(n+1)$ is the upper bound for the reachability index of the P2DGM.

3. Problem solution

Solution of the problem is based on the following lemma

LEMMA. Let

$$\det\left[I_n z_1 z_2 - A_0 - A_1 z_1 - A_2 z_2\right] = z_1^n z_2^n - \sum_{\substack{i=0 \\ i+j \neq 2n}}^{n} \sum_{i=0}^{n} d_{ij} z_1^i z_2^j. \tag{9}$$

Then the transition matrices T_{ij} (defined by (5)) satisfy the equations

$$T_{n+k,0} = A_2^{n+k} = \sum_{i=0}^{n-1} d_{i0}^k A_2^i, k = 0, 1, \ldots \tag{10a}$$

$$T_{0,n+l} = A_1^{n+l} = \sum_{j=0}^{n-1} d_{0j}^l A_1^j, l = 0, 1, \ldots \tag{10b}$$

$$T_{n+k,n+l} = \sum_{\substack{i=0 \\ i+j \neq 2n}}^{n} \sum_{j=0}^{n} d_{ij} T_{i+k,j+k} \text{ for } k, l = 1, 2 \tag{10c}$$

P r o o f. The relations (10a) and (10b) follow from the Cayley-Hamilton theorem applied to A_2 and A_1, respectively.

Taking into account that

$$\left[I_n z_1 z_2 - A_0 - A_1 z_1 - A_2 z_2\right]^{-1} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} T_{ij} z_1^{-(i+1)} z_2^{-(j+1)} \tag{11}$$

we may write

$$\sum_{i=0}^{n} \sum_{j=0}^{n} H_{ij} z_1^i z_2^j = \left(z_1^n z_2^n - \sum_{\substack{i=0 \\ i+j \neq 2n}}^{n} \sum_{i=0}^{n} d_{ij} z_1^i z_2^j \right) \cdot \left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} T_{ij} z_1^{-(i+1)} z_2^{-(j+1)} \right) \tag{12}$$

where $\sum_{i=0}^{n} \sum_{j=0}^{n} H_{ij} z_1^i z_2^j$ is the adjoint matrix to the matrix $[I_n z_1 z_2 - A_0 - A_1 z_1 - A_2 z_2]$.

From comparison of the matrix coefficients at the same powers of $z_1^{-k} z_2^{-l}$ for $k, l = 0, -1, -2, \ldots., k+l < 0$ of the equality (12) we obtain (10c).

THEOREM 3. If the P2DGM (1) is reachable then it is reachable in at most $2(n+1)$ steps ($h \leqslant n, k \leqslant n$), i.e.

$$I_R \leqslant 2(n+1) \quad (h \leqslant n, k \leqslant n). \tag{13}$$

P r o o f. If the P2DGM (1) is reachable then by Theorem 2 the reachablity matrix (7) contains an $n \times n$ monomial matrix for $h+k \leqslant 2(n+1)$ since by the equation (10) the columns M_i^1, M_j^2 and M_{ij} of (7) for $h+k \leqslant 2(n+1)$ ($h \geqslant n$, $k \geqslant n$) are linear combinations of the columns of the matrix R_{hk} for $h+k \leqslant 2(n+1)$ ($h \leqslant n$, $k \leqslant n$).

Example. Consider the P2DGM with

$$A_0 = 0, \; A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B_0 = 0, \; B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; B_2 = 0. \tag{14}$$

Using (5) and (7) we obtain

$$T_{i0} = \begin{cases} A_2 & \text{for } i = 1 \\ 0 & \text{for } i > 2 \end{cases}, \; T_{20} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$T_{0j} = \begin{cases} A_1 & \text{for } j = 1 \\ 0 & \text{for } j > 2 \end{cases}, \quad T_{02} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$T_{11} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad T_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$T_{13} = 0, \; T_{14} = 0$$

$$T_{21} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \; T_{22} = I_4, \; T_{23} = A_1,$$

$$T_{24} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{31} = 0, \; T_{32} = A_2, \; T_{33} = T_{11}, \; T_{34} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T_{41} = 0, \; T_{42} = T_{20}, \; T_{43} = T_{21}, \; T_{44} = I_4$$

and

$$R_{13} = \left[M_0, M_1^1, M_1^2, M_2^2, M_3^2, M_{11}, M_{12}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_{31} = \left[M_0, M_1^1, M_2^1, M_3^1, M_1^2, M_{11}, M_{21}\right] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_{22} = \left[M_0, M_1^1, M_2^1, M_1^2, M_2^2, M_{11}, M_{12}, M_{21}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_{33} = \left[M_0, M_1^1, M_2^1, M_3^1, M_1^2, M_2^2, M_3^2, M_{11}, M_{12}, M_{13}, M_{21}, M_{22}, M_{23}, M_{31}, M_{32}\right] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From Theorem 2 it follows that the P2DGM with (14) is not reachable for $h + h \leqslant n = 4$ and it is reachable for $h + k = 6 > n^2/4$. The reachability index of the system satisfies the condition (13), i.e. $I_R = h + k = 6 < 2(n + 1) = 10$.

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