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Numerical error bound of optimal control for homogeneous linear systems

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In this article we focus on the balanced truncation linear quadratic regulator (LQR) with constrained states and inputs. For closed-loop, we want to use the LQR to find an optimal control that minimizes the objective function which called "the quadratic cost function" with respect to the constraints on the states and the control input. In order to do that we have used formal asymptotes for the Pontryagin maximum principle (PMP) and we introduce an approach using the so called *The Hamiltonian Function* and the underlying algebraic Riccati equation. The theoretical results are validated numerically to show that the model order reduction based on open-loop balancing can also give good closed-loop performance.

Key words: time-invariant systems, quadratic cost function, linear quadratic regulator, agebraic Riccati equation, Hamiltonian function, L_2 norm

1. Introduction

Balanced model reduction of linear control systems have attracted a lot of attention in the last decades, both in terms of the development of the theory as well as in terms of concrete applications to problems in science, engineering, chemical and biological phenomena [3]. More precisely the semi-discretization of partial differential equation describing physical phenomena lead to the well-known representation of a linear time-invariant (LTI) system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du, \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D^{p\times m}$ are constant matrices. The order n of the system ranges from a few tens to several hundreds as in control problems

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for large flexible space structures. A common feature of the model used is that it is high dimensional and display a variety of time scales. If the time scales in the system are well separated, it is possible to eliminate the fast degrees of freedom and to derive low-ordered reduced models, using averaging and homogenization techniques. Homogenization of linear control systems has been widely studied by various authors using different mathematical tools [2, 4, 10, 16]. In many applications, however, an explicit smallness parameter that characterizes the time scales present in the dynamics is not available, so that perturbative methods such as averaging or homogenization can not be applied.

Balanced model reduction going back to Moore [18] provides a rational basis for perturbation approximation [24] and includes easily computable error bounds [12]; see also [3, 25] and the reference given there. The general idea of balanced model reduction is to restrict the system to the subspace of easily controllable and observable states which can be determined by the Hankel singular values associated with the system. All these methods give the stable reduced systems and guarantee the upper bound of the error reduction,i.e that is $x_0 = 0$.

In fact a number of methods have been presented in the literature to reduce the the order of infinite dimentional linear time-invariant systems such as balanced truncation [12], Hankel norm approximation [24] and singular perturbation approximation [20]. All these methods give the stable reduced systems and guarantee the upper bound of the error reduction.

Although balanced truncation and singular perturbation approximation methods give the same of the upper bound of error reduction in the case when the dynamical system is homogeneous, but the characteristics of both methods are contrary to each other [17].

It has been shown that the reduced systems by balanced truncation have a smaller error at high frequencies, and tend to be larger at low frequencies. Furthermore, the reduced systems through the singular perturbation approximation method behave otherwise, i.e. the error goes to zero at low frequencies and tend to be large at high frequencies.

The balanced truncation and Hankel norm approximation techniques have been generalized to infinite dimensional systems [6, 23]. Curtain and Glover [6] generalized the balanced truncation techniques to infinite-dimensional systems and the upper bound of the error reduction can be found in [13].

In [7], it has been shown that the reduced systems through balanced truncation method in infinite dimensional systems preserve the behavior of the original system in infinite frequency. More often this condition is not desirable in applications. Therefore, it is necessary to improve the singular perturbation approximation method so that it can be applied to infinite dimensional systems.

Many of the properties of the singular perturbation approximation method can be connected through balanced reciprocal system as shown in [20].

For finite time-horizon optimal problems, among the most actively investigated singularly perturbed optimal control problems is the linear quadratic regulator problems. Most of these approaches are based on the singularly perturbed differential Riccati equation. An alternative approach via boundary value problems is presented in [21]. Its relationship with the Riccati approach is analyzed in [22].

In this article we introduce one of the most important methods in control problems, that is *The Linear Quadratic Regulator*(LQR). We are interested in the case of the linear quadratic regulator with constrained states and inputs. For closed-loop, we want to use the LQR to find an optimal control that minimizes the objective function which called “the quadratic cost function” with respect to the constraints on the states and the control input. In order to do that we have used formal asymptotes for the Pontryagin maximum principle (PMP) and we introduce an approach using the so called *The Hamiltonian Function* and the underlying algebraic Riccati equation. The theoretical results are validated numerically to show that the model order reduction based on open-loop balancing can also give good closed-loop performance.

The paper is organized as follows: In Section 2 the linear quadratic regulator (LQR) is introduced. Section 3 contains the main theoretical results of the optimal control for the original and reduced systems using the balanced truncation method. Numerical results which shows the validity of theoretical results are presented in section 4 and conclusions are drawn in section 5.

2. Linear quadratic regulator optimal control (LQR)

We start by considering the following continuous linear dynamical system defined as:

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \\ x(0) &= x_0, \end{aligned} \tag{2}$$

where A, B and C are a constant matrices and x, u are the state and the input of the system respectively and $x(0)$ represents the initial condition [8,9].

We assume that the linear system described by equation (2) is controllable and observable.

The quadratic cost function J is defined by the following equation:

$$J = \frac{1}{2}\int_0^\infty \left(y^T y + u^T R u\right) \mathrm{d}t \tag{3}$$

or, equivalently

$$J = \frac{1}{2} \int_0^{\infty} \left(x^T Q x + u^T R u \right) \mathrm{d}t, \tag{4}$$

where $Q = C^T C \geqslant 0$ is a positive semi definite matrix representing the cost penalty of the states and $R > 0$ is a positive definite matrix that represents the cost penalty of the input.

We want to find an optimal control u that minimizes the quadratic cost function J subject to the constraint

$$\dot{x} = Ax + Bu.$$

The optimal control can be denoted by u^* such that:

$$J(u^*) \leqslant J(u), \quad \forall u \in L^2$$

and the constraint equation $\dot{x} = Ax + Bu$ has a solution.

If we substitute the value of u^* in the constraint equation, we have that:

$$\dot{x} = Ax + Bu^*$$

and the optimal solution of this equation is denoted by x^*.

Now, we introduce an approach that depends on the Hamiltonian function defined in the following form:

$$H = \frac{1}{2} \left(x^T Q x + u^T R u \right) + \lambda^T (Ax + Bu), \tag{5}$$

where $\lambda \in \Re^n$ is called the costate variable.

The following theorem describes the way in which we can find the optimal control that minimizes the quadratic cost function J in equations (3) and (4).

Theorem 1 *[14, 19] (Maximum Principle) If x^*, u^* is optimal (or a solution of the LQR), then there exists a solution $\lambda^* \in \Re^n$ such that:*

$$\dot{x} = \frac{\partial H}{\partial \lambda}, \tag{6}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \tag{7}$$

and the minimality condition of the Hamiltonian

$$H(x^*, u^*, \lambda^*) \leqslant H(x^*, u, \lambda^*)$$

holds for all $u \in \Re^m$.

For more details on the proof (see [14, 19]).

If H is a differentiable function, then to minimize H with respect to u we can find our optimal control input.

The following condition must be true to find such u that is:

$$\frac{\partial H}{\partial u} = 0 \tag{8}$$

if we solve equation (8), we obtain the following control:

$$u = -R^{-1}B^T\lambda. \tag{9}$$

From theorem (1) and equation (9), we have the following canonical differential equations that form a linear system (or Hamiltonian system) written as:

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial \lambda}\\ &= Ax - BR^{-1}B^T\lambda, \qquad x(0) = x_0,\\ \dot{\lambda} &= -\frac{\partial H}{\partial x}\\ &= -Qx - A^T\lambda.\end{aligned} \tag{10}$$

Since the terminal cost is not defined, then there is no constraint on the final value of λ.

This is a coupled system, linear in x and λ, of order $2n \times 2n$.

These control equations can be written in matrix form as:

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix}\begin{pmatrix} x \\ \lambda \end{pmatrix}. \tag{11}$$

It is not easy to solve the system described in equation (11), so we guess the solution of this system or the relation between x and λ in the form:

$$\lambda = Px, \tag{12}$$

where $P \in \Re^{n\times n}$.

We introduce now an important differential equation in the linear quadratic regulator problem that is called *Matrix Riccati Equation* (MRE) and to derive this equation, we start from equation (12) and use (10) in the following way:

$$\begin{aligned}\lambda &= Px,\\ \dot{\lambda} &= \dot{P}x + P\dot{X},\\ -Qx - A^T\lambda &= \dot{P}x + P(Ax - BR^{-1}B^T\lambda),\\ -Qx - A^TPx &= \dot{P}x + PAx - PBR^{-1}B^TPx,\end{aligned}$$

$$\dot{P}x + PAx + A^TPx - PBR^{-1}B^TPx + Qx = 0.$$

From the final step, we obtain the MRE written as:

$$\dot{P} = -PA - A^T P + PBR^{-1}B^T P - Q. \tag{13}$$

Since we have an infinite time horizon, there is no information about the terminal cost and hence λ has no constraint. In this case the steady state solution P of a so called Algebraic Riccati Equation (ARE) can be used instead of $P(t)$ [19].

In case when the time approaches infinity, we have:

$$\lim_{t\to\infty} \dot{P} = 0.$$

By using the limit above, we get another differential equation called *Algebraic Riccati Equation* (ARE), written as:

$$PA + A^T P - PBR^{-1}B^T P + Q = 0, \tag{14}$$

where P is the unique positive-definite solution.

We want now to find a state feedback control u that can be used to move any state x to the origin, so we let the system evolve in a closed-loop [11, 19].

If we find the solution P of the ARE (14), then the optimal control u that can be used to minimize the quadratic cost function J is written as:

$$u = -R^{-1}B^T Px. \tag{15}$$

By substituting equation (15) into the original system described by equation (2), we get the following equation:

$$\dot{x} = \left(A - R^{-1}B^T P\right) x. \tag{16}$$

Since the matrix $A - BK$ is stable, we have closed-loop poles formed by the eigenvalues of this matrix [11].

If we solve equation (16) and find the optimal solution x, then we can find our optimal control u that can be used to find a minimum value of the quadratic cost function J described in equations (3) and (4).

The *LQR* method can be illustrated by the following algorithm:

1. We start with the linear dynamical system:
$$\dot{x} = Ax + Bu,$$
$$y = Cx,$$
$$x(0) = x_0.$$
2. We assume that this system is controllable.
3. We define the quadratic cost function as:
$$J = \frac{1}{2}\int_0^\infty \left(x^T Q x + u^T R u\right) \mathrm{d}t.$$
4. We choose $Q = Q^T \geqslant 0$ such that $Q = C^T C$ and $R = R^T > 0$.
5. We find the constant solution P of the ARE:
$$PA + A^T P - PBR^{-1}B^T P + Q = 0.$$
6. We find the optimal control u such that:
$$u = -R^{-1}B^T P x.$$
7. We write the original system in the form:
$$\dot{x} = (A - R^{-1}B^T P)x.$$

3. Optimal control for a reduced system using the balanced truncation method

In [8], we applied the singular perturbation linear quadratic regulator to find an optimal control for the reduced system.

In this section we introduce an approach to find the optimal control of the reduced system using the Balance Truncation optimal control.

Consider the full linear time-invariant dynamical system defined by the following form:

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u, \\ y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix}. \end{aligned} \tag{17}$$

We can rewrite the original system in equation (17) in another form as:

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}z + B_1u, \\ \epsilon\dot{z} &= \epsilon A_{21}x + A_{22}z + \epsilon B_2u. \end{aligned} \tag{18}$$

If we apply the balanced truncation method to reduce the original system described by equation (18), we get the following reduced system form:

$$\begin{aligned} \dot{x}_r &= A_{11}x_r + B_1u_r\,, \\ y_r &= C_1x_r\,. \end{aligned} \tag{19}$$

From equation (19), to find an optimal control for the reduced system, we start by defining the quadratic cost function J for the original system (17) as:

$$J = \frac{1}{2}\int_0^\infty \left(y^Ty + u^TRu\right)\mathrm{d}t \tag{20}$$

or equivalently

$$J = \frac{1}{2}\int_0^\infty \left(x^TQx + u^TRu\right)\mathrm{d}t, \tag{21}$$

where $Q = C^TC \geqslant 0$ and $R > 0$.

Our optimal control u for the original system is defined as:

$$u = -R^{-1}\begin{pmatrix} B_1^T & B_2^T \end{pmatrix} P \begin{pmatrix} x \\ z \end{pmatrix}. \tag{22}$$

The matrix P is the solution of the following Algebraic Riccati Equation:

$$PA + A^TP - PBR^{-1}B^TP + Q = 0. \tag{23}$$

The next step now is to find a reduced Riccati equation for the full Riccati equation (23) when $\epsilon = 0$.

To avoid the unboundness when $\epsilon = 0$, we choose the solution P in the form:

$$P = \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix}. \tag{24}$$

By substituting equation (24) into equation (23), we get:

$$\begin{aligned}&\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} \\ &+\begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & \dfrac{1}{\epsilon}A_{22}^T \end{pmatrix}\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \\ &-\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} R^{-1}\begin{pmatrix} B_1^T & B_2^T \end{pmatrix}\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \\ &+\begin{pmatrix} C_1^T \\ C_2^T \end{pmatrix}\begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0.\end{aligned} \tag{25}$$

After solving equation (25), we obtain the following equations:

$$\begin{aligned}0 &= P_{11}A_{11} + \epsilon P_{12}A_{21} + A_{11}^T P_{11} \\ &\quad + \epsilon A_{21}^T P_{12}^T - (P_{11}B_1 + \epsilon P_{12}B_2)R^{-1}(B_1^T P_{11} + \epsilon B_2^T P_{12}^T) + C_1^T C_1,\end{aligned} \tag{26}$$

$$\begin{aligned}0 &= P_{11}A_{12} + P_{12}A_{22} + \epsilon A_{11}^T P_{12} \\ &\quad + \epsilon A_{21}^T P_{22} - (P_{11}B_1 + \epsilon P_{12}B_2)R^{-1}(\epsilon B_1^T P_{12} + \epsilon B_2^T P_{22}) + C_1^T C_2,\end{aligned} \tag{27}$$

$$\begin{aligned}0 &= \epsilon P_{12}^T A_{11} + \epsilon P_{22}A_{21} + A_{12}^T P_{11} \\ &\quad + A_{22}^T P_{12}^T - (\epsilon P_{12}^T B_1 + \epsilon P_{22}B_2)R^{-1}B_1^T P_{11} + \epsilon B_2^T P_{12}^T) + C_2^T C_1,\end{aligned} \tag{28}$$

$$\begin{aligned}0 &= \epsilon P_{12}^T A_{12} + P_{22}A_{22} + \epsilon A_{12}^T P_{12} \\ &\quad + A_{22}^T P_{22} - (\epsilon P_{12}^T B_1 + \epsilon P_{22}B_2)R^{-1}\epsilon B_1^T P_{12} + \epsilon B_2^T P_{22}) + C_2^T C_2 .\end{aligned} \tag{29}$$

Now, if we set $\epsilon = 0$ in equations (26)–(29), we obtain the following reduced system Riccati Equations:

$$\overline{P}_{11}A_{11} + A_{11}^T\overline{P}_{11}^T - \overline{P}_{11}B_1R^{-1}B_1^T\overline{P}_{11} + C_1^TC_1 = 0, \tag{30}$$

$$\overline{P}_{11}A_{12} + \overline{P}_{12}A_{22} + C_1^TC_2 = 0, \tag{31}$$

$$A_{21}^T\overline{P}_{11} + A_{22}^T\overline{P}_{12} + C_2^TC_1 = 0, \tag{32}$$

$$\overline{P}_{22}A_{22} + A_{22}^T\overline{P}_{22} + C_2^TC_2 = 0. \tag{33}$$

Assumption 1 *The pair* (A_{11}, B_1) *is controllable and* $\overline{P}_{11}$ *is a unique positive semidefinite solution of equation (30) such that:*

$$A_{11} - B_1 R^{-1} B_1^T \overline{P}_{11}$$

is stable.

According to [8], we can use $\overline{P}_{ij}$ instead of P_{ij} to rewrite the feedback control in equation (22) as:

$$\begin{aligned} u &= -R^{-1} \begin{pmatrix} B_1^T & B_2^T \end{pmatrix} \begin{pmatrix} \overline{P}_{11} & \epsilon \overline{P}_{12} \\ \epsilon \overline{P}_{12}^T & \epsilon \overline{P}_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \\ &= -R^{-1} \left(B_1^T \overline{P}_{11} + \epsilon B_2^T \overline{P}_{12} \right) x - R^{-1} \left(\epsilon B_1^T \overline{P}_{12} + \epsilon B_2^T \overline{P}_{22} \right) z. \end{aligned} \tag{34}$$

Using equation (34), we obtain a new form of the original system described by equation (18) such that:

$$\begin{aligned} \dot{x} &= \left(A_{11} - B_1 R^{-1} \left(B_1^T \overline{P}_{11} + \epsilon B_2^T \overline{P}_{12} \right) \right) x \\ &\qquad + \left(A_{12} - B_1 R^{-1} \left(\epsilon B_1^T \overline{P}_{12} + \epsilon B_2^T \overline{P}_{22} \right) \right) z, \\ \epsilon \dot{z} &= \left(\epsilon A_{21} - \epsilon B_2 R^{-1} \left(B_1^T \overline{P}_{11} + \epsilon B_2^T \overline{P}_{12} \right) \right) x \\ &\qquad + \left(A_{22} - \epsilon B_2 R^{-1} \left(\epsilon B_1^T \overline{P}_{12} + \epsilon B_2^T \overline{P}_{22} \right) \right) z. \end{aligned} \tag{35}$$

If the above system is asymptotically stable and equation (34) holds, then we have a solution $x(t)$ and $z(t)$ for this system with $O(\epsilon)$ of the optimal solution [15].

We are going now to define the quadratic cost function for the reduced order model system described in equation (34).

Let $\bar{J}$ be the quadratic cost function of the reduced system in equation (19) defined as:

$$\bar{J} = \frac{1}{2} \int_0^\infty \left(\bar{y}^T \bar{y} + \bar{u}^T \bar{R} \bar{u} \right) \mathrm{d}t \tag{36}$$

or equivalently,

$$\bar{J} = \frac{1}{2} \int_0^\infty \left(\bar{x}^T \bar{Q} \bar{x} + \bar{u}^T \bar{R} \bar{u} \right) \mathrm{d}t, \tag{37}$$

where $\bar{Q} = \bar{C}^T \bar{C} \geqslant 0$ and $\bar{R} = R > 0$.

The optimal feedback control for the reduced order model is defined as:

$$\bar{u} = -\bar{R}^{-1} \bar{B}^T \overline{P} \bar{x}, \tag{38}$$

where $\overline{P}$ is the solution of the Algebraic Riccati Equation for the reduced order model and given as:

$$\overline{P}A_1 + A_1^T\overline{P} - \overline{P}B_1\bar{R}^{-1}B_1^T\overline{P} + C_1^TC_1 = 0. \tag{39}$$

From theorem [1, 8], we see that the two solutions $\overline{P}_{11}$ and $\overline{P}$ are both identical.

Hence we conclude that $\overline{P}_{11}$ is the reduced Riccati Equation (30) and it is the same as $\overline{P}$ which is the solution of the reduced system.

By substituting the feedback control equation (38) into the reduced system (19), we get:

$$\dot{\bar{x}} = \left(A_{11} - B_1R^{-1}B_1^T\overline{P}\right) x_r\,, \tag{40}$$

where we have assumed that the matrix $(A_{11} - \bar{B}R^{-1}B_1^T\overline{P})$ is stable.

If we solve equation (40) of the reduced system, then we can use the solution $x(t)$ to find the optimal control. This optimal control can be used to find the optimality of $\bar{J}$.

4. Numerical illustration

In this section we include all results obtained by the balanced truncation technique to determine the order of the reduced model. We consider the Building model system from the SLICOT library [5] with $n = 48$ degrees of freedom. We start by applying the standard balanced truncation to the system with homogeneous initial conditions $x = 0$. The optimal controls U_1 for the original system and u_r for the reduced order system are computed and the size of the reduced model is $r_s = 2$. The optimal control is computed by using the results in section (3). The solution of the Riccati equation P of the full system is computed and used to find the value of U_1. We apply the balanced truncation approach presented in section (3) to find the solution of the Riccati equation P_r of the reduced system. Since the first block P_{11} of P is equal to the value of P_r, so we can extend P using P_r as the first block and the rest blocks are zero to obtain a new solution of the Riccati equation denoted by $\widetilde{P}_{11}$.

Figure (1) represents the plots of the two optimal controls U_1, u_r and $(U_1 - u_r)$ using the balanced truncation method.

Finally, Table (1) contains the values of $\|U_1 - u_r\|_{L_2}$ and $\|P_{11} - \widetilde{P}_{11}\|_{L_2}$ by applying the balanced truncation to the building model.

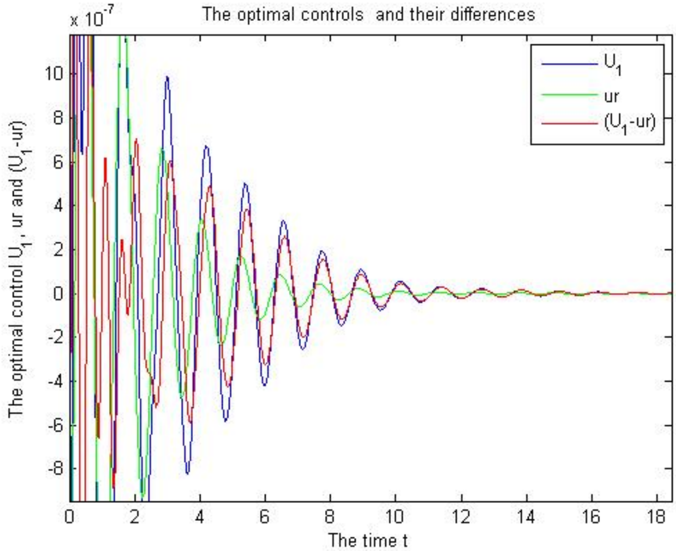

Figure 1: The optimal controls of the building model

Table 1: The L^2 norm of $(U_1 - u_r)$ and $(P_{11} - \widetilde{P}_{11})$ of the building model

r_s	$\|U_1 - u_r\|_{L_2}$ BT	$\left\|P_{11} - \widetilde{P}_{11}\right\|_{L_2}$
2	8.0741×10^{-15}	3.4098×10^{-9}
4	1.1778×10^{-15}	3.2719×10^{-9}
6	3.6256×10^{-16}	3.7787×10^{-9}
8	8.7808×10^{-17}	1.3664×10^{-9}
10	9.0881×10^{-17}	1.2910×10^{-9}
12	1.6477×10^{-18}	1.1948×10^{-9}
18	1.9097×10^{-19}	3.5177×10^{-10}
20	5.2767×10^{-20}	1.7128×10^{-10}
26	1.6078×10^{-21}	4.0415×10^{-12}

5. Conclusions

Many physical, mechanical and artificial processes can be described by dynamical systems, which can be used for simulation or control. In this article, we have discussed the balanced truncation of the linear quadratic regulator (LQR) systems on open-loop balancing of controllability and observability properties.

In fact, for open-loop control problems, the L^2 error bound for both the balanced truncation (BT) and the singular perturbation approximation (SPA) for systems with homogeneous initial condition is the same. For finite time-horizon optimal problems, among the most actively investigated singular perturbed optimal control problems is the linear quadratic regulator problems. Most of these approaches are based on the singularly perturbed differential Riccati equation. In this work, our main approach is to use the LQR to find an optimal control that minimizes the quadratic cost function with respect to the constrains on the states and the control input. This approach shows clearly that the balanced truncation gives good closed-loop performance in comparison to the singular perturbation. Moreover, we use the solutions of the Riccati equations to find the L^2 norm of the difference between the optimal control for both original and reduced systems. The credibility of this approach has been tested by numerical experiment, illustrating that the optimal control of the reduced system can be used to approximate the optimal control of the original system. Even though our approach remained purely formal, we have given some numerical evidence that open-loop balancing can give good closed-loop performance.

References

[1] A. Daraghmeh and N. Qatanani: Error bound for non-zero initial condition using the singular perturbation approximation method. *Mathematics*, **6** (2018), 1–16.

[2] O. Alvarez and M. Bardi: Viscosity solutions methods for singular perturbations in deterministic and stochastic control. *SIAM Journal on Control and Optimization*, **40** (2002), 1159–1188.

[3] A.C. Antoulas: *Approximation of large-scale dynamical systems*. Advances in design and control, Society for Industrial and Applied Mathematics, 2005.

[4] A. Bensoussan and G. Blankenship: *Singular perturbations in stochastic control*. In: Singular Perturbations and Asymptotic Analysis in Control Systems, Springer, 1987, 171–260.

[5] Y. Chahlaoui and P. Van Dooren: *Benchmark examples for model reduction of linear time invariant dynamical systems*. In: Dimension Reduction of Large-Scale Systems, P. Benner, D.C. Sorensen, and V. Mehrmann, eds., vol. 45 of Lect. Notes Comput. Sci. Eng., Springer, Berlin, 2005, 379–392.

[6] R.F. Curtain and K. Glover: *Balanced realisations for infinite dimensional systems*. In: Operator theory and system, Proc. Workshop Amesterdam (1986), 87–104.

[7] —: *An Introduction to Infinite-Dimensional Systems*. Springer-Verlag. New York, 1995.

[8] A. Daraghmeh: *Model Order Reduction of Linear Control Systems: Comparison of Balanced Truncation and Singular Perturbation Approximation with Application to Optimal Control*. PhD Thesis, Berlin Freie Universität,Germany, 2016.

[9] A. Daraghmeh, C. Hartmann, and N. Qatanan: Balanced model reduction of linear systems with nonzero initial conditions: Singular perturbation approximation. *Applied Mathematics and Computation*, **353** (2019), 295–307.

[10] L.C. Evans: The perturbed test function method for viscosity solutions of nonlinear pde. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, **111** (1989), 359–375.

[11] S.A. Ghoreishi, M.A. Nekoui, and S.O. Basiri: Optimal design of lqr weighting matrices based on intelligent optimization methods. *International Journal of Intelligent Information Processing*, **2** (2011).

[12] K. Glover: All optimal Hankel-norm approximations of linear multivariable systems and their l_∞-error bounds†. *International Journal of Control*, **39** (1984), 1115–1193.

[13] K. Glover, R.F. Curtain, and J.R. Partington: Realisation and approximation of linear infinite-dimensional systems with error bounds. *SIAM Journal on Control and Optimization*, **26** (1988), 863–898.

[14] G.E. Knowles: *An Introduction to Applied Optimal Control*. Mathematics in Science and Engineering 159, Academic Press, 1981.

[15] P.V. Kokotovic, R.E. O'malley, and P. Sannuti: Singular perturbations and order reduction in control theory: an overview. *Automatica*, **12** (1976), 123–132.

[16] P.-L. Lions and P.E. Souganidis: Correctors for the homogenization of Hamilton-Jacobi equations in the stationary ergodic setting. *Communications on Pure and Applied Mathematics*, **56** (2003), 1501–1524.

[17] Y. Liu and B.D. Anderson: Singular perturbation approximation of balanced systems. *International Journal of Control*, **50** (1989), 1379–1405.

[18] B. Moore: Principal component analysis in linear systems: Controllability, observability, and model reduction. *IEEE Transactions on Automatic Control*, **26** (1981), 17–32.

[19] R. M. Murray: *Optimization-based control*. California Institute of Technology, CA, 2009.

[20] G. Muscato, G. Nunnari, and L. Fortuna: Singular perturbation approximation of bounded real balanced and stochastically balanced transfer matrices. *International Journal of Control*, **66** (1997), 253–270.

[21] R. O'Malley, Jr.: The singularly perturbed linear state regulator problem. *SIAM Journal on Control*, **10** (1972), 399–413.

[22] R.E. O'Malley Jr: On two methods of solution for a singularly perturbed linear state regulator problem. *SIAM Review*, **17** (1975), 16–37.

[23] A. Sasane: *Hankel norm approximation for infinite-dimensional systems*, vol. 277. Springer Science & Business Media, 2002.

[24] S. Skogestad and I. Postlethwaite: *Multivariable feedback control: analysis and design*, vol. 2. Wiley New York, 2007.

[25] K. Zhou, J.C. Doyle, and K. Glover: *Robust and optimal control*. Prentice Hall, 1998.