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Energy-optimal current distribution in an electrical network – controlling by the differential or the integral systems

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Abstract. In the complex RLC network, apart from the currents flows arising from the normal laws of Kirchhoff, other distributions of current, resulting from certain optimization criteria, may also be received. This paper is the development of research on distribution that meets the condition of the minimum energy losses within the network called energy-optimal distribution. Optimal distribution is not reachable itself, but in order to trigger it off, it is necessary to introduce the control system in current-dependent voltage sources vector, entered into a mesh set of a complex RLC network. For energy-optimal controlling, to obtain the control operator, the inversion of **R**(s) operator is required. It is the matrix operator and the dispersive operator (it depends on frequency). Inversion of such operators is inconvenient because it is algorithmically complicated. To avoid this the operator **R**(s) is replaced by the **R'** operator which is a matrix, but non-dispersive one (it does not depend on *s*). This type of control is called the suboptimal control. Therefore, it is important to make appropriate selection of the **R'** operator and hence the suboptimal control. This article shows how to implement such control through the use of matrix operators of multiple differentiation or integration. The key aspect is the distribution of a single rational function $H(s)$ in a series of 's' or 's^{-1}'. The paper presents a new way of developing a given, stable rational transmittance with real coefficients in power series of 's/s^{-1}'. The formulas to determine values of series coefficients (with 's/s^{-1}') have been shown and the conditions for convergence of differential/integral operators given as series of 's/s^{-1}' have been defined.

Key words: principle of minimum energy losses, optimal and suboptimal control, power series, operators of multiple differentiation or integration, decomposition of a rational function.

1. Introduction. Energy-optimal distribution and control systems

In DC circuits there is the minimum energy principle, according to which the currents distribution in a complex network are such that the total energy losses are minimal [1, 2]. However, this rule usually does not work in the sinusoidal current circuits [3]. On the other hand, in non-sinusoidal signals domain, the term "reactive power" makes no sense, which means that this term should not be used during testing the quality of electrical energy distribution in the network [4, 5]. However, the compensation problems aimed at resetting the indicator of reactive power can be solved as optimization tasks consisting in minimizing energy losses in the network or as related tasks of minimizing the RMS value of currents [6, 7]. Study [8] showed that in the complex RLC network, besides the currents flows arising from the normal laws of Kirchhoff – called current divider – other distributions of current, resulting from certain optimization criteria, may also be received through appropriate controls.

The distribution that meets the condition of the minimum energy losses within the network was examined (energy-optimal distribution). In Fig. 1 the RLC network with power given as a vector of current signals $\boldsymbol{i}_0$ is shown. Distribution of mesh currents within the network is determined by the vector of current signals $\boldsymbol{i}$.

The network is characterized by the so-called internal operators matrix $\mathbf{Z}(s)$ ($s = d/dt$), contact operators matrix $\mathbf{Z}_0(s)$ and external operators matrix $\mathbf{Z}_{00}(s)$.

The equations of network operator assume the form:

$$\begin{aligned} \mathbf{Z}\mathbf{i} \; - \mathbf{Z}_0\mathbf{i}_0 \; &= \mathbf{0} \\ -\mathbf{Z}_0^{\mathrm{T}}\mathbf{i} + \mathbf{Z}_{00}\mathbf{i}_0 &= \mathbf{u}_0 \end{aligned} \tag{1}$$

($\mathbf{0}$ – zero vector (or zero operator), T – a sign of transposition).

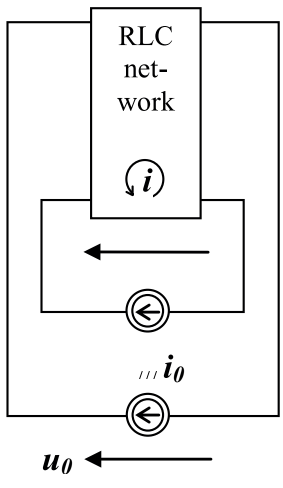

Fig. 1. The complex network with multicurrent power; $\boldsymbol{i}$ – internal mesh currents vector; $\boldsymbol{i}_0$ external current vector

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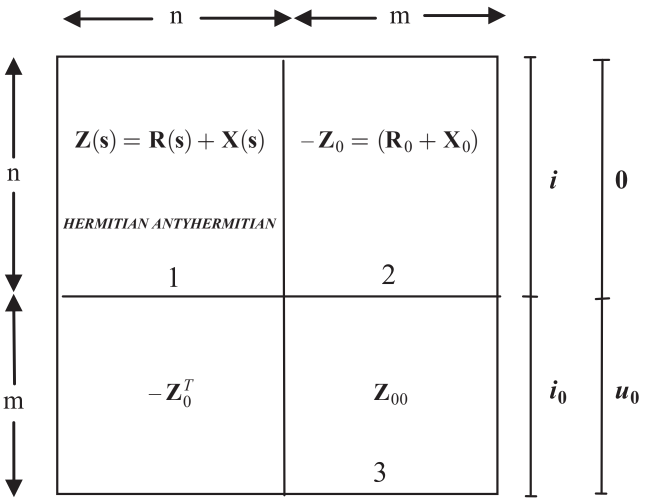

Fig. 2. Scheme the system of equations (1);
1 – internal operators matrix, 2 – contact operators matrix,
3 – external operators matrix, **0** – vector (or operator) zero,
T – a sign of transposition

In Fig.2 the structure of system of equations (1) is illustrated. In this figure the sizes of the matrix and vectors are shown.

Current divider and energy-optimal current distribution are described by the two matrix-similar systems of operator equations:

$$\mathbf{Z}(s)\mathbf{i} - \mathbf{Z}_0(s)\mathbf{i}_0 = \mathbf{0} \tag{2}$$

$$\mathbf{R}(s)\mathbf{i} - \mathbf{R}_0(s)\mathbf{i}_0 = \mathbf{0} \tag{3}$$

where the matrix operators of impedance $\mathbf{Z}(s)$, $\mathbf{R}(s)$ and $\mathbf{Z_0}(s)$, $\mathbf{R_0}(s)$ are related in the way that:

$$\mathbf{Z}(s) = \mathbf{R}(s) + \mathbf{X}(s) \tag{4}$$

where:

$$\mathbf{Z}(-s) = \mathbf{R}(s); \qquad \mathbf{X}(-s) = -\mathbf{X}(s) \tag{5}$$

which makes that distribution (4) on Hermitian and skew-Hermitian part is a unique one:

$$\begin{aligned} \mathbf{R}(s) &= \frac{1}{2}[\mathbf{Z}(s) + \mathbf{Z}(-s)]; \\ \mathbf{X}(s) &= \frac{1}{2}[\mathbf{Z}(s) - \mathbf{Z}(-s)]. \end{aligned} \tag{6}$$

In this way the systems of equations (2) – ($\mathbf{Z}$, $\mathbf{Z_0}$) type and (3) – ($\mathbf{R}$, $\mathbf{R_0}$) type are identical in matrix way, but in an operator way equations (3) are the Hermitian variant of equations (2).

Solutions of the systems of equations (2) and (3) are: mesh currents vector of current divider

$$\mathbf{i}^{dz} = [\mathbf{Z}(s)]^{-1}\mathbf{Z}_0(s)\mathbf{i}_0 \tag{7}$$

and energy-optimal mesh currents distribution as a vector

$$\mathbf{i}^{opt} = [\mathbf{R}(s)]^{-1}\mathbf{R}_0(s)\mathbf{i}_0 \tag{8}$$

Optimal distribution itself is not as reachable as the distribution of current divider but in order to trigger it off, it is necessary to carry out the optimal control by the control operator $\mathbf{X}_c(s)$, generating signal of the voltage source $\mathbf{e}^c$ appropriately distributed in the internal meshes of network:

$$\mathbf{e}^c = \mathbf{X}_c(s)\mathbf{i_0} \tag{9}$$

where [8]:

$$\mathbf{X}_c(s) = \mathbf{X}(s)[\mathbf{R}(s)]^{-1}\mathbf{R_0}(s) - \mathbf{X_0}(s). \tag{10}$$

So, $\mathbf{X_c}(s)$ is a skew-Hermitian matrix operator processing the signal-vector $\mathbf{i_0}$ in the voltage signal-vector $\mathbf{e}^c$:

$$\mathbf{X}_c(-s) = -\mathbf{X}_c(s)$$

For DC networks, distributions $\mathbf{i}^{dz}$ and $\mathbf{i}^{opt}$ overlap because matrix operators $\mathbf{Z}(s)$ and $\mathbf{R}(s)$ overlap for $s = 0$ and it is consistent with the principle that in the DC circuits, currents distribution is such that the total energy losses are minimal [1, 2].

It also appears that energy-optimal distributions and current divider distributions can be the same without control when the deviation operator disappears:

$$\boldsymbol{\Delta}(s) = [\mathbf{R}(s)]^{-1}\mathbf{R_0}(s) - [\mathbf{Z}(s)]^{-1}\mathbf{Z_0}(s) \tag{11}$$

which is related to the optimal control operator by the formula:

$$\mathbf{X}_c(s) = \mathbf{Z}(s)\boldsymbol{\Delta}(s). \tag{12}$$

From equations (10) and (12) the following theorem of equivalence is derived:

$$\begin{array}{ccc} \boldsymbol{\Delta} = \mathbf{0} & & \mathbf{X}\mathbf{R}^{-1}\mathbf{R}_0 = \mathbf{X}_0 \\ \Updownarrow & \Leftrightarrow & or \\ \mathbf{X}_c = \mathbf{0} & & \mathbf{R}\mathbf{X}^{-1}\mathbf{X}_0 = \mathbf{R}_0\,. \\ \text{for each } s & & \end{array} \tag{13}$$

Networks fulfilling the condition (13) reach the energy-optimal current distribution of $\mathbf{i_0}$ without control. In study [8] such networks were called naturally energy-optimal.

Example 1.
A ladder structure – a long line in a discrete model (Fig. 3). The figure shows a circuit which is powered from both sides.

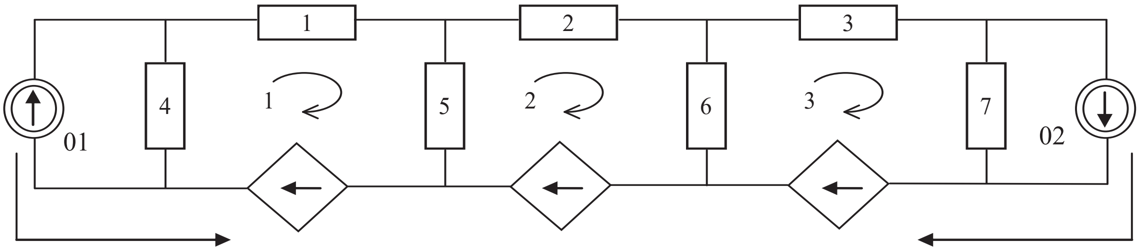

Fig. 3. A ladder system powered from two sides and optimal control sources distributed in the meshes of the ladder circuit

In the diagram, inner meshes 1, 2, 3 and external 01, 02 was selected. The structure of the system of mesh equations is:

Z	1	2	3	01	02 ($-\mathbf{Z}_0$)			
1	1, 4, 5	–5		–4		1 ($\boldsymbol{i}$)		
2	–5	2, 5, 6	–6			2		
3		–6	3, 6, 7		–7	3	=	
	X			X		01 ($\boldsymbol{i}_0$)		x
						02		x

The structure of the current divider equations $\mathbf{Zi} - \mathbf{Z_0 i_0} = \mathbf{0}$ is visible above. Empty places indicate 'zero' operators or signals, places with 'x' contain operators or signals that do not take part in solving the system of equations.

For the branch structure of RLC type:

$\mathbf{r}$ $\quad s\mathbf{L}$ $\quad s^{-1}\boldsymbol{\Sigma}$

is obtained

$$\mathbf{Z}(\mathrm{s}) = \mathbf{R}(\mathrm{s}) + \mathbf{X}(\mathrm{s}) = \mathbf{r} + \mathrm{s}\mathbf{L} + \mathrm{s}^{-1}\boldsymbol{\Sigma}$$

where:
r – matrices of mesh resistances
L – matrices of mesh inductances
Σ – matrices of mesh elastances (the inverse of the capacity).

The optimal control operator have the following form:

$$\mathbf{X}_c(s) = s\left(\mathbf{L}\mathbf{r}^{-1}\mathbf{r}_0 - \mathbf{L}_0\right)$$

or

$$\mathbf{X}_c(s) = s^{-1}\left(\boldsymbol{\Sigma}\mathbf{r}^{-1}\mathbf{r}_0 - \boldsymbol{\Sigma}_0\right)$$

or

$$\mathbf{X}_c(s) = s\left(\mathbf{L}\mathbf{r}^{-1}\mathbf{r}_0 - \mathbf{L}_0\right) + s^{-1}\left(\boldsymbol{\Sigma}\mathbf{r}^{-1}\mathbf{r}_0 - \boldsymbol{\Sigma}_0\right).$$

The condition of the naturally energy-optimal network ($\boldsymbol{\Delta} = \mathbf{0}$) takes the form of the following matrix structure:

L (**Σ**)			**r**		-1	$\mathbf{r}_0$			$\mathbf{L}_0$ ($\boldsymbol{\Sigma}_0$)	
1, 4, 5	–5		1, 4, 5	–5		4			4	
–5	2, 5, 6	–6	–5	2, 5, 6	–6			=		
	–6	3, 6, 7		–6	3, 6, 7		7			7

Figure 3 also shows optimal control implemented using voltage sources controlled by the currents 01, 02 distributed in the meshes of the ladder circuit.

2. Suboptimal control. Decomposition of the matrix-rational impedance operators in a power series of s and s^{-1}

As shown in equation (10) to designate the key control operator $\mathbf{X}_c(s)$ for the approach presented here, one needs to carry out a complicated operation to reverse the dispersive matrix operator (it depends on s) $\mathbf{R}(s)$. This can be avoided by using so-called suboptimal control, defined by the operator [9]:

$$\mathbf{e}^{sub} = \left(\Delta\mathbf{R}(s) + \mathbf{X}_{sub}(s)\right)\mathbf{i}_0 \tag{14}$$

where:

$$\Delta\mathbf{R}(s) = \mathbf{R}(s)(\mathbf{R}')^{-1}\mathbf{R}'_0 - \mathbf{R}_0(s) \tag{15}$$

is the differential operator of mismatch resistance and:

$$\mathbf{X}_{sub}(s) = \mathbf{X}(s)(\mathbf{R}')^{-1}\mathbf{R}'_0 - \mathbf{X}_0(s) \tag{16}$$

is the suboptimal control operator.

Independent of s the matrix operator $\mathbf{R}'$ will replace the dispersive operator $\mathbf{R}(s)$.

As a result of such a simplification the full control operator $[\Delta\mathbf{R}(s) + \mathbf{X}_{sub}(s)]$ ceases to be skew-Hermitian, which makes that the property of skew-Hermitian operators is lost, namely energy-neutral feature [10, 11].

Therefore, making appropriate selection of the $\mathbf{R}'$ operator is important and hence the suboptimal control. One of the many possibilities of solving this issue is the distribution of a matrix rational function in the power series of s or s^{-1} [12–15]. It provides a simple way to separate impedance operators into Hermitian and skew-Hermetian parts. As a result, one can determine non-dispersive resistance components as zero terms of the appropriate power series.

The matrix-rational operators type $\mathbf{Z}(s)$ and $\mathbf{Z_0}(s)$ in the system of equations (2) (see also Fig. 2) are decomposed in a power series of s:

$$\begin{aligned}\mathbf{Z}(s) &= \mathbf{R}(s) + \mathbf{X}(s) = \\ &= \mathbf{r} + \sum_{n=1}^{\infty}\mathbf{r}_n s^{2n} + s\left(\mathbf{L} + \sum_{n=1}^{\infty}\mathbf{L}_n s^{2n}\right) \\ \mathbf{Z}_0(s) &= \mathbf{R}_0(s) + \mathbf{X}_0(s) = \\ &= \mathbf{r}^0 + \sum_{n=1}^{\infty}\mathbf{r}_n^0 s^{2n} + s\left(\mathbf{L}^0 + \sum_{n=1}^{\infty}\mathbf{L}_n^0 s^{2n}\right)\end{aligned} \tag{17}$$

or in a power series of s^{-1}:

$$\begin{aligned}\mathbf{Z}(s) &= \mathbf{R}(s) + \mathbf{X}(s) = \\ &= \mathbf{r} + \sum_{n=1}^{\infty}\mathbf{r}_n s^{-2n} + s^{-1}\left(\boldsymbol{\Sigma} + \sum_{n=1}^{\infty}\boldsymbol{\Sigma}_n s^{-2n}\right) \\ \mathbf{Z}_0(s) &= \mathbf{R}_0(s) + \mathbf{X}_0(s) = \\ &= \mathbf{r}^0 + \sum_{n=1}^{\infty}\mathbf{r}_n^0 s^{-2n} + s^{-1}\left(\boldsymbol{\Sigma}^0 + \sum_{n=1}^{\infty}\boldsymbol{\Sigma}_n^0 s^{-2n}\right)\end{aligned} \tag{18}$$

where:

$\mathbf{r}, \mathbf{r}_n, \mathbf{r}^0, \mathbf{r}_n^0$ – resistance matrices;

$\mathbf{L}, \mathbf{L}_n, \mathbf{L}^0, \mathbf{L}_n^0$ – inductance matrices;

$\boldsymbol{\Sigma}, \boldsymbol{\Sigma}_n, \boldsymbol{\Sigma}^0, \boldsymbol{\Sigma}_n^0$ – elastance matrices (the inverse of the capacity).

With a view to the suboptimal control (see formulas (14, 15 and 16)), assuming the nondispersive operator $\mathbf{R'}$ as $\mathbf{r}$ ($\mathbf{R'} = \mathbf{r}$), the following is obtained:

– the suboptimal control operators

$$\begin{aligned}\mathbf{X}_{sub}(s) &= \mathbf{X}(s)\mathbf{r}^{-1}\mathbf{r}^0 - \mathbf{X}^0(s) = s\Big[\left(\mathbf{L}\mathbf{r}^{-1}\mathbf{r}^0 - \mathbf{L}^0\right) + \\ &+ \sum_{n=1}^{\infty}s^{2n}\left(\mathbf{L}_n\mathbf{r}^{-1}\mathbf{r}^0 - \mathbf{L}_n^0\right)\Big]\end{aligned} \tag{19}$$

for s decomposition, or

$$\begin{aligned}\mathbf{X}_{sub}(s) = s^{-1}\Big[&\left(\boldsymbol{\Sigma}\mathbf{r}^{-1}\mathbf{r}^0 - \boldsymbol{\Sigma}^0\right) + \\ &+ \sum_{n=1}^{\infty}s^{-2n}\left(\boldsymbol{\Sigma}_n\mathbf{r}^{-1}\mathbf{r}^0 - \boldsymbol{\Sigma}_n^0\right)\Big]\end{aligned} \tag{20}$$

for s^{-1} decomposition,

– and the differential operators of mismatch resistance

$$\begin{aligned}\Delta\mathbf{R}(s) &= \mathbf{R}(s)\mathbf{r}^{-1}\mathbf{r}^0 - \mathbf{R}_0(s) = \left(\mathbf{r} + \sum_{n=1}^{\infty}\mathbf{r}_n s^{2n}\right)\mathbf{r}^{-1}\mathbf{r}^0 - \mathbf{r}^0 - \\ &- \sum_{n=1}^{\infty}\mathbf{r}_n^0 s^{2n} = \sum_{n=1}^{\infty}s^{2n}\left(\mathbf{r}_n\mathbf{r}^{-1}\mathbf{r}^0 - \mathbf{r}_n^0\right)\end{aligned}$$

or

$$\Delta\mathbf{R}(s) = \sum_{n=1}^{\infty}s^{-2n}\left(\mathbf{r}_n\mathbf{r}^{-1}\mathbf{r}^0 - \mathbf{r}_n^0\right).$$

They all are the matrix operators of multiple differentiation or integration wherein $\mathbf{X}_{sub}(s)$ operators are odd operators (skew-Hermitian) and $\Delta\mathbf{R}(s)$ operators are even operators (Hermitian).

Decomposition of a single rational function $H(s)$ in a power series of s, which is essential in further proceedings, is carried out as follows:

$$\begin{aligned}H(s) &= \frac{b_0 + b_1 s + b_2 s^2 + \ldots + b_{N-1}s^{N-1}}{a_0 + a_1 s + a_2 s^2 + \ldots + a_N s^N} = \\ &= x_0 + x_1 s + x_2 s^2 + x_3 s^3 + \ldots\end{aligned}$$

The set of searched coefficients $\{x_n\}_{n=1}^{\infty}$ meets the system of equations:

$$\begin{aligned}&a_0 x_0 = b_0 \\ &a_0 x_1 + a_1 x_0 = b_1 \\ &a_0 x_2 + a_1 x_1 + a_2 x_0 = b_2 \\ &\ldots \qquad \ldots \\ &a_0 x_{N-1} + a_1 x_{N-2} + a_2 x_{N-3} + \ldots + a_{N-1}x_0 = b_{N-1} \\ &\ldots \qquad \ldots \\ &a_0 x_{n+N} + a_1 x_{n+N-1} + a_{2\,n+N-2} + \ldots + a_N x_n = 0\end{aligned} \tag{21}$$

From the first N – equations of equations system (21) the initial values of sequence $\{x_n\}_{n=0}^{\infty}$ are determined:

$$\{x_0, x_1, \ldots, x_{N-1}\}.$$

The rest of equations are recursive equations from which the remainder values of the sequence $\{x_n\}_{n=0}^{\infty}$ are determined by substituting:

$$x_n = p^{-n}.$$

This leads to a characteristic equation of the system with the rational transmittance $H(s)$:

$$\left\{a_0 + a_1 p + a_2 p^2 + \ldots + a_N p^N = 0\right\}. \tag{22}$$

The roots of characteristic equation (22):

$$\{p_1, p_2, p_3, \ldots, p_N,\}$$

are simultaneously poles of $H(s)$ (the eigenvalues). The general solution of the recursive equation (21) has the form:

$$x_n = \sum_{m=1}^{N} c_m p_m^{-n} \tag{23}$$

where a set of constant $\{c_m\}_{m=1}^{N}$ is determined from the the linear equations:

$$\sum_{m=1}^{N} c_m p_m^{-n} = x_n \qquad \text{for } n = 0, 1, 2, ..., N-1 \tag{24}$$

The same rational function can be decomposed in a series 's^{-1}':

$$H(s) = \frac{b_{N-1} + b_{N-2}s^{-1} + b_{N-3}s^{-2} + \ldots + b_0 s^{-(N-1)}}{a_N + a_{N-1}s^{-1} + a_{N-2}s^{-2} + \ldots + a_0 s^{-N}} s^{-1} =$$
$$= \left(y_0 + y_1 s^{-1} + y_2 s^{-2} + y_3 s^{-3} + \ldots\right)s^{-1}.$$

The searched coefficients set of the series $\{y_n\}_{n=0}^{\infty}$ meets N – equations of initial conditions:

$$\begin{aligned}
&a_N y_0 = b_{N-1}\\
&a_N y_1 + a_{N-1} y_0 = b_{N-2}\\
&a_N y_2 + a_{N-1} y_1 + a_{N-2} y_0 = b_{N-3}\\
&\ldots \qquad\qquad \ldots\\
&a_N y_{N-1} + a_{N-1} y_{N-2} + a_{N-2} y_{N-3} + \ldots + a_1 y_0 = b_0
\end{aligned} \tag{25}$$

from which the values of $\{y_n\}_{n=0}^{N-1}$ and recursive equation are determined:

$$a_N y_{n+N} + a_{N-1} y_{n+N-1} + a_{N-2} y_{n+N-2} + \ldots + a_0 y_n = 0$$

which, after substituting $x_n = p^n$ transforms into the characteristic equation (22) so in the same equation as for the 's' decomposition. Thus the general form of the series $\{y_n\}_{n=0}^{\infty}$ takes a form:

$$y_n = \sum_{m=1}^{N} d_m p_m^n \tag{26}$$

where the weighting coefficients $\{d_m\}_{m=1}^{N}$ satisfy the system of linear equations:

$$\sum_{m=1}^{N} p_m^n d_m = y_n \qquad \text{for } n = 0, 1, 2, ..., N-1 \tag{27}$$

and the $\{p_m\}_{m=1}^{n}$ is the set of poles of the rational function $H(s)$.

Example 2.
A special role is played by the 2nd order system, for which:

$$H(s) = \frac{b_0 + b_1 s}{a_0 + a_1 s + a_2 s^2} = s^{-1}\frac{b_1 + b_0 s^{-1}}{a_2 + a_1 s^{-1} + a_0 s^{-2}}$$

with real coefficients, but such that a_0, a_1, a_2 have the same sign ($a_0 \neq 0$, $a_2 \neq 0$). The 's', 's^{-1}' decompositions then proceed according to the following scheme:

$$H(s) = x_0 + x_1 s + x_2 s^2 + \ldots =$$
$$= \left(y_0 + y_1 s^{-1} + y_2 s^{-2} + \ldots\right)s^{-1}$$

The common characteristic equation and poles can be expressed in the form:

$$a_0 + a_1 p + a_2 p^2 = 0$$

hence

$$p = \frac{a_1}{2} + j\sqrt{a_0 a_2 - \left(\frac{a_1}{2}\right)^2}$$
$$\{p_1, p_2\} = \{p, p^*\} \text{ so that } |p| = \sqrt{a_0 a_2}$$

Note
When:

$$p = -\frac{a_1}{2} + j\sqrt{(-1)\left[\left(\frac{a_1}{2}\right)^2 - a_0 a_2\right]} =$$
$$= -\frac{a_1}{2} \pm j^2\sqrt{\left(\frac{a_1}{2}\right)^2 - a_0 a_2} =$$
$$= -\frac{a_1}{2} \pm \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0 a_2} < 0.$$

Initial conditions and the general solution of recursive equations for sequences $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ take the form:

$$\begin{matrix} a_0 x_0 = b_0 \\ a_0 x_1 + a_1 x_0 = b_1 \end{matrix} \;\rightarrow\; \begin{matrix} x_0 = \dfrac{b_0}{a_0} \\ x_1 = \dfrac{a_0 b_1 - a_1 b_0}{a_0^2} \end{matrix}$$

for the 's' decomposition, and

$$\begin{matrix} a_2 y_0 = b_1 \\ a_2 y_1 + a_1 y_0 = b_0 \end{matrix} \;\rightarrow\; \begin{matrix} y_0 = \dfrac{b_1}{a_2} \\ y_1 = \dfrac{a_2 b_0 - a_1 b_1}{a_2^2} \end{matrix}$$

for the 's^{-1}' decomposition.
Hence:

$$x_n = c_1 p^{-n} + c_2 (p^*)^{-n}$$

wherein the coefficients c_1, c_2 satisfy the system of equations

$$\begin{matrix} c_1 + c_2 = x_0 \\ p^{-1}c_1 + (p^*)^{-1}c_2 = x_1 \end{matrix} \rightarrow c_{1,2} = \frac{W_{1,2}}{W}$$

$$W_1 = \begin{vmatrix} x_0 & 1 \\ x_1 & (p^*)^{-1} \end{vmatrix} = (p^*)^{-1}x_0 - x_1$$

$$W_2 = \begin{vmatrix} 1 & x_0 \\ p^{-1} & x_1 \end{vmatrix} = x_1 - p^{-1}x_0 = -W_1^*$$

$$W = (p^*)^{-1} - p^{-1} = -W^*.$$

So occurs:

$$c_1^* = \frac{W_1^*}{W^*} = \frac{-W_2}{-W} = c_2$$

On the other hand:

$$y_n = d_1 p^n + d_2 (p^*)^n$$

where:

$$\begin{matrix} d_1 + d_2 = y_0 \\ pd_1 + p^* d_2 = y_1 \end{matrix} \rightarrow d_{1,2} = \frac{W_{1,2}}{W}$$

$$W_1 = p^* y_0 - y_1$$

$$W_2 = -(py_0 - y_1) = -W_1^*$$

$$W = p^* - p = -W^*$$

and hence: $d_1^* = d_2$.

Thus, the common expansion of s/s^{-1} will take the following form:

$$\begin{aligned} H(s) &= \sum_{n=0}^{\infty}\left(cp^{-n} + c^*(p^*)^{-n}\right)s^n = \\ &= \sum_{n=0}^{\infty}\left[2|c||p|^{-n}\cos(\angle c - n\angle p)^{-n}\right]s^n = \\ &= \sum_{n=0}^{\infty}\left(c\left(\frac{s}{p}\right)^n + c^*\left(\frac{s}{p^*}\right)^n\right) = \\ &= s^{-1}\sum_{n=0}^{\infty}\left(dp^n + d^*(p^*)^n\right)s^{-n} = \\ &= s^{-1}\sum_{n=0}^{\infty}\left[2|d||p|^n\cos(\angle d + n\angle p)\right]s^{-n} = \\ &= s^{-1}\sum_{n=0}^{\infty}\left(d\left(\frac{p}{s}\right)^n + d^*\left(\frac{p^*}{s}\right)^n\right). \end{aligned}$$

The general s/s^{-1} expansion of a rational function of any order has the following form:

$$\begin{aligned} H(s) &= \sum_{n=0}^{\infty}\left(\sum_{m=1}^{N} c_m p_m^{-n}\right)s^n = \\ &= \sum_{m=1}^{N} c_m \sum_{n=0}^{\infty}\left(\frac{s}{p_m}\right)^n = \\ &= s^{-1}\sum_{n=0}^{\infty}\left(\sum_{m=1}^{N} d_m p_m^n\right)s^{-n} = \\ &= s^{-1}\sum_{m=1}^{N} d_m \sum_{n=0}^{\infty}\left(\frac{p_m}{s}\right)^n. \end{aligned} \tag{28}$$

3. The convergence of s/s^{-1} series

The convergence of operators (differential/integral) set by series 's/s^{-1}' (28) needs to be tested by a harmonic signal forcing, substituting $s = j\omega$.

Hence the convergence conditions obtained:

– for the series 's'

$$\left|\frac{\omega}{p_m}\right| < 1 \quad \rightarrow \quad \omega < \min_{1 \le m \le N}\{|p_m|\} \tag{29}$$

– and for the series 's^{-1}'

$$\left|\frac{p_m}{\omega}\right| < 1 \quad \rightarrow \quad \omega > \max_{1 \le m \le N}\{|p_m|\}. \tag{30}$$

Particularly for the 1st and the 2nd order systems:

$$\frac{b_0}{a_0 + s}; \quad \frac{b_0 + b_1 s}{a_0 + a_1 s + a_2 s^2}: \qquad a_0, a_1, a_2 > 0$$

whose poles are appropriate:

$$p = -a_0$$

for the 1st order system and

$$\Delta = a_1^2 - 4a_0 a_2 = (-1)4\left(a_0 a_2 - \left(\frac{a_1}{2}\right)^2\right) \rightarrow$$

$$\rightarrow p_{1,2} = -\frac{a_1}{2} \pm j\sqrt{a_0 a_2 - \left(\frac{a_1}{2}\right)^2} \quad \div \quad |p_{1,2}| = \sqrt{a_0 a_2}$$

for the 2nd order system.

Hence, the following convergence conditions for the series 's' and 's^{-1}' are obtained:

Order 1: $\omega < a_0$ $\omega > a_0$

Order 2: $\omega < \min|p|$ $\omega > \max|p|$

where:

$$\min|p| = \frac{a_1}{2} - \sqrt{\Delta};$$

$$\max|p| = \frac{a_1}{2} + \sqrt{\Delta};$$

and
$$\sqrt{\Delta} = \sqrt{\left(\frac{a_1}{2}\right)^2 - a_0 a_2}$$

appropriate to the 's' and 's^{-1}' series for real poles, and:

Order 2: $\omega < \sqrt{a_0 a_2}$ $\omega > \sqrt{a_0 a_2}$

for the complex poles and appropriate to the 's' and 's^{-1}'series.

The appropriate location of the poles on the plane is illustrated in Fig. 4.

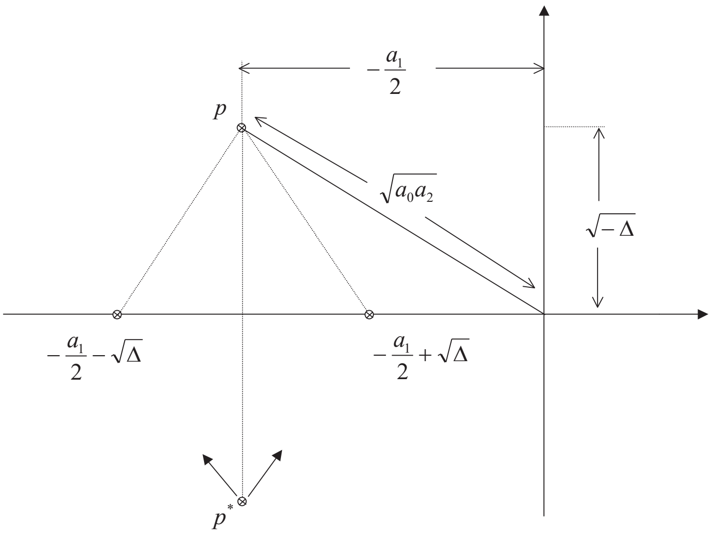

Fig. 4 The location of poles of the system on the plane

The 's' series, as a multiple differentiation system is convergent in the low frequency band, while the multiple integration system described by the 's^{-1}' power series is convergent in the high frequency band. It means that the frequency range of convergence for the 's' decomposition is bound above and an appropriate range of convergence for the 's^{-1}' decomposition from the bottom.

4. Conclusions

In order to obtain the energy-optimal currents distribution, it is necessary to determine the optimal control operator. The operation of inverse a matrix-dispersive Hermitian operator $\mathbf{R}(s)$ is considered, which consists of two parts: a matrix operator inversion and a rational dispersive operators inversion. However, to avoid executing this second step, so called suboptimal control with the operator in which the resistive components are nondispersive, is suggested. It can be achieved by decomposing all impedance operators participating in the issue in power series of variables 's' or 's^{-1}'.

It allows for a simple separation of impedance operators on the part of Hermitian and skew-Hermitian, and also for finding non-dispersive resistance components as zero terms of appropriate power series. A key aspect is the decomposition of a single rational function $H(s)$ in a power series of 's' or 's^{-1}'. To examine this the following is proposed.

Theorem. The given, stable rational transmittance

$$H(s) = \frac{b_0 + b_1 s + b_2 s^2 + \ldots + b_{N-1} s^{N-1}}{a_0 + a_1 s + a_2 s^2 + \ldots + a_N s^N}$$

with real coefficients can be decomposed in power series of 's/s^{-1}' type:

$$H(s) = \sum_{n=0}^{\infty} x_n s^n = s^{-1} \sum_{n=0}^{\infty} y_n s^{-n}$$

which complete series coefficients $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ are determined by formulas:

$$x_n = \sum_{m=1}^{N} c_m p_m^{-n}, \qquad y_n = \sum_{m=1}^{N} d_m p_m^{n}$$

where $\{p_1, p_2, p_3, \ldots, p_N,\}$ is the set of single poles of the function $H(s)$, ie. the roots of characteristic equation:

$$a_0 + a_1 s + a_2 s^2 + \ldots + a_N s^N = 0$$

and the constant weight $\{c_m\}_{m=1}^{N}$ and $\{d_m\}_{m=1}^{N}$ are determined from the system of linear equations

$$\sum_{m=1}^{N} p_m^{-n} c_m = x_n, \qquad \sum_{m=1}^{N} p_m^{n} d_m = y_n$$

for $n = 0, 1, 2, \ldots, N-1$
where the initial values of series, ie. $\{x_n\}_{n=0}^{N-1}$, $\{y_n\}_{n=0}^{N-1}$, are calculated from the bottom triangle systems of linear equations (21) and (25).

References

[1] C.A. Desoer, "The maximum power transfer theorem for n-ports". *IEEE Trans*., vol. CT-20, 228–230, (1979).

[2] R.A. Rohrer, "Optimal matching: A new approach to the matching problem for real invariant one port networks". *IEEE Trans.,* vol. CT-15, 118–124, (1968).

[3] M. Siwczyński, A. Drwal, and S. Żaba, "Minimum-energetic sinusoidal signals distribution in electrical circuits", *Wiadomości Elektrotechniczne*, no. 9, 22–25 (2014), [in Polish].

[4] L.S. Czarnecki, "Discussion on a uniform concept of reactive power of nonsinusoidal currents in a time-domain", *Przegląd Elektrotechniczny*, vol. 85, no. 6, 164–166, (2009).

[5] A.P. Rens, "Validation of popular nonsinusoidal power theories for the analysis and management of modern power systems", North-West University, Potchefstroom Campus, 2006.

[6] J. Walczak, and M. Pasko, "The minimization of losses of active power and the symmetrization of power flow in the nonsinusoidal systems", *Jakość i Użytkowanie Energii Elektrycznej*, vol. 5, no. 1, 55–59, (1999), [in Polish].

[7] L.S. Czarnecki, "Currents' Physical Components (CPC) concept: a fundamental for power theory". *Przegląd Elektrotechniczny*, vol. 84, no. 6, 28–37, (2008).

[8] M. Siwczyński, A. Drwal, and S. Żaba, "Energy-optimal current distribution in a complex linear electrical network with pulse or periodic voltage and current signals. Optimal control", *Bull. Pol. Ac.: Tech.*, 64 (1), 45–50, (2016).

[9] M. Siwczyński, A. Drwal, and S. Żaba, "Energy-optimal current distribution in a complex linear electrical network with pulse or periodic voltage and current signals. Suboptimal control". *Measurement Automation Monitoring*, vol. 62, no. 4, 125–128, (2016).

[10] M.A. Krasnosielskij, F.A. Lipszyc, and A.V. Sobolev, "Pozitywnyje liniejnyje sistiemy". *Science*, Moscow, 1985, [in Russian].

[11] E.N. Rosenvasser and S. K. Volovodov, "Operatornyje mietody i kolebatielnyje processy". *Science*, Moscow, 1985, [in Russian].

[12] D.V. Lee, "On the power-series expansion of a rational function", *Acta Arithmetica*, vol. 57, no. 3, 229–255, (1992).

[13] P.B. Laval, "Representation of functions as power series", Kennesaw State University, USA, 2008.

[14] A. Straub, "Multivariate apery numbers and supercongruences of rational functions". *Algebra Number Theory*, no. 8, 1985–2007, (2014).

[15] M. Houben, "Congruences for Coefficiences of Power Series Expansions of Rational Functions". Utrecht University, Niederlads, 2016.