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Nonlinear Interaction of Modes in a Planar Flow of a Gas with Viscous and Thermal Attenuation

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The nonlinear interaction of wave and non-wave modes in a gas planar flow are considered. Attention is mainly paid to the case when one sound mode is dominant and excites the counter-propagating sound mode and the entropy mode. The modes are determined by links between perturbations of pressure, density, and fluid velocity. This definition follows from the linear conservation equations in the differential form and thermodynamic equations of state. The leading order system of coupling equations for interacting modes is derived. It consists of diffusion inhomogeneous equations. The main aim of this study is to identify the principle features of the interaction and to establish individual contributions of attenuation (mechanical and thermal attenuation) in the solution to the system.

Keywords: nonlinear wave theory; nonlinear acoustics; coupling dynamic equations; Burgers equation; diffusion equation.

1. Introduction

Nonlinear acoustics studies not only distortions of sound itself but also interaction of different types of fluid motion. The variety of interactions includes excitation of the non wave modes (entropy and vortex ones) by waves, scattering of waves on wires and thermal inhomogeneities which may associate with the non wave modes, and scattering of waves by waves. The problems concerning interaction of modes go far beyond the fluid dynamics and belong to the general wave theory. The similar problems arise in electrodynamics, radiophysics, and solid state theory (Askaryan, 1966; Leble, Perelomova, 2018). In solids, fast and slow waves may propagate, as well as in the magnetic fluids. This increases diversity of interactions. Heterogeneity of a fluid always complicates definition of wave modes and theoretical description of nonlinear fluid dynamics. The features of interaction depend on the ratio of magnitudes of specific perturbations. Particularly, self-action of intense wave mode leads to its nonlinear distortions in the course of propagation.

The main goal of this study is to derive and analyse simplified system of equations responsible for weakly nonlinear interaction of modes in the case of the dominant wave mode. We consider the simplest planar flow with mechanical and thermal losses, which allows to make general conclusions in more complex flows. The key issue is to determine the modes of a flow. The analysis makes use of the linear definition of modes and properties of projecting. The linear projecting of the total field of perturbations into specific modes can be provided algorithmically. It has been worked out and applied by the author in various examples of fluid flows. Definition of the modes in the correspondent to conservation equations spectral problem is the foundation of the theory. In a one dimensional flow, a mode is determined by the links of velocity and two thermodynamic variables (excess pressure and density, for example) for every type of possible motion in a fluid: two acoustic and entropy modes. In a three dimensional flow, there are five modes in general: two acoustic, two vortex, and one entropy mode (Chu, Kovasznay, 1958; Pierce, 2019). Involving in consideration relaxation processes, ionised media, multi-phase flows, boundaries may increase the number of modes and make their definition fairly difficult (Leble, Perelomova, 2018; Perelomova, 2015; 2018). In spite of this, the procedure of definition is algorithmic in all cases, that is, it consists of a certain sequence of actions.

Once the modes are determined accordingly to the linear links of perturbations, the system of conservation laws in the differential form splits by means of projecting into a system of nonlinear dynamic equations which govern specific perturbations. The system accounts for weakly nonlinear interactions between modes. In the leading order, it is equivalent to the initial system of conservation laws, but it is much more convenient for the approximate solution in many applications. The advantage of the method of projecting is in consideration of every mode individually and making use of the first order partial differential equations with respect to time, which are much simpler for analytical or numerical solution than the higher order equations. The advantage is also in possibility of instantaneous description of modes interaction, without estimation of average over time and reference to strictly periodic processes (Hamilton, Blackstock, 1998; Pierce, 2019). Examples of various applications of projecting in the fluid mechanics are discussed by Leble and Perelomova (2018) and referenced there papers. Projecting is an algorithmic and direct method which is exceptionally successful in complex problems of nonlinear fluid flow. Most problems of nonlinear acoustics presuppose the dominant sound, and perturbations in other modes are comparatively small (Makarov, Ochmann, 1996; Hamilton, Blackstock, 1998). Over the temporal and spatial domains, where it holds true, excitation of other modes is described by the PDEs of the first order regarding time with an acoustic source quadratic in the leading order. The interaction of modes takes place exclusively in the nonlinear flow with attenuation, and all source terms which reflect interaction, are nonlinear. They are proportional to viscous or thermal attenuation. Velocity in the dominant wave mode in a Newtonian flow satisfies the Burgers dynamic equation (Rudenko, Soluyan, 1977).

The main problem is to solve the system of PDEs with the inhomogeneity representing the quadratic terms of the dominant mode and proper interpretation of results. The perturbations specifying sound are quasi isentropic, if they are determined by the linear links. They need to be corrected by quadratic terms going to studies of nonlinear effects associating with sound which are also quadratic in the leading order. The modes determined by linear links of perturbations are nearly progressive or quasi stationary only in the frames of a linear flow. The directivity property fails even in a weakly nonlinear flow. As for the self action of the dominant mode, when the linear dynamic equation is supplemented by a quadratic term of the dominant perturbation, the directivity property is preserved. It turns out that the excited perturbations in the leading order consist of parts which propagate with their own linear speed and speed of the dominant mode. This must be paid attention to in evaluations of total secondary fields. This study clarifies the postulate and takes into account viscous and thermal attenuation, that is, it considers solutions to the diffusion inhomogeneous equations. The individual contributions of thermal conduction and shear viscosity in the context of excitation of the secondary modes are discussed.

2. Foundations of weakly nonlinear projecting

In this section, we remind briefly the definition of modes and relating to them projectors in accordance to the author's results (Leble, Perelomova, 2018; Perelomova, 2003; 2006). We start from the set of conservation equations in the planar flow of a thermoconducting Newtonian fluid in the differential form. They are: the momentum equation, the energy balance equation, and the continuity equation:

$$\begin{aligned}
&\rho\left(\frac{\partial v}{\partial t}+v\frac{\partial \rho}{\partial x}\right)+\frac{\partial p}{\partial x}=\frac{4\mu}{3}\frac{\partial^2 v}{\partial x^2},\\
&\rho\left(\frac{\partial e}{\partial t}+v\frac{\partial e}{\partial x}\right)+p\frac{\partial v}{\partial x}=\chi\Delta T+\frac{4\mu}{3}\left(\frac{\partial v}{\partial x}\right)^2,\\
&\frac{\partial \rho}{\partial t}+\frac{(\partial \rho v)}{\partial x}=0,
\end{aligned} \tag{1}$$

where x, t are the spatial coordinate and time, and ρ, p, v, e, T denote density, pressure, velocity, internal energy, and temperature of a fluid. χ, μ are thermal conductivity and shear viscosity, both assumed to be constants. Thermodynamic equations of state of an ideal gas supplement the system (1). Its internal energy and temperature are related as

$$e=C_vT=\frac{p}{(\gamma-1)\rho},$$

with C_v denoting the heat capacity under constant volume per unit mass. $\gamma=C_p/C_v$ is the ratio of specific heats. Fluids obeying the equations of state differing from that for an ideal gas, may be readily considered (Leble, Perelomova, 2018). We make use of excess quantities which are denoted by primes $p'=p-p_0, \rho'=\rho-\rho_0$. The equilibrium quantities are indicated by the lower index 0. A gas is static in equilibrium, $v_0=0$.

A linear fluid flow, that is, a flow of infinitely small magnitude of perturbations, is represented by the linearised version of the system (1) with zero right-hand side, which may be rearranged as

$$\frac{\partial \psi}{\partial t}+L\psi=0, \tag{2}$$

where

$$\psi=\begin{pmatrix} v\\ p'\\ \rho'\end{pmatrix},\qquad L=\begin{pmatrix} -\delta_1\dfrac{\partial^2}{\partial x^2} & \dfrac{1}{\rho_0}\dfrac{\partial}{\partial x} & 0\\ \gamma p_0\dfrac{\partial}{\partial x} & \frac{\gamma\delta_2}{\gamma-1}\dfrac{\partial^2}{\partial x^2} & -\dfrac{c_0^2\delta_2}{\gamma-1}\dfrac{\partial^2}{\partial x^2}\\ \rho_0\dfrac{\partial}{\partial x} & 0 & 0\end{pmatrix},$$

and

$$\delta_1 = \frac{4\mu}{3\rho_0}, \qquad \delta_2 = \frac{\chi}{\rho_0}\left(\frac{1}{C_v} - \frac{1}{C_p}\right), \qquad c_0 = \sqrt{\frac{\gamma p_0}{\rho_0}}.$$

The vectors as follows describe all possible types of perturbations in a linear flow:

$$\psi_i = \begin{pmatrix} v_i \\ p_i \\ \rho_i \end{pmatrix}, \qquad i = 1,2,3,$$

$$\psi_1 = \begin{pmatrix} 1 \\ c_0\rho_0 + \rho_0 \dfrac{\beta - 2\delta_2}{2}\dfrac{\partial}{\partial x} \\ \dfrac{\rho_0}{c_0} + \dfrac{\beta}{2c_0^2}\dfrac{\partial}{\partial x} \end{pmatrix} v_1,$$

$$\psi_2 = \begin{pmatrix} 1 \\ -c_0\rho_0 + \rho_0 \dfrac{\beta - 2\delta_2}{2}\dfrac{\partial}{\partial x} \\ -\dfrac{\rho_0}{c_0} + \dfrac{\beta}{2c_0^2}\dfrac{\partial}{\partial x} \end{pmatrix} v_2, \tag{3}$$

$$\psi_3 = \begin{pmatrix} -\dfrac{\delta_2}{(\gamma-1)\rho_0}\dfrac{\partial}{\partial x} \\ 0 \\ 1 \end{pmatrix} \rho_3.$$

The vectors represent modes with the ordering numbers i ($i = 1,2,3$), where v_1, v_2 are velocities of a fluid which associate with the first and second wave modes, ρ_3 is an excess density which specifies the entropy mode, and $\beta = \delta_1 + \delta_2$ denotes the total attenuation. They correspond to the well known dispersion relations (Rudenko, Soluyan, 1977):

$$\omega_1 = c_0 k + i\frac{\beta k^2}{2}, \qquad \omega_2 = -c_0 k + i\frac{\beta k^2}{2}, \qquad \omega_3 = i\frac{\delta_2 k^2}{\gamma - 1}, \tag{4}$$

where k is the wave number. Three linearly independent vectors (3) reflect three kinds of a fluid motion in a planar flow: two first are acoustic ones, propagating in the positive and negative directions of axis x, and the third one is the entropy mode. The reference quantities v_1, v_2, ρ_3 determine the overall perturbations v, p', ρ' in a one-to-one way:

$$\psi = \sum_{i=1}^{3} \psi_i.$$

The specific velocities may be extracted from the vector of total perturbations by means of projecting rows:

$$M_i\psi = v_i, \qquad i = 1,2,3. \tag{5}$$

These rows take the forms

$$\begin{aligned} M_1 &= \begin{pmatrix} \dfrac{1}{2} - a^* & \dfrac{1}{2c_0\rho_0} - \dfrac{\delta_2}{2b^*}\dfrac{\partial}{\partial x} & \dfrac{\delta_2}{2c^*}\dfrac{\partial}{\partial x} \end{pmatrix}, \\ M_2 &= \begin{pmatrix} \dfrac{1}{2} + a^* & -\dfrac{1}{2c_0\rho_0} - \dfrac{\delta_2}{2b^*}\dfrac{\partial}{\partial x} & \dfrac{\delta_2}{2c^*}\dfrac{\partial}{\partial x} \end{pmatrix}, \\ M_3 &= \begin{pmatrix} 0 & \dfrac{\delta_2}{b^*}\dfrac{\partial}{\partial x} & -\dfrac{\delta_2}{c^*}\dfrac{\partial}{\partial x} \end{pmatrix}, \end{aligned} \tag{6}$$

where

$$a^* = \frac{\delta_1 - \delta_2}{4c_0}\frac{\partial}{\partial x}, \qquad b^* = (\gamma - 1)c_0^2\rho_0, \qquad c^* = (\gamma-1)\rho_0.$$

Obviously,

$$\sum_{i=1}^{3} M_i = (1 \quad 0 \quad 0).$$

The important property of projecting rows M_i is to extract the linear dynamic equation for v_i when they apply at Eq. (2). For example,

$$M_1\left(\frac{\partial\psi}{\partial t} + L\psi\right) = \frac{\partial v_1}{\partial t} + c_0\frac{\partial v_1}{\partial x} - \frac{\beta}{2}\frac{\partial^2 v_1}{\partial x^2} = 0.$$

The projecting rows (6) lead to the system of dynamic equations accounting for nonlinear interaction of modes. When they apply at Eq. (1), they yield the leading order dynamic equations with cross quadratic nonlinear terms forming the "forces" responsible for interaction between modes. Meaning the dominance of sound, only terms belonging to the acoustic modes can be considered. They are of order of the squared Mach number, M^2. Hence, the dominant wave modes must theirselves be corrected in order to hold adiabaticity within accuracy up to terms proportional to M^2. In particular, if the rightwards progressive mode is dominant, the relations take the forms

$$\psi_1 = \begin{pmatrix} 1 \\ c_0\rho_0 + \dfrac{(\beta - 2\delta_2)\rho_0}{2}\dfrac{\partial}{\partial x} \\ \dfrac{\rho_0}{c_0} + \dfrac{\beta\rho_0}{2c_0^2}\dfrac{\partial}{\partial x} \\ 1 \end{pmatrix} v_1 + \begin{pmatrix} 0 \\ \dfrac{(\gamma+1)\rho_0}{4} \\ -\dfrac{(\gamma-3)\rho_0}{4c_0^2} \end{pmatrix} v_1^2. \tag{7}$$

Equation (7) in fact is the leading order series of the links which specify the simple wave (Rudenko, Soluyan, 1977). Additionally, they take into account attenuation. We make use of relation (7) going to study the weakly nonlinear flow. They have impact on the coupling of modes and dynamic equations taking into account interaction between modes. Velocity specifying the first acoustic mode satisfies the Burgers equation

$$\frac{\partial v_1}{\partial t} + c_0\frac{\partial v_1}{\partial x} + \frac{(\gamma+1)}{2}v_1\frac{\partial v_1}{\partial x} - \frac{\beta}{2}\frac{\partial^2 v_1}{\partial x^2} = 0. \tag{8}$$

It may be obtained by applying of M_1 at the system (1) for ψ_1 taken alone among all nonlinear terms. Equation (8) may be rearranged into the linear diffusion equation and hence may be solved analytically with the proper initial and boundary conditions (Rudenko, Soluyan, 1977).

3. Excitation of the secondary perturbations in the field of one dominant sound mode

To derive an equation which takes into account coupling of the first and second sound modes, let us suppose that the first mode is dominant. Hence, we consider exclusive contribution of v_1 among all quadratic terms forming a "driving force". Applying M_2 at Eqs (1) and making use of ψ_1 (Eq. (7)) and ψ_2 (Eqs (3)), we arrive at the leading order dynamic equation:

$$\frac{\partial v_2}{\partial t} - c_0\frac{\partial v_2}{\partial x} + \frac{(\gamma+1)c_0}{2\rho_0}v_2\frac{\partial v_2}{\partial x} - \frac{\beta}{2}\frac{\partial^2 v_2}{\partial x^2} = -\frac{\beta(3\gamma-5)}{8c_0^2}\left(\frac{\partial v_1}{\partial x}\right)^2 - \frac{\beta(\gamma+1)}{8c_0^2}v_1\frac{\partial^2 v_1}{\partial x^2} = F_2(x,t). \tag{9}$$

The "acoustic force" in the right hand side of equation F_2 is nonlinear (quadratic in the frames of accepted accuracy) and proportional to the total attenuation. This reflects the necessary conditions for modes coupling, namely, nonlinearity and attenuation. It is important to note that both nonlinear distortions of dominant mode and nonlinear excitation of the perturbations in the second mode depend exclusively on the total attenuation, not on its parts associating with mechanical and thermal attenuation. Excitation of an excess density in the entropy mode is described by equation

$$\frac{\partial \rho_3}{\partial t} - \frac{\delta_2}{\gamma-1}\frac{\partial^2 \rho_3}{\partial x^2} = \frac{\beta(\gamma-1)\rho_0}{2c_0^2}\left(v_1\frac{\partial^2 v_1}{\partial x^2} - \left(\frac{\partial v_1}{\partial x}\right)^2\right) = F_3(x,t). \tag{10}$$

An excess density ρ_3 is used as a reference variable since the velocity associating with the entropy mode is fairly small, and specific excess pressure equals zero in the leading order. The solution to Eq. (10) depends on both δ_1 and δ_2. We will consider solutions to Eqs (9) and (10) at the infinite axis x which satisfy zero initial conditions,

$$v_2(x,0) = 0, \qquad \rho_3(x,0) = 0. \tag{11}$$

3.1. *Preliminary remarks concerning the structure of excited perturbations at the infinite axis*

Equations (9) and (10) are instantaneous. The preliminary conclusions of nonlinear interaction may be drawn out from the simplified dynamic equations

$$\frac{\partial v_2}{\partial t} - c_0\frac{\partial v_2}{\partial x} = F_2(x,t), \tag{12}$$

$$\frac{\partial \rho_3}{\partial t} = F_3(x,t), \tag{13}$$

following from Eqs (9) and (10) if attenuation and nonlinearity in the left hand linear sides of equations are omitted. An exact solution to Eq. (12) with accounting for Eq. (11) is

$$v_2(x,t) = \int_0^t F_2(c_0(t-\tau)+x,\tau)\,\mathrm{d}\tau. \tag{14}$$

For the first approximation, velocity in the dominant mode is a function of $x-c_0t$, $F_2(x-c_0t)$. In this case, the solution is

$$v_2(x,t) = \frac{1}{2c_0}\left(\Phi_2(x+c_0t) - \Phi_2(x-c_0t)\right), \tag{15}$$

where Φ_2 is a primitive function to F_2. So, v_2 in general consists of planar waves of the same shape propagating with velocities c_0 and $-c_0$. The solution to (13), which satisfies zero initial condition, takes the form

$$\rho_3(x,t) = \frac{1}{c_0}\left(\Phi_3(x) - \Phi_3(x-c_0t)\right), \tag{16}$$

if F_3 is a function of $x-c_0t$, and Φ_3 is a primitive function to it.

3.1.1. *Harmonic excitation*

Let us consider an example of harmonic solution to the wave equation without account for nonlinearity and attenuation, that is,

$$v_1 = V_0\sin(kc_0t - kx), \tag{17}$$

where the wave number k relates to the frequency ω as $k=\omega/c_0$. The "acoustic forces" for this exciter equal

$$F_2 = -\frac{\beta k^2V_0^2[\gamma-3+2(\gamma-1)\cos(2k(x-c_0t))]}{8c_0}, \qquad F_3 = -\frac{\beta k^2V_0^2(\gamma-1)\rho_0}{2c_0^2}. \tag{18}$$

The solutions to the Cauchy problem with zero initial conditions in the dimensionless values

$$X = kx, \qquad T = kc_0t, \qquad B = \frac{\beta k}{c_0}, \qquad M = \frac{V_0}{c_0} \tag{19}$$

take the forms

$$\frac{v_2}{c_0} = -\frac{BM^2(\gamma-3)T}{8} - \frac{BM^2(\gamma-1)}{16}\cdot\left(\sin(2T-2X)+\sin(2T+2X)\right), \tag{20}$$

$$\frac{\rho_3}{\rho_0} = -\frac{BM^2(\gamma-1)T}{2}. \tag{21}$$

Oscillation frequency of perturbations in the excited wave mode doubles due to nonlinear effects, and non oscillating parts of perturbations enlarge in time poportionally to T.

3.1.2. Gaussian impulse

The next example concerns an impulse

$$v_1 = Mc_0 \exp((X - T)^2). \tag{22}$$

This case corresponds to impulsive "acoustic forces" F_2 and F_3:

$$\begin{aligned} F_2 &= -\frac{Bkc_0^2M^2e^{-2(X-T)^2}}{4}\big((\gamma-1)(8(X-T)^2) - (\gamma+1)\big), \\ F_3 &= -B(\gamma-1)kc_0M^2\rho_0e^{-2(X-T)^2}. \end{aligned} \tag{23}$$

The impulse excites perturbations as follows:

$$\begin{aligned} \frac{v_2}{c_0} &= \frac{BM^2}{32}\Big[8(\gamma-1)((T-X)\exp(-2(T-X)^2) \\ &\quad + (T+X)\exp(-2(T+X)^2)) \\ &\quad \cdot (\gamma-3)\sqrt{2\pi}(\mathrm{Er}f(\sqrt{2}(X-T)) \\ &\quad - \mathrm{Er}f(\sqrt{2}(X+T)))\Big], \\ \frac{\rho_3}{\rho_0} &= -\frac{d^*}{2\sqrt{2}}, \end{aligned} \tag{24}$$

where

$$d^* = BM^2(\gamma-1)\sqrt{\pi}\Big[\mathrm{Er}f(\sqrt{2}X) - \mathrm{Er}f\left(\sqrt{2}(X-T)\right)\Big].$$

We may conclude that v_2 is symmetric with respect to the straight line $X = 0$, and ρ_3 is symmetric with respect to the line $X = T/2$. The head front of ρ_3 moves with the speed c_0, and its back front is motionless. As for the summary perturbation of density excited by the dominant sound, it is a sum of individual parts, $\rho_2 \approx -\frac{\rho_0 v_2}{c_0}$ and ρ_3. In all evaluations, $\gamma = 1.4$. They were carried out in Mathematica. The total excited perturbations of density and pressure are negative and diverge along the axis x with different speeds. They are shown in Fig. 1. All figures have been plotted in Mathematica. In view of links established by ψ_3, the "acoustic force" producing velocity which associates with this mode equals $-\frac{\delta_2}{(\gamma-1)\rho_0}\frac{\partial F_3}{\partial x}$, that is, is the quantity of order β^2 and hence is negligible.

3.1.3. The shock wave

As an exciter, the stationary solution of the Burgers Eq. (8) is considered:

$$v_1 = Mc_0 \tanh(\mathrm{Re}\,(T - X)), \tag{25}$$

where Re is the Reynolds number which expresses the ratio of nonlinear and viscous effects,

$$\mathrm{Re} = \frac{M(\gamma-1)}{B}. \tag{26}$$

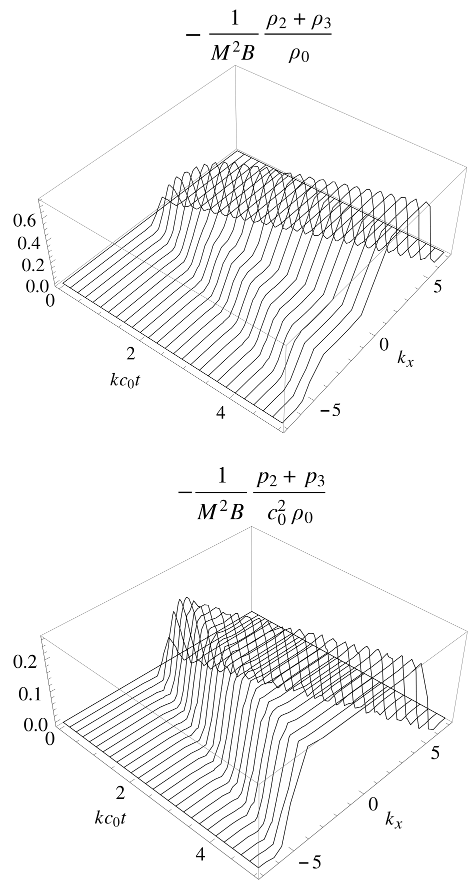

Fig. 1. Summary perturbations of density (top panel) and pressure (bottom panel) in secondary modes excited by the dominant wave mode. Case of impulsive excitation (22).

The "acoustic forces" in this case take the forms

$$\begin{aligned} F_2 &= \frac{(\gamma-1)c_0^2kM^3\mathrm{Re}}{8\cosh^4(\mathrm{Re}\,(T-X))} \\ &\quad \cdot [4 - 4\gamma + (\gamma+1)\cosh(2\mathrm{Re}\,(T-X))], \\ F_3 &= -\frac{(\gamma-1)^2c_0kM^3\mathrm{Re}\rho_0\cosh(2\mathrm{Re}\,(T-X))}{2\cosh^4(\mathrm{Re}\,(T-X))}. \end{aligned}$$

Hence, the shapes of excited perturbations depend on the Reynolds number, but their magnitudes are proportional to M^3, not to M^2B. Evaluations in Mathematica lead to the following excited perturbations:

$$\begin{aligned} \frac{v_2}{c_0} &= \frac{M^3}{48}(\gamma-1)\big\{[9 - 7\gamma + 2(3-\gamma)\cosh(2\mathrm{Re}\,(T-X))] \\ &\quad \cdot \cosh^{-2}[\mathrm{Re}\,(T-X)]\tanh[\mathrm{Re}\,(T-X)] \\ &\quad + [9 - 7\gamma + 2(3-\gamma)\cosh(2\mathrm{Re}\,(T+X))] \\ &\quad \cdot \cosh^{-2}[\mathrm{Re}\,(T+X)]\tanh[\mathrm{Re}\,(T+X)]\big\}, \\ \frac{\rho_3}{\rho_0} &= -\frac{M^3}{6}(\gamma-1)^2(\cosh^{-3}(\mathrm{Re}\,(T-X)) \\ &\quad \cdot \sinh(3\mathrm{Re}\,(T-X)) + \cosh^{-3}(\mathrm{Re}X)\sinh(3\mathrm{Re}X)). \end{aligned}$$

The excited secondary perturbations are shown in Fig. 2.

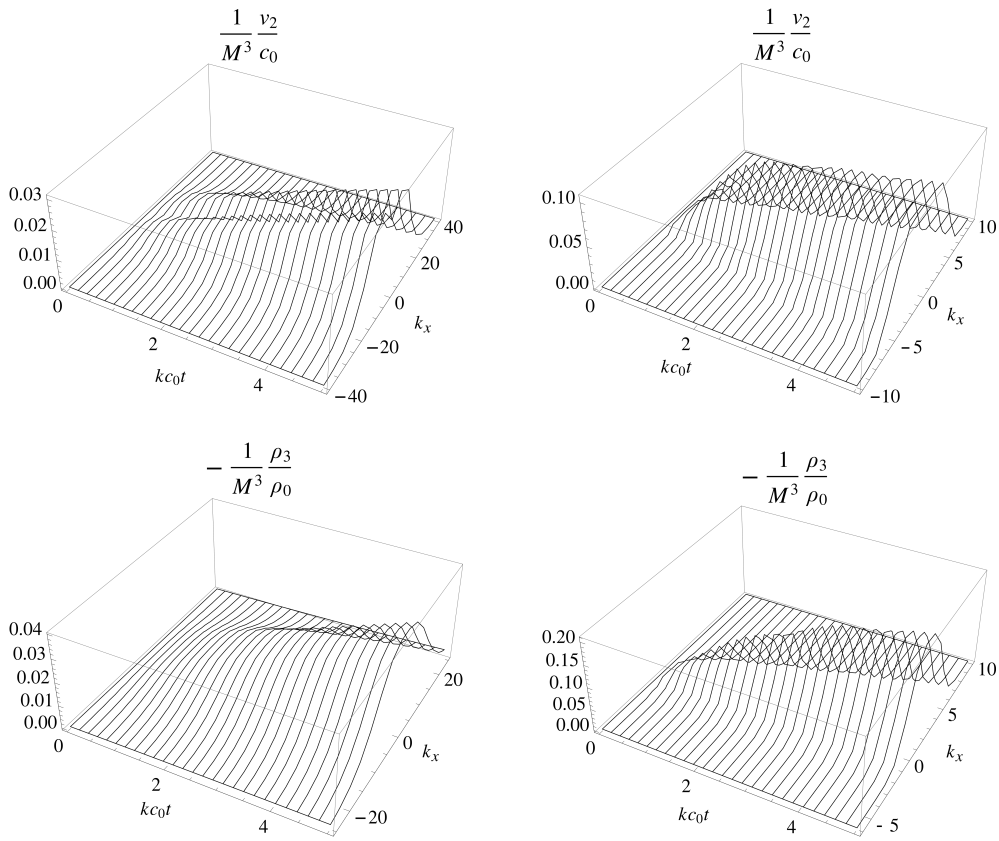

Fig. 2. Top row: velocity in the second mode excited by the shock wave (25). Bottom row: excess density in the third mode excited by the shock wave (25). Re = 0.1 (left panels) and Re = 1 (right panels).

3.2. Solutions to the diffusion equation at the infinite axis

One may expect that the main features of intermode's excitation are similar to those discovered in the previous subsection. The Burgers Eq. (8) readily rearranges into the linear diffusion equation by means of the Hopf-Cole transformation. Equation (9) may be also rearranged into the inhomogeneous diffusion equation. We focus on the equation with the omitted nonlinearity in the left hand side,

$$\frac{\partial v_2}{\partial t} - c_0\frac{\partial v_2}{\partial x} - \frac{\beta}{2}\frac{\partial^2 v_2}{\partial x^2} = F_2(x,t). \tag{27}$$

Let us consider the Cauchy problem with zero initial condition at the infinite axis x. It is convenient to rearrange the equation in the variables $x_1 = x + c_0 t$, t:

$$\frac{\partial v_2}{\partial t} - \frac{\beta}{2}\frac{\partial^2 v_2}{\partial x_1^2} = F_2(x_1 - c_0 t, t). \tag{28}$$

The solution to Eq. (28) takes the form

$$v_2(x_1,t) = \int\limits_0^t\int\limits_{-\infty}^{\infty}\frac{e^{-\frac{(x_1-\xi)^2}{2\beta(t-\tau)}}}{\sqrt{2\beta(t-\tau)}}F_2(\xi - c_0\tau,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{29}$$

The solution to (10) which satisfies the zero initial condition, is

$$\rho_3(x,t) = \sqrt{\gamma-1}\int\limits_0^t\int\limits_0^{\infty}\frac{e^{-\frac{(x-\xi)^2(\gamma-1)}{4\delta_2(t-\tau)}}}{2\sqrt{\pi\delta_2(t-\tau)}}F_3(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau. \tag{30}$$

3.2.1. Harmonic excitation

In the case of harmonic dominant perturbations (17), evaluations for "forces" given by Eqs (18) in Mathematica, result in

$$\begin{aligned}\frac{v_2}{c_0} &= -\frac{BM^2(\gamma-1)}{8(4+B^2)}\left[Be^{2BT}\cos(4T-2X_1) - B\cos(2X_1)\right.\\ &\quad\left. + 2e^{2BT}\sin(4T-2X_1) + 2\sin(2X_1)\right]\\ &\quad - \frac{BM^2(\gamma-3)T}{8}\\ &\approx -\frac{BM^2(\gamma-3)T}{8} - \frac{BM^2(\gamma-1)}{16}\\ &\quad\cdot\left(e^{2BT}\sin(2T-2X) + \sin(2T+2X)\right),\end{aligned} \tag{31}$$

where $X_1 = kx_1$. In view of weak attenuation over the sound period, $B \ll 1$. The conclusion is as before, that is, the excited perturbation includes a part following the dominant mode and a part propagating with its own linear speed. At large number of sound periods, $T \gg 1$, the difference in solutions Eqs (31) and (20) enhances in their parts following the dominant mode. At small times, Eqs (31) and (20) coincide in the leading order. The solution ρ_3 coincides with Eq. (21). There is no difference between the solutions to diffusion equation, Eqs (30) and (13), due to spatial homogeneity of ρ_3.

3.2.2. Gaussian impulse

As for the impulse excitation Eq. (22), the "acoustic forces" are determined by Eqs (23). The numerical solutions to Eq. (29) are shown in Fig. 3 in the cases of weak and strong attenuation. There is only a small difference between the cases of weak and strong attenuation as for the shapes of surfaces. Obviously, magnitudes of excited perturbations are proportional to M^2B. The surface in the left panel of Fig. 3 almost overlaps the surface established by Eq. (24).

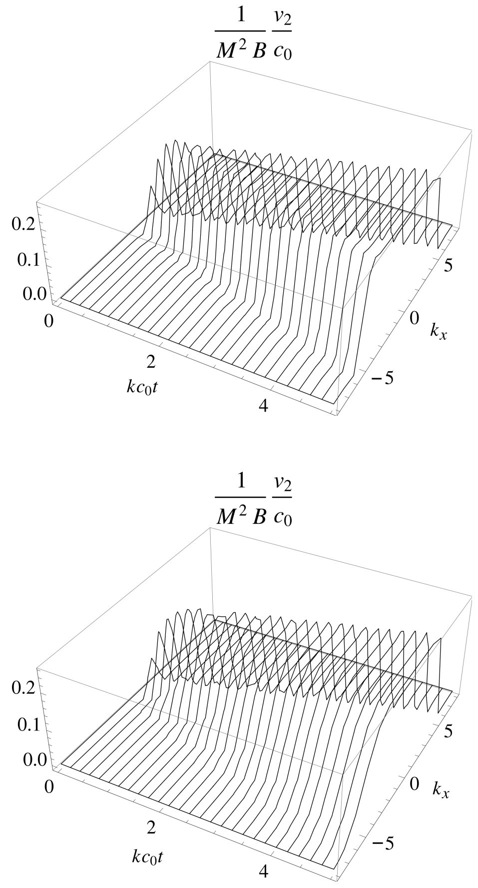

Fig. 3. Velocity in the second mode which is excited by the Gaussian impulse (22). $B = 0.001$ (top panel) and $B = 0.3$ (botoom panel).

An excess density in the entropy mode is negative. This corresponds to losses in acoustic energy and positive excess temperature produced during isobaric acoustic heating. Excitation of the entropy mode depends on β but actually very weakly on δ_2 individually. This is shown in Fig. 4.

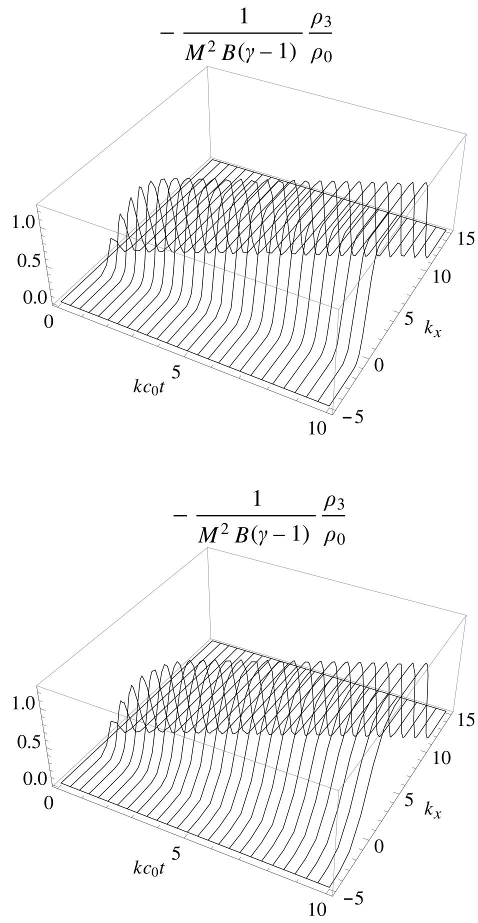

Fig. 4. Excess density in the entropy mode which is excited by the Gaussian impulse (22). $\frac{\delta_2 c_0 k}{\gamma-1} = 0.001$ (top panel) and $\frac{\delta_2 c_0 k}{\gamma-1} = 0.1$ (bottom panel).

The plots in Figs 3 and 4 represent Eqs (29) and (30). Both double integrals reduce analytically to integrals over time by means of Mathematica. The integrands are too long to fit in the text. The latter integrals were evaluated numerically.

4. The dominant entropy mode

Similarly, other kinds of excitation may be considered. Among them, there are problems relating to generation of bulk flows and the entropy mode in bounded volumes and resonators, where perturbations in the counterpropagating wave modes are of the same order. In this case, quadratic cross terms form the "acoustic forces". As usual, these terms are averaged over the sound period. In the leading order, the counterpropagating periodic waves do not nonlinearly interact. This has been proved by Kaner *et al.* (1977) and Ruderman (2013). This considerably simplifies evaluations but allows to consider only periodic exciters and averaged secondary perturbations. Non wave modes also may be dominant. Applying M_3 from Eqs (6) at the system (1) and taking into account links (3), one arrives at the instantaneous system of equations for excitation of the wave perturbations by the dominant entropy mode:

$$\frac{\partial v_1}{\partial t} + c_0 \frac{\partial v_1}{\partial x} = F_1(x) = \frac{\delta_2 c_0}{2\rho_0^2(\gamma-1)} \frac{\partial^2 \rho_3^2}{\partial x^2},$$
$$\frac{\partial v_2}{\partial t} - c_0 \frac{\partial v_2}{\partial x} = F_2(x) = F_1(x) = \frac{\delta_2 c_0}{2\rho_0^2(\gamma-1)} \frac{\partial^2 \rho_3^2}{\partial x^2}. \tag{32}$$

At the infinite axis x it has solutions

$$v_1 = \frac{1}{c_0}\left(\Phi_1(x) - \Phi_1(x - c_0 t)\right),$$

$$v_2 = -\frac{1}{c_0}\left(\Phi_1(x) - \Phi_1(x + c_0 t)\right)$$

satisfying zero initial conditions,

$$v_1(x, t = 0) = v_2(x, t = 0) = 0,$$

where Φ_1 is the primitive function to F_1,

$$\Phi_1 = \frac{\delta_2 c_0}{2\rho_0^2(\gamma-1)} \frac{\partial \rho_3^2}{\partial x}.$$

Other specific perturbations develop in the following way, in accordance to the links (3):

$$p_1 = \rho_0\left(\Phi_1(x) - \Phi_1(x - c_0 t)\right),$$

$$p_2 = \rho_0\left(\Phi_1(x) - \Phi_1(x + c_0 t)\right),$$

$$\rho_1 = \frac{\rho_0}{c_0^2}\left(\Phi_1(x) - \Phi_1(x - c_0 t)\right),$$

$$\rho_2 = \frac{\rho_0}{c_0^2}\left(\Phi_1(x) - \Phi_1(x + c_0 t)\right).$$

The main conclusion is that the excitation is possible only in fluids with thermal conduction. The magnitudes of the secondary counterpropagating perturbations are equal.

5. Concluding remarks

Equations which take into account distortions of wave caused by different obstacles, including thermal inhomogeneities of a medium and wires, are of permanent interest in the wave theory. As usual, the linear wave equation is supplied by some inhomogeneity and takes the form

$$\frac{\partial^2 v}{\partial t^2} - c_0^2 \Delta v - \beta \frac{\partial}{\partial t} \Delta v = F(x, y, z, t). \tag{33}$$

The solution to it is a sum of the general solution to the homogeneous equation and a partial solution to the inhomogeneous equation. Often, the inviscid version of Eq. (33) with zero β is paid attention to (Tikhonov, Samarski, 2011). Two initial conditions are necessary in the one dimensional flow along the infinite axis,

$$v(x, 0) = \phi(x),$$

$$\frac{\partial v}{\partial t}(x, 0) = \psi(x).$$

The initial conditions relate to the total velocity v. Equation (33) does not differentiate between branches of a sound, and does not make use of difference in magnitude of their perturbations. The proper interpretation of terms belonging to different branches is not obvious. Also, velocity associating with the entropy and vortex modes is not considered by the wave Eq. (33) at all. Equation (33) does not provide information about perturbations of thermodynamic variables (pressure, density). Correct determination of the source function F is an important issue.

Hence, essence, meaning and applicability of Eq. (33) and the system (9), (10) are fairly different. Equations (9) and (10) are derived with account for the ratio of magnitudes of specific perturbations, and they consider excitation of both secondary modes, first of them being the wave mode, and the second one being the entropy mode. Equations (9) and (10) determine all individual and total perturbations in view of the links (3), not only velocity in the wave processes. They describe nonlinear interaction of modes in the frames of the method which has been worked out by the author. The modes are understood as relations between specific perturbations in a linear flow. With respect to weakly nonlinear flow, these relations still define modes. We mainly consider the rightwards propagating sound as the dominant mode in this study. Equations (9) and (10) are PDE equations which contain the first order derivatives with respect to time and account for nonlinearity and attenuation in the "forces" of interaction. They require two initial conditions (11) (or maybe non zero ones) for every secondary mode individually. The method allows to follow individual dynamics of the modes (including the entropy mode) and the total perturbations in pressure and density, in contrast to the inhomogeneous wave equation – Eq. (33). One of the main conclusions is that the property of directivity of the secondary modes is broken. This should be taken into account in evaluations of the total perturbations. In particular, perturbations in the secondary wave mode consist of parts which propagate with speeds c_0 and $-c_0$. This follows from Eq. (14) which in turn supposes that the dominant perturbations propagate with the speed c_0. In nonlinear flows with attenuation and dispersion, the dominant mode may be stationary with speed $\widetilde{c}$ different from c_0 (the shock waves in Newtonian flows and solitons in dispersive flows). If so, the secondary perturbations are given by formulas correcting Eqs (15) and (16):

$$v_2 = \frac{1}{c_0 + \widetilde{c}} \left(\Phi_2(x + c_0 t) - \Phi_2(x - \widetilde{c}t) \right),$$

$$\rho_3 = \frac{1}{\widetilde{c}} \left(\Phi_3(x) - \Phi_3(x - \widetilde{c}t) \right).$$

We have considered individual contributions of thermal conduction and mechanical viscosity in a flow with a dominant sound. It turns out that the thermal conduction influences the shape of excited perturbations only weakly. In this context, it may be ignored, Eq. (10) is significantly simplified and may be solved by direct integration of the "acoustic force" over time. Solution to Eq. (9) may be obtained without the linear therm proportional to the total attenuation in its linear part with the exception of large times of harmonic excitation. The conclusions give hope for simple solutions in more complex cases of nonlinear interactions in a flow. In spite of the fact that the theory is based on the modes which are determined accordingly to linear links between perturbations, modes may be redefined in accordance to directivity properties in a weakly nonlinear flow. That can be carried out by summing up parts propagating with similar speeds.

The theoretical results may be useful for technical and medical applications dealing with intense sound. In particular, they may be useful in evaluations of perturbations in the reflected sound wave and variations in temperature associating with the entropy mode. The theory considers time limited exciters and allows to follow development of the secondary perturbations. This is of importance in therapeutic applications of high intensity focused ultrasound (HIFU), where the temperature of tissue should be strictly controlled (Duck *et al.*, 1998). The theory may be useful in exciter's selection meeting the conditions and goals of treatment.

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