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Optimal sliding control of mobile manipulators

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Abstract. The paper addresses optimal control problem of mobile manipulators. Dynamic equations of those mechanisms are assumed herein to be uncertain. Moreover, unbounded disturbances act on the mobile manipulator whose end-effector tracks a desired (reference) trajectory given in a task (Cartesian) space. A computationally efficient class of two-stage cascaded (hierarchical) control algorithms based on both the transpose Jacobian matrix and transpose actuation matrix, has been proposed. The offered control laws involve two kinds of non-singular terminal sliding mode (TSM) manifolds, which were also introduced in the paper. The proposed class of cooperating sub-controllers is shown to be finite time stable by fulfilment of practically reasonable assumptions. The performance of the proposed control strategies is illustrated on an exemplary mobile manipulator whose end-effector tracks desired trajectory.

Key words: mobile manipulator, trajectory tracking, finite time control.

1. Introduction

Mobile manipulators are robotic systems whose range of operating the non-holonomic platforms in the work space is, in fact, not bounded. The holonomic manipulator, attached to the platform, makes is possible to accomplish by the end-effector various manipulation tasks such as tracking of the desired (reference trajectories) usually specified in work (Cartesian) space. For modern systems of control of such mechanisms, a requirement is placed on both high precision and stability of the task accomplishment. Due to the fact that desired trajectories are most often given in the Cartesian (task) coordinates of the work space, application of known control techniques in joint coordinates requires first solving the inverse kinematic problem (see e.g. [1]). The process of kinematic inversion is, in general, time consuming and becomes particularly complicated when the Cartesian trajectory forces kinematic and/or algorithmic singularities [2–4]. Consequently, a controller to be designed should accurately track desired trajectory despite potential singularities appearing during the mobile manipulator movement, uncertain dynamic equations and unknown external disturbing signals. Moreover, this controller should generate at least absolutely continuous steering signals to avoid undesirable chattering effects. On account of the challenges posed to modern controllers in a context of their design, three main approaches can be distinguished in the literature.

The first of them uses formulation of an extended (augmented) task space (including also input-output linearisation techniques) in the problem of inverse kinematics, analysed in works [5–9]. The proposed controllers in [5–9] require knowledge of the inverse matrix to an extended Jacobian matrix, which may contain algorithmic and/or kinematic singularities. The control algorithms from works [5–7] require full knowledge of dynamic equations. In turn, studies, [8, 9] utilize regression matrices of dynamic equations as well as discontinuous terms in controllers to compensate for both parametric uncertainties and (unknown) bounded external disturbances. Furthermore, control laws from [5–9] are not also optimal in any sense.

The second approach to the control of mobile manipulators, analysed in works [12–18], is based on the use of (generalized) pseudo-inverse of the Jacobian matrix. Although, control algorithms derived from the pseudo-inverse of the Jacobian matrix are attractive and further investigated, they also have some shortcomings. Namely, generated steering signals are at most sub-optimal. Works [12–14] assume full knowledge of dynamic equations. In turn, studies [15–18] require the knowledge of regression matrix whose numerical implementation seems to be both time-consuming and complex. Moreover, generated torques in [18] are only bounded functions, which in the limit become discontinuous. Furthermore, control strategies based on (generalized) pseudo-inverses are not, in general, repeatable (see e.g. works [19, 20]).

The third approach, proposed in [21, 22], is based on application of a gradient of some potential functions. Nevertheless, the algorithms from [21, 22] provide discontinuous steering signals and solve only point-to-point control problems in the task space.

The present work is a significant generalization of the results recently published in [23–26]. Namely, works [23–26] solve the problem of control in a finite time for only holonomic and uncertain dynamic systems, in particular, for stationary robotic manipulators. On the other hand, the present study introduces a new class of stable finite time controllers for mobile manipulators (with uncertain dynamics) whose platforms are subject to non-holonomic constraints. In order to eliminate the aforementioned shortcomings of the controllers known from the lit-

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erature, two kinds of non-singular terminal sliding mode (TSM) manifolds, defined by non-linear integral equations both of the second order with respect to the task errors and the first order with respect to reduced mobile manipulator acceleration, are introduced herein. The proposed controller has a two-stage (hierarchical) structure consisting of two sub-controllers, which utilize the introduced integral manifolds. The task of the first sub-controller (utilizing only the kinematic equations) is to track desired trajectory by the the-effector with simultaneous minimization of some criterion function reflecting a given kinematic characteristics. In turn, the task of the second sub-controller (taking into account uncertain dynamics and unknown disturbances) is a dynamic compensation of the error between actual value of reduced mobile manipulator acceleration and a reference acceleration provided by the first sub-controller. Both sub-controllers use (also introduced in the paper) new dynamic version of the classic (static) computed torque presented e.g. in works [27, 28]. By fulfilment of reasonable assumptions regarding the ranks of both Jacobian matrix and the actuation one, the proposed combined control scheme is shown to be finite-time stable. In this context, controller analysed in our most recent article [29] differs significantly from that proposed herein. Namely, the structure of controller from [29] is more complex as compared to that proposed herein. Furthermore, control law from [29] requires the inverse of actuation matrix whereas our second sub-controller needs only its transpose. Moreover, the control law proposed herein generates at least absolutely continuous mappings thus avoiding the undesirable chattering effect. The remainder of the paper is organized as follows. Section 2 formulates the task of optimal tracking of desired trajectory, prescribed in work (Cartesian) space. Section 3 sets up a class of cooperating sub-controllers solving the problem of the optimal trajectory tracking in a finite time. A computer example of an optimal trajectory tracking by mobile manipulator operating in a two dimensional work space is given in Section 4. Finally, concluding remarks are drawn in Section 5. Throughout the work, $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ denote minimal and maximal eigenvalue of matrix $(\cdot)$.

2. Problem formulation

Let us consider a mobile manipulator with non-holonomic platform. Its location in global coordinate system $OX_1X_2X_3$ is described by vector of generalized coordinates $x \in \mathbb{R}^l$ (see the platform posture in Fig. 1 given by variables $x_{1,c}$, $x_{2,c}$, $x_{3,c}$, θ and angles ϕ_1, ϕ_2 of driving wheels, where θ is the orientation angle of the platform with respect to $OX_1X_2X_3$; $x_{1,c}, x_{2,c}$ denote coordinates of the platform centre; $x_{3,c}$ stands for the height of the platform; R is the radius of the wheel; $(x_{1,c} + \mathrm{a}\cos(\theta)$, $x_{2,c} + \mathrm{a}\sin(\theta)$, $x_{3,c} + \mathrm{c})^T$ denotes the point in coordinate system $OX_1X_2X_3$ at which the holonomic manipulator is fasten to the platform; $2W$ stands for the platform width and $2L$ is its length; y_1, y_2, y_3 stand for joint angles of the holonomic manipulator; l_1, l_2, l_3 are the lengths of the arm), where $l \geq 2$ and the posture of holonomic manipulator attached to the platform is defined by vector of joint (generalized) coordinates

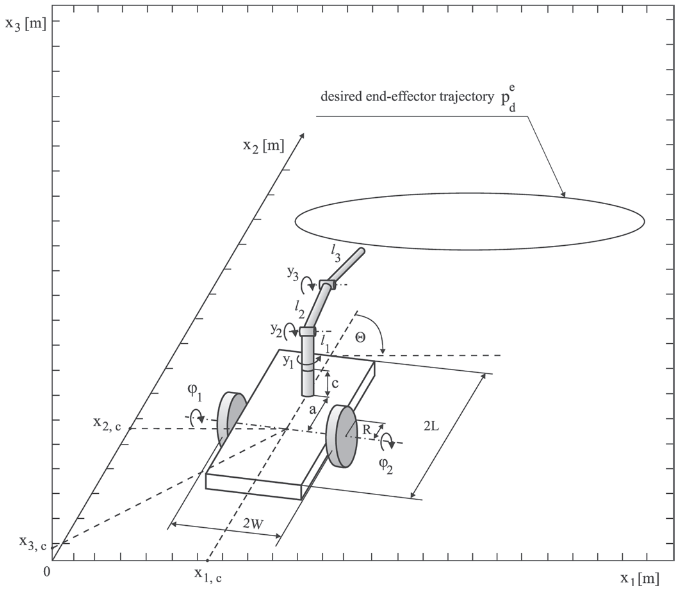

Fig. 1. Kinematic scheme of the mobile manipulator and the trajectory tracking task to be accomplished

$y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$; n is the number of kinematic pairs of the holonomic manipulator. The movement of the mobile platform is subject to $1 \leq k < l$ non-holonomic constraints usually expressed in a Pfaffian form

$$A(x)\dot{x} = 0, \tag{1}$$

where $A(x)$ denotes the $(k \times l)$ matrix of full rank (i.e., $\mathrm{rank}(A(x)) = k$), which depends analytically on x.

Non-holonomic constraints for the platform of $(2,0)$ type from Fig. 1 with no lateral and longitudinal slip of both wheels can be described as follows

$$A(x)\dot{x} = \begin{bmatrix} \sin(\theta) & -\cos(\theta) & 0 & 0 & 0 \\ \cos(\theta) & \sin(\theta) & W & -R & 0 \\ \cos(\theta) & \sin(\theta) & -W & 0 & -R \end{bmatrix} \dot{x} = 0, \tag{2}$$

where $x = (x_{1,c}, x_{2,c}, \theta, \phi_1, \phi_2)^T$; $l = 5$; $k = 3$.

Let $\mathrm{Ker}(A(x))$ be a null space generated by vector fields $a_1(x), \ldots, a_{l-k}(x)$, respectively. Hence, differential constraint (1) may be equivalently expressed as drift-less dynamic system of the form

$$\dot{x} = N(x)\alpha, \tag{3}$$

where $N(x) = [a_1(x), \ldots, a_{l-k}(x)]$; $\mathrm{rank}(N(x)) = l - k$ and $\alpha = (\alpha_1, \ldots, \alpha_{l-k})^T$ denotes vector of quasi-velocities (introduced in work [31]). Let us note that

$$A(x)N(x) = 0. \tag{4}$$

For the mobile manipulator depicted in Fig. 1, equation (3) takes the following form:

$$\dot{x} = N(x)\alpha = \begin{bmatrix} \cos(\theta) & \cos(\theta) \\ \sin(\theta) & \sin(\theta) \\ \frac{1}{W} & -\frac{1}{W} \\ \frac{2}{R} & 0 \\ 0 & \frac{2}{R} \end{bmatrix} \alpha, \tag{5}$$

where $\alpha = (\alpha_1, \alpha_2)^T$. Taking onto account the last two components of (5), we have $\dot{\phi} = (\dot{\phi}_1, \dot{\phi}_2)^T = \frac{2}{R}\alpha$.

Let us note that the choice of coordinates $x_{1,c}$, $x_{2,c}$, $x_{3,c}$ leads to simple forms of matrices $A(x)$, and $N(x)$, respectively (see formulas (2) and (5)). Nevertheless, linearization of the mobile platform kinematics is not possible for coordinates $x_{1,c}, x_{2,c}, x_{3,c}$. In order to avoid this inconvenience, we can introduce another vector x describing the location of the platform from Fig. 1 (see e.g. [36]), which equals $x = (x'_{1,c}, x'_{2,c}, \theta, \phi_1, \phi_2)^T$, where $x'_{1,c} = x_{1,c} + a\cos(\theta)$, $x'_{2,c} = x_{2,c} + a\sin(\theta)$. If this is the case, matrices $A(x)$ and $N(x)$ take the following forms:

$$A(x) = \begin{bmatrix} \sin(\theta) & -\cos(\theta) & a & 0 & 0 \\ \cos(\theta) & \sin(\theta) & W & -R & 0 \\ \cos(\theta) & \sin(\theta) & -W & 0 & -R \end{bmatrix} \tag{6}$$

and

$$N(x) = \begin{bmatrix} Y(W\cos(\theta) - a\sin(\theta)) & Y(W\cos(\theta) + a\sin(\theta)) \\ Y(W\sin(\theta) + a\cos(\theta)) & Y(W\sin(\theta) - a\cos(\theta)) \\ Y & -Y \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{7}$$

where $Y = \frac{R}{2W}$, respectively. Let us observe that $\alpha = \dot{\phi}$ in such a case.

Location and orientation of the end-effector with respect to the global coordinate system $OX_1X_2X_3$ is described by the kinematic equation of the mobile manipulator

$$p_e = f_e(q), \tag{8}$$

where $p_e \in \mathbb{R}^m$ denotes the coordinates of the end-effector; $q = \begin{pmatrix} x \\ y \end{pmatrix}$ is mobile manipulator configuration; $f_e : \mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}^m$ represents m-dimensional (in general, non-linear with respect to q) mapping and m is the dimension of the task (work) space. Combining $\dot{q}$, $\ddot{q}$ and (3), one obtains the following expressions:

$$\dot{q} = Cz, \qquad \ddot{q} = C\dot{z} + \dot{C}z, \tag{9}$$

where $C = \begin{bmatrix} N(x) & 0 \\ 0 & \mathbb{I}_n \end{bmatrix}$; $z = \begin{pmatrix} \alpha \\ \dot{y} \end{pmatrix} \in \mathbb{R}^{l+n-k}$ is reduced mobile manipulator velocity; $\mathbb{I}_n$ denotes the $(n \times n)$ identity matrix. On account of the fact that mobile manipulator considered in the work is a redundant mechanism with respect to a task to be accomplished, the following inequality holds true $l + n \geq m + k$. Consequently, there exists a possibility to augment vector of the end-effector coordinates, describing the classic (conventional) trajectory tracking, by additional task coordinates (specified by the user) of the following form:

$$p_a = f_a(q), \tag{10}$$

where $f_a : \mathbb{R}^{l+n} \to \mathbb{R}^{l+n-m-k}$ is at least triply continuously differentiable mapping with respect to q. From the practical point of view, it is particularly interesting to generate trajectory $q = q(t)$ in such a was as to minimize some objective function $\mathscr{F}(q)$, which is assumed to be at least four times continuously differentiable with respect to q. This function may represent some measure of kinematic characteristics realized in such a way that redundant degrees of freedom are utilized to fulfil additional goals: collision avoidance, steering to a desired mobile manipulator posture, singularity avoidance, etc. The general form of f_a, proposed e.g. in [2] for holonomic systems and generalized in [7] for the non-holonomic ones, may be expressed as

$$f_a = \mathscr{N}(q)\frac{\partial \mathscr{F}(q)}{\partial q}, \tag{11}$$

where $\mathscr{N}$ denotes $(l+n-m-k) \times (l+n)$ orthogonal complementary matrix to

$$\mathrm{j}(q) = \begin{bmatrix} \mathrm{j}_1(q) \\ \cdot \\ \mathrm{j}_{m+k}(q) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_e(q)}{\partial q} \\ A \quad 0_{k \times n} \end{bmatrix},$$

i.e., $\mathrm{j}\mathscr{N}^T = 0$; $0_{k\times n}$ stands for the $(k \times n)$ zero matrix. Let us note that j is an auxiliary Jacobian matrix related to necessary condition of minimum of criterion function $\mathscr{F}$, which is subject to both holonomic constrains (8) and non-holonomic ones (1) (see details in [7]). Consequently, equalities $\mathscr{N}(q)\frac{\partial \mathscr{F}(q)}{\partial q} = 0$ present $l + n - m - k$ transversality conditions which together with (8) and (1) lead to determining the optimal configuration q corresponding to a current location p_e of the end-effector. Without loss of generality, the following criterion function $\mathscr{F}(q)$ is assumed in further analysis:

$$\mathscr{F}(q) = \frac{c_{\mathscr{F}}}{2}\langle q - q_{rest},\, K_{\mathscr{F}}(q - q_{rest})\rangle, \tag{12}$$

where $\langle\ ,\ \rangle$ denotes scalar product of vectors; q_{rest} is a preferred mobile manipulator posture; $c_{\mathscr{F}}$ stands for a positive constant; $K_{\mathscr{F}}$ is a positive definite diagonal weighting matrix. Let us note that taking into account criterion function (12) into optimization problem with equality constraints (1), (8) results in fulfilment of sufficient condition for a local (in general) optimality of trajectory $q = q(t)$, $t \geq 0$. If this is the case, the Hessian H for $\mathscr{F}$ given by (12) and constraints $f_e(q) - p_e = 0$ and $A(x)\dot{x} = 0$

equals

$$H = \mathcal{N}\left(c_{\mathcal{F}}K_{\mathcal{F}} + \frac{\sum_{i=1}^{m+k} l_i \frac{\partial \mathrm{j}_i(q)}{\partial q} + \left(\sum_{i=1}^{m+k} l_i \frac{\partial \mathrm{j}_i(q)}{\partial q}\right)^T}{2}\right)\mathcal{N}^T,$$

where l_i denote the Lagrange multipliers for regular (by assumption) constrained optimization problem (1), (8) and (12), $i = 1,\ldots,m+k$. Matrix

$$\frac{\sum_{i=1}^{m+k} l_i \frac{\partial \mathrm{j}_i(q)}{\partial q} + \left(\sum_{i=1}^{m+k} l_i \frac{\partial \mathrm{j}_i(q)}{\partial q}\right)^T}{2}$$

is symmetric and norm bounded. Let us note that for sufficiently large values of elements of diagonal matrix $c_{\mathcal{F}}K_{\mathcal{F}}$, matrix

$$c_{\mathcal{F}}K_{\mathcal{F}} + \frac{\sum_{i=1}^{m+k} l_i \frac{\partial \mathrm{j}_i(q)}{\partial q} + \left(\sum_{i=1}^{m+k} l_i \frac{\partial \mathrm{j}_i(q)}{\partial q}\right)^T}{2}$$

becomes symmetric and positive definite. Consequently, taking into account the assumed regularity of constraints $f_e(q) - p_e = 0$, $A(x)\dot{x} = 0$ (full rank of matrix j and hence full rank of $\mathcal{N}$), we deduce that matrix H is symmetric and positive definite too, what implies (local) optimality of trajectory $q(t)$. Concatenating $f_e(q)$ with $f_a(q)$, one obtains generalized kinematic-differential mappings which relate q with augmented task coordinates $p = \begin{pmatrix} p_e \\ p_a \end{pmatrix}$

$$p = f(q), \qquad \dot{p} = Jz, \tag{13}$$

where $f = \begin{pmatrix} f_e \\ f_a \end{pmatrix}$ and $J = \frac{\partial f}{\partial q}C$ is the $(l+n-k)\times(l+n-k)$ extended Jacobian matrix. The task, accomplished by the mobile manipulator, is to track both desired end-effector trajectory $p_d^e(t) \in \mathbb{R}^m$, $t \in [0,\ \infty)$ and auxiliary (user specified) trajectory $p_d^a(t) \in \mathbb{R}^{l+n-m-k}$, which for f_a given by relation (11) equals $p_d^a(t) = 0$. Vector functions $p_d^e(\cdot)$ and $p_d^a(\cdot)$ are assumed to be at least triply continuously differentiable with respect to time. Introducing the task tracking error $e = \begin{pmatrix} e^e \\ e^a \end{pmatrix} = f(q) - p_d(t)$, where $p_d = \begin{pmatrix} p_d^e \\ p_d^a \end{pmatrix}$; $e^e = (e_1^e,\ldots,e_m^e)^T = f_e - p_d^e$; $e^a = (e_1^a,\ldots,e_{l+n-m-k}^a)^T = f_a - p_d^a$, the finite time control problem in the task space may be formally expressed by means of the following equations:

$$\lim_{t\to T} e(t) = 0, \qquad \lim_{t\to T} \dot{e}(t) = 0, \qquad \lim_{t\to T} \ddot{e}(t) = 0, \tag{14}$$

where $0 \le T$ denotes a finite time of convergence of $f(q)$ to p_d and $e(t) = \dot{e}(t) = \ddot{e}(t) = 0$ for $t \ge T$. Let us note that the left-sided equation of (14) presents for $\mathcal{F}$ given by (12) and $t \ge T$ a necessary and sufficient condition of minimum. In further analysis, J is assumed to be full rank in the operation region of the end-effector, i.e.

$$\mathrm{rank}(J(q)) = l+n-k. \tag{15}$$

The dynamics of a mobile manipulator described in generalized coordinates q is given by the following equation [33]:

$$M'(q)\ddot{q} + P'(q,\ \dot{q}) + G'(q) + D' + [A(x)\ 0_{k\times n}]^T\lambda = B'v, \tag{16}$$

where $M'(q)$ denotes the $(n+l)\times(n+l)$ positive definite inertia matrix; $P'(q,\ \dot{q})$ is the $(n+l)$-dimensional vector representing centrifugal and Coriolis forces; $\dot{q}$ denotes the mobile manipulator velocity; $G'(q)$ stands for the $n+l$-dimensional vector of generalized gravity forces; D' represents $(l+n)$-dimensional external disturbing signal; $0_{k\times n}$ denotes the $k\times n$ zero matrix; λ is the k-dimensional vector of Lagrange multipliers (reaction forces acting on the platform) corresponding to non-holonomic constraints (1); $B' = \begin{bmatrix} B'' & 0 \\ 0 & \mathbb{I}_n \end{bmatrix}$; B'' stands for the $l\times(l-k)$ matrix indicating which state variables of the platform are directly driven by the actuators (its elements equal 1 for state variables directly driven by the actuators and 0 otherwise); $\mathbb{I}_n$ denotes the $n\times n$ identity matrix, $\mathbb{R}^{n+l-k} \ni v$ is the vector of controls (torques/forces) and $(q^T, \dot{q}^T)^T$ denotes the state vector of the mobile manipulator. Let us note that state coordinates of dynamic equations (16) are subject to non-holonomic constraints (1). Moreover, matrix B' is both actuator deficient (dimension of q equals $l+n$ and the number of independent actuators is equal to $l+n-k$) and not square. Furthermore, dynamic equations (16) include an unknown vector of Lagrange multipliers λ. Consequently, it is extremely hard in such a case to determine control v accomplishing the robot task (14). The aim of introducing auxiliary velocities α and consequently the vector of reduced velocity z is both to eliminate reaction forces λ from dynamic equations (16) and to reduce their dimensionality. Replacing $\dot{q}$ and $\ddot{q}$ from (16) by reduced velocity z and acceleration $\dot{z}$ (see expressions (9)), we significantly simplify our control problem (14). Premultiplying left-sided the dynamic equations (16) by C^T and then using the equality $C^T[A(x)\ 0_{k\times n}]^T = 0$ (see equality (4)), we obtain dynamic equations of the mobile manipulator in the following reduced form [14, 15], which is convenient for our control purposes:

$$M(q)\dot{z} + P(q,\ z)z + G(q) + D(t,\ q,\ z) = B(q)v, \tag{17}$$

where $M = C^TM'C$ denotes the $(l+n-k)\times(l+n-k)$ positive definite symmetric inertia matrix; $P = C^T(M'\dot{C}z + P')$ is the $(l+n-k)$-dimensional vector representing reduced centrifugal and generalized Coriolis forces; $G = C^TG'$ stands for reduced gravity forces acting on the mobile manipulator; $D = C^TD'$ represents $(l+n-k)$-dimensional external disturbing signal whose time derivative $\dot{D}$ is (by assumption) locally bounded Lebesgue measurable mapping; $B = C^TB'$ denotes the

$(l+n-k)\times(l+n-k)$ actuation matrix (describing the configuration variables of the platform and holonomic manipulator which are directly driven by the actuators). Let us note that λ is eliminated from (17) and actuation matrix B is square.

In such a context, reduced state vector $(q^T, z^T)^T$ corresponding to dynamic equations (17) of the mobile manipulator schematically shown in Fig. 1 equals

$$(x_{1,c}, x_{2,c}, \theta, \phi_1, \phi_2, y_1, y_2, y_3, \alpha_1, \alpha_2, \dot{y}_1, \dot{y}_2, \dot{y}_3)^T \in \mathbb{R}^{13},$$

where

$$q = \left(x_{1,c}, x_{2,c}, \theta, \phi_1, \phi_2, y_1, y_2, y_3\right)^T \in \mathbb{R}^8;$$
$$z = \left(\alpha_1, \alpha_2, \dot{y}_1, \dot{y}_2, \dot{y}_3\right)^T \in \mathbb{R}^5$$

whereas state vector $(q^T, \dot{q}^T)^T$ corresponding to dynamic equations (16) is equal to

$$(x_{1,c}, x_{2,c}, \theta, \phi_1, \phi_2, y_1, y_2, y_3, \dot{x}_{1,c}, \dot{x}_{2,c}, \dot{\theta}, \dot{\phi}_1, \dot{\phi}_2, \dot{y}_1, \dot{y}_2, \dot{y}_3) \in \mathbb{R}^{16}.$$

Moreover, let us also note that for $\alpha' = (\vartheta, \dot{\theta})^T$, where ϑ is a linear velocity and $\dot{\theta}$ denotes angular velocity of the platform, vector $\dot{x} = (\dot{x}_{1,c}, \dot{x}_{2,c}, \dot{\theta}, \dot{\phi}_1, \dot{\phi}_2)^T$ is equal to $\dot{x} = N'(x)\begin{pmatrix}\vartheta \\ \dot{\theta}\end{pmatrix}$, where $N' = \begin{bmatrix}\cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \\ 1 & \frac{W}{R} \\ 1 & -\frac{W}{R}\end{bmatrix}$. Hence, $A(x)N'(x) \neq 0$. Consequently, vector λ can not be eliminated from dynamic equations (16) for $\alpha' = (\vartheta, \dot{\theta})^T$.

Without loss of generality of considerations, $\|D\|$ and $\|\dot{D}\|$ are assumed to be upper bounded as follows

$$\|D\| \leq \beta^0(t,\, q,\, z), \qquad \|\dot{D}\| \leq \beta^1(t,\, q,\, z), \tag{18}$$

where $\beta^0(\cdot), \beta^1(\cdot)$ denote non-negative time and state dependent functions, which are locally bounded and Lebesgue measurable. Taking into account equations (9), (17), our aim is to determine at least absolutely continuous vector function of controls $v(\cdot)$ such that

$$\ddot{q} = C\dot{z} + \dot{C}z, \tag{19}$$
$$M\dot{z} = Bv - (Pz + G + D) \tag{20}$$

and trajectory $q = q(t)$ corresponding to the solution of differential equations (19), (20), fulfils task constraints (14). In further analysis, useful properties of kinematic equations (13) are given, which will be used by designing our controller. For the revolute kinematic pairs of the holonomic manipulator and function $\mathscr{F}$ given by (12), the following inequalities are satisfied:

$$\|J\|_F, \left\|\frac{\partial J}{\partial q}\right\|_F, \left\|\frac{\partial^2 J}{\partial q^2}\right\|_F \leq w_1 + w_2\|q - q_{rest}\|, \tag{21}$$

where $\|\cdot\|_F$ is the Frobenius (Euclidean) matrix norm; w_1, w_2 denote positive coefficients (construction parameters of the mobile manipulator dependent of configuration q). Moreover, based on (15), we deduce that there exists a constant $a > 0$ such that

$$0 < a \leq \lambda_{\min}(JJ^T). \tag{22}$$

Our aim is to determine at least absolutely continuous controls v for kinematic task (14). For this purpose, expressions (17), (19) are once differentiated with respect to time, what results in the following system of differential equations:

$$\dddot{q} = C\ddot{z} + 2\dot{C}\dot{z} + \ddot{C}z, \tag{23}$$
$$M\ddot{z} = B\dot{v} + \dot{B}v - \dot{M}\dot{z} - \frac{d}{dt}(Pz + G + D). \tag{24}$$

Let us also differentiate task error equation $e = f - p_d$ once with respect to time thus obtaining

$$\dot{e} = \frac{\partial f}{\partial q}\dot{q} - \dot{p}_d. \tag{25}$$

From (9) and (13), it follows that

$$\dot{e} = \frac{\partial f}{\partial q}Cz - \dot{p}_d = Jz - \dot{p}_d. \tag{26}$$

By double differentiating equality (26) with respect to time, we have

$$\dddot{e} = J\ddot{z} + \ddot{J}z + 2\dot{J}\dot{z} - \dddot{p}_d. \tag{27}$$

Relations (23), (24), (27) and the Lyapunov stability theory will be used in the next section to determine the solution of the (locally) optimal control problem (14), (17).

3. Two-stage cascaded sliding controller of mobile manipulator

The idea of the proposed control law utilizes two cooperating systems. The first one is a kinematic controller of the second order with respect to e whose task is to determine reduced reference acceleration v_{ref} which fulfils relations (14). The task of the second (dynamic) sub-controller is to compensate uncertain dynamics as well as unknown disturbances in such a way as to reduce the error between v_{ref} and actual reduced acceleration $\dot{z}$ of the mobile manipulator to zero in a finite time.

3.1. Optimal kinematic controller. The controller to be proposed uses reformulated equations (19), (23) and (27) to the following form (without taking into account dynamic equations):

$$\ddot{q} = Cv_{ref} + \dot{C}z, \tag{28}$$
$$\dddot{q} = C\dot{v}_{ref} + 2\dot{C}v_{ref} + \ddot{C}z, \tag{29}$$
$$\dddot{e} = J\dot{v}_{ref} + \ddot{J}z + 2\dot{J}v_{ref} - \dddot{p}_d, \tag{30}$$

where $v_{ref} = \dot{z}$ denotes reduced reference acceleration to be determined, for which, relations (14) are fulfilled. In order to find v_{ref}, let $s = (s_1, \ldots, s_{l+n-k})^T \in \mathbb{R}^{l+n-k}$ be a vector sliding variable. The following non-singular TSM manifold $\mathscr{S}_k$ is introduced:

$$\mathscr{S}_k = \Big\{(s(t),\, \ddot{e}(t),\, \ddot{e},\, \dot{e},\, e) \colon s(t) = \ddot{e}(t) + \int_0^t \Big(\lambda_2 \ddot{e}^{3/5} + \lambda_2 \lambda_1^{3/5}(\dot{e}^{9/7} + \lambda_0^{9/7} e)^{1/3}\Big)\, \mathrm{d}\tau\Big\}, \tag{31}$$

where $s(t),\ \ddot{e}(t) \in \mathbb{R}^{n+l-k}$; $\ddot{e} \in AC\left([0,\, \infty),\, \mathbb{R}^{n+l-k}\right)$; $AC(\cdot)$ denotes a class of absolutely continuous functions; $\dot{e} \in C^1\left([0,\, \infty),\, \mathbb{R}^{n+l-k}\right)$; $e \in C^2\left([0,\, \infty),\, \mathbb{R}^{n+l-k}\right)$; $\lambda_0 = diag(\lambda_{0,1}, \ldots, \lambda_{0,l+n-k})$; $\lambda_1 = diag(\lambda_{1,1}, \ldots, \lambda_{1,l+n-k})$; $\lambda_2 = diag(\lambda_{2,1}, \ldots, \lambda_{2,l+n-k})$; $\lambda_{i,j}$ denote positive coefficients (controller gains); $i = 0:2$; $j = 1:l+n-k$. The potency of both e, $\dot{e}$, $\ddot{e}$ and λ_0, λ_1, λ_2 is defined component-wise. From the definition of $\mathscr{S}_k$ in (31), it follows that $\dim(\mathscr{S}_k) = \infty$. Moreover, $\operatorname{codim}(\mathscr{S}_k) = n+l-k$. Set $\mathscr{S}_k \subset \mathbb{R}^{n+l-k} \times \mathbb{R}^{n+l-k} \times AC \times C^1 \times C^2$ is also called an embedded manifold in $\mathbb{R}^{n+l-k} \times AC \times C^1 \times C^2$.

Let us note that equality

$$\ddot{e}(t) + \int_0^t \Big(\lambda_2 \ddot{e}^{3/5} + \lambda_2 \lambda_1^{3/5}(\dot{e}^{9/7} + \lambda_0^{9/7} e)^{1/3}\Big)\, \mathrm{d}\tau = 0$$

is equivalent to a known homogeneous triple integral system of negative degree equal to $-\dfrac{2}{9}$ whose finite-time stability was proved in [34]. As was also shown in [34], the exponents in coefficients λ_0, λ_1 and λ_2 in (31) ensure the finite-time convergence to the origin for $s(t) = 0$.

In what follows, we give a useful result [23–25].

Lemma 1. If $s(t) = 0$ for $t \geq T_k$, where $T_k < \infty$ then task errors $(e,\ \dot{e},\ \ddot{e})$ converge in a finite-time to the origin $(e,\ \dot{e},\ \ddot{e}) = (0,\ 0,\ 0)$.

Using the methodology of dynamically computed torque, introduced in our works [23–26], we propose a sub-controller which determines v_{ref} from the following relation:

$$\dot{v}_{ref} = J^T u_{ref}\,, \tag{32}$$

where $u_{ref} \in \mathbb{R}^{l+n-k}$ is a new reference acceleration to be determined. Substituting the right-hand side of (32) for $\dot{v}_{ref}$ from (30), task error dynamic equation dependent of u_{ref} is obtained as follows

$$\dddot{e} = JJ^T u_{ref} + \ddot{J}z + 2\dot{J}v_{ref} - \dddot{p}_d\,. \tag{33}$$

In order to find u_{ref} and consequently to satisfy equality constraints (14), the following kinematic control law is proposed:

$$u_{ref}(t, q, z, v_{ref}, s) = \begin{cases} -\dfrac{c}{a}\dfrac{s}{\|s\|}(\mathscr{W} + c_0) & \text{for } s \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{34}$$

where

$$\mathscr{W} = \left\|\lambda_2 \ddot{e}^{3/5} + \lambda_2 \lambda_1^{3/5}(\dot{e}^{9/7} + \lambda_0^{9/7} e)^{1/3} - \dddot{p}_d\right\| + (w_1 + w_2)\|q - q_{rest}\|\left(w_3^k \|v_{ref}\|\|z\| + w_4^k \|z\|^3\right);$$

w_3^k, w_4^k denote positive coefficients (construction parameters of the mobile manipulator); c, c_0 are controller gains to be specified further on. Based on (32) and (34), we can determine v_{ref} (in the Filippov sense [32]) from the following differential equation:

$$\dot{v}_{ref} = J^T u_{ref}(t,\ q,\ z,\ v_{ref},\ s). \tag{35}$$

Existence of the solution of (35) has been shown in work [26]. On account of the fact that the right-hand side of equation (35) is not a Lipschitz mapping, the solution to (35) is assumed in further analysis to be unique. Our aim is to give conditions on controller gains λ_0, λ_1, λ_2, c and c_0, which guarantee fulfilment of equalities (14). Applying the Lyapunov stability theory, we propose the following result.

Theorem 1. If matrix J fulfils inequalities (22), λ_0, λ_1, λ_2, $c_0 > 0$ and $c \geq 1$ then control scheme (34), (35) results in a finite time stable convergence of task errors $(e,\ \dot{e},\ \ddot{e})$ to the origin $(e,\ \dot{e},\ \ddot{e}) = (0,\ 0,\ 0)$. Moreover, control law (34), (35) (locally) minimizes criterion function (12).

Proof. Consider the following Lyapunov function candidate:

$$V = \frac{1}{2}\langle s,\ s\rangle. \tag{36}$$

Differentiating (36) with respect to time and taking into account definition (31) results in the following expression:

$$\dot{V} = \Big\langle s,\ \dddot{e} + \lambda_2 \ddot{e}^{3/5} + \lambda_2 \lambda_1^{3/5}(\dot{e}^{9/7} + \lambda_0^{9/7} e)^{1/3}\Big\rangle. \tag{37}$$

Based on (33), one obtains

$$\dot{V} = \langle s, JJ^T u_{ref}\rangle + \Big\langle s, 2\dot{J}v_{ref} + \ddot{J}z - \dddot{p}_d + \lambda_2 \ddot{e}^{3/5} + \lambda_2 \lambda_1^{3/5}(\dot{e}^{9/7} + \lambda_0^{9/7} e)^{1/3}\Big\rangle. \tag{38}$$

Inserting the right-hand side of (34) into (38) results in

$$\dot{V} = -\Big\langle s, JJ^T \frac{c}{a}\frac{s}{\|s\|}(\mathscr{W} + c_0)\Big\rangle + \Big\langle s, 2\dot{J}v_{ref} + \ddot{J}z - \dddot{p}_d + \lambda_2 \ddot{e}^{3/5} + \lambda_2 \lambda_1^{3/5}(\dot{e}^{9/7} + \lambda_0^{9/7} e)^{1/3}\Big\rangle. \tag{39}$$

On account of (22), we get

$$\dot{V} \leq -c\|s\|(\mathscr{W} + c_0) + \Big\langle s, 2\dot{J}v_{ref} + \ddot{J}z - \dddot{p}_d + \lambda_2 \ddot{e}^{3/5} + \lambda_2 \lambda_1^{3/5}(\dot{e}^{9/7} + \lambda_0^{9/7} e)^{1/3}\Big\rangle. \tag{40}$$

Let us estimate the second scalar product in (40). Taking into account inequalities (21), we have after simple calculations that

$$\left\langle s,\, 2\dot{J}v + \dot{J}z - \dddot{p}_d + \lambda_2 \ddot{e}^{3/5} + \lambda_2\lambda_1^{3/5}(\dot{e}^{9/7} + \lambda_0^{9/7} e)^{1/3} \right\rangle \leq \|s\|\mathscr{W}. \tag{41}$$

Consequently, based on the assumption $c \geq 1$ from Theorem 1, one easily obtains that

$$\dot{V} \leq -c\|s\|(\mathscr{W} + c_0) + \|s\|\mathscr{W} \leq -cc_0\|s\|. \tag{42}$$

Since $cc_0 > 0$, inequality (42) proves that TSM $s = 0$ is attainable in a finite time less or equal to $\frac{\sqrt{2V(0)}}{cc_0}$. Consequently, from Lemma 1, it follows that the origin $(e,\ \dot{e},\ \ddot{e}) = (0,\ 0,\ 0)$ is attainable in a finite time T. □

If the control problem is only to track trajectory p_d^e (without taking into account objective function $\mathscr{F}$) then mobile manipulator becomes strictly redundant mechanism with $l + n > m + k$. In such a case, we define sliding variable s^e as $s^e(t) = \ddot{e}^e(t) + \int_0^t \left(\lambda_{2,e}(\ddot{e}^e)^{3/5} + \lambda_{2,e}\lambda_{1,e}^{3/5}((\dot{e}^e)^{9/7} + \lambda_{0,e}^{9/7} e^e)^{1/3}\right) \mathrm{d}\tau$, where $\lambda_{0,e}$, $\lambda_{1,e}$, $\lambda_{2,e}$ are positive controller gains. Applying s^e, we propose the following simplified control law:

$$\dot{v}_{ref}^e = (j^e)^T u_{ref}^e(t,\ q,\ z,\ v_{ref}^e,\ e^e,\ \dot{e}^e, \ddot{e}^e, s_e), \tag{43}$$

where $j^e = \frac{\partial f_e}{\partial q} C$ and

$$u_{ref}^e = \begin{cases} -\frac{c_e}{a_e} \frac{s^e}{\|s^e\|}(\mathscr{W}_e + c_0^e) & \text{for } s^e \neq 0 \\ 0, & \text{otherwise,} \end{cases} \tag{44}$$

$\mathscr{W}_e = \left\| \lambda_{2,e}(\ddot{e}^e)^{3/5} + \lambda_{2,e}\lambda_{1,e}^{3/5}\left((\dot{e}^e)^{9/7} + \lambda_{0,e}^{9/7} e^e\right)^{1/3} - \dddot{p}_d^e \right\| + (w_1 + w_2\|q - q_{rest}\|)\left(w_3^k\|v_{ref}^e\|\|z\| + w_4^k\|z\|^3\right)$; $c^e > 1$, c_0^e are positive controller gains; a_e fulfils inequality $0 < a_e \leq \lambda_{\min}(j^e(j^e)^T)$.

3.2. Dynamic sub-controller of the mobile manipulator. The aim of dynamic controller is to compensate uncertain dynamics and unknown (globally) unbounded external disturbances such that the dynamic tracking error E and its time derivative $\dot{E}$, defined below

$$E = z - \int_0^t v_{ref}\,\mathrm{d}\tau, \qquad \dot{E} = \dot{z} - v_{ref}, \tag{45}$$

stably converge to the origin $(E,\ \dot{E}) = (0,\ 0)$ in a finite time. Let us note that tracking errors E, $\dot{E}$ equal identically zero when mobile manipulator dynamics is neglected (see relation $\dot{z} - v_{ref} = 0$ immediately after formula (30)). Moreover, taking into account mobile manipulator dynamics implies non-zero tracking errors (45). Our task is to find at least absolutely continuous control vector v reducing E and $\dot{E}$ to zero in a finite time. For this purpose, (24) is expressed in the following compact form:

$$\ddot{z} = M^{-1}B\dot{v} + \mathscr{R}(t,\ q,\ z,\ v), \tag{46}$$

where $\mathscr{R} = M^{-1}\left(\dot{B}v - \dot{M}\dot{z} - \frac{\mathrm{d}}{\mathrm{d}t}(Pz + G + D)\right)$. Partially inspired by the control methodology borrowed from the stationary robotic manipulators (see e.g. [25]), we propose to seek v as follows

$$\dot{v} = B^T u, \tag{47}$$

where $u \in \mathbb{R}^{l+n-k}$ is a new control to be determined further on. Replacing $\dot{v}$ in (46) by the right side of (47), we obtain expression dependent of u

$$\ddot{z} = M^{-1}BB^T u + \mathscr{R}. \tag{48}$$

The aim is to find input signal $u(t)$ and consequently control v such that vector $z(t)$ exactly tracks $\int_0^t v_{ref}\,\mathrm{d}\tau$. Therefore, let us differentiate twofold error equation E with respect to time thus obtaining

$$\ddot{E} = \ddot{z} - \dot{v}_{ref} = \ddot{z} - J^T u_{ref}. \tag{49}$$

Inserting the right-hand side of (48) into (49), we obtain error dynamic equation which is dependent of u

$$\ddot{E} = M^{-1}BB^T u + \mathscr{R} - J^T u_{ref}. \tag{50}$$

Let $S = (S_1, \ldots, S_{l+n-k})^T \in \mathbb{R}^{l+n-k}$ be a sliding vector variable. In order to find control law which reduces E and $\dot{E}$ to zero in a finite time subject to dynamic equations (20), the following sliding vector mode manifold $\mathscr{S}_d$ is proposed:

$$\mathscr{S}_d = \left\{ (S(t),\ \dot{E},\ E) : S(t) = = \dot{E}(t) + \int_0^t \left(\Lambda_0 E^{\alpha_1} + \Lambda_1(\dot{E})^{\alpha_2}\right) \mathrm{d}\tau \right\}, \tag{51}$$

where $S(t)$, $\dot{E}(t) \in \mathbb{R}^{n+l-k}$; $\dot{E} \in AC\left([0, \infty),\ \mathbb{R}^{n+l-k}\right)$; $\alpha_1 = \frac{n_1}{n_2}$; n_1, n_2 are positive odd numbers which fulfil the following inequalities: $n_1 < n_2 < 2n_1$; $\alpha_2 = \frac{2\alpha_1}{1 + \alpha_1}$; $\Lambda_0 = diag(\Lambda_{0,1}, \ldots, \Lambda_{0,l+n-k})$; $\Lambda_1 = diag(\Lambda_{1,1}, \ldots, \Lambda_{1,l+n-k})$; $\Lambda_{i,j}$ are positive gain coefficients and $i = 0, 1$; $j = 1, 2, \ldots, l + n - k$. In what follows, we give useful result [24].

Lemma 2. If $S(t) = 0$ for $t \geq T_d$, where $0 \leq T_d < \infty$ then dynamic tracking errors $(E, \dot{E})$ of (51) stably converge in finite time to the origin $(E, \dot{E}) = (0, 0)$.

Differentiating S in (51) and then replacing $\ddot{E}$ by the right-hand side of (50), one obtains expression

$$\dot{S} = M^{-1}BB^T u + \mathscr{U}(q,\ z,\ v,\ t,\ E,\ \dot{E}), \tag{52}$$

where $\mathscr{U} = \mathscr{R} - J^T\ u_{ref} + \Lambda_0\ E^{\alpha_1} + \Lambda_1\ (\dot{E})^{\alpha_2}$. In further analysis, we upper estimate the Euclidean norm of expression $M\mathscr{U} + \frac{\dot{M}S}{2}$ obtaining the following inequality:

$$\left\| M\mathscr{U} + \frac{\dot{M}S}{2} \right\| \leq \chi(t,\ q,\ z,\ v,\ E,\ \dot{E}), \tag{53}$$

where $\chi = w_3\|v\|\|z\| + w_4\|z\|^3 + w_5(\|z\| + \|z\|\beta_0) + w_6\beta_1 + w_7\|\Lambda_0\ E^{\alpha_1} + \Lambda_1\ (\dot{E})^{\alpha_2} - J^T\ u_{ref}\| + w_8\|z\|\|S\| + w_9\|z\|\|\dot{z}\|$; w_3,..., w_9 are known positive coefficients (construction parameters dependent of configuration q). Moreover, actuation matrix B is assumed in further analysis to have full rank. Hence, there exists a positive number A such that

$$0 < A \leq \lambda_{\min}(BB^T). \tag{54}$$

In order to fulfil equalities $(E, \dot{E}) = (0, 0)$ in a finite time, the following dynamic sub-controller is proposed:

$$u(t,q,z,v,E,\dot{E},S) = \begin{cases} -\dfrac{C}{A}\dfrac{S}{\|S\|}(\chi + C_0) & \text{for } S \neq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{55}$$

where C, C_0 denote positive controller gains to be specified further on. Based on (55) and (47), we can find absolutely continuous control vector v by solving (in the Filippov sense [32]) the following differential equation:

$$\dot{v} = B^T\ u(t,\ q,\ z,\ v,\ E,\ \dot{E},\ s). \tag{56}$$

The aim of the further considerations is to give conditions on controller gains Λ_0, Λ_1, C and C_0, which guarantee fulfilment of equality $(E, \dot{E}) = (0, 0)$. Applying the Lyapunov stability theory, we offer the following result.

Theorem 2. If actuation matrix B fulfils (54) and Λ_0, Λ_1, $C_0 > 0$ as well as $C \geq 1$ then control scheme (55), (56) allows convergence of dynamic tracking errors $(E,\ \dot{E})$ to the origin $(E, \dot{E}) = (0, 0)$ in a finite time.

Proof. Consider a Lyapunov function candidate

$$V = \frac{1}{2}\langle S,\ MS \rangle. \tag{57}$$

The time-derivative of (57) equals

$$\dot{V} = \langle S,\ M\dot{S} \rangle + \left\langle S,\ \frac{\dot{M}S}{2} \right\rangle. \tag{58}$$

Let us note that M is positive definite and symmetric matrix. Replacing $\dot{S}$ in (58) by the right-hand side of (52), one obtains the expression

$$\dot{V} = \langle S,\ BB^T u \rangle + \left\langle S,\ M\mathscr{U} + \frac{1}{2}\dot{M}S \right\rangle. \tag{59}$$

Inserting the right-hand side of (55) into (59) results in

$$\dot{V} = \left\langle S,\ -BB^T \frac{C}{A}\frac{S}{\|S\|}(\chi + C_0) \right\rangle + \left\langle S,\ M\mathscr{U} + \frac{1}{2}\dot{M}S \right\rangle. \tag{60}$$

Based on (54), we obtain

$$\dot{V} \leq -\|S\|C(\chi + C_0) + \left\langle S,\ M\mathscr{U} + \frac{1}{2}\dot{M}S \right\rangle. \tag{61}$$

In the next step, we upper estimate scalar product $\left\langle S,\ M\mathscr{U} + \frac{1}{2}\dot{M}S \right\rangle$. On account of (53), we get

$$\left\langle S,\ M\mathscr{U} + \frac{1}{2}\dot{M}S \right\rangle \leq \|S\|\ \chi. \tag{62}$$

Hence, utilizing the assumption $C \geq 1$ from Theorem 2, we can easily obtain that

$$\begin{aligned} \dot{V} &\leq -C\|S\|(\chi + C_0) + \|S\|\chi \leq -C\|S\|(\chi + C_0) + C\|S\|\chi \\ &\leq -C\ C_0\|S\|. \end{aligned} \tag{63}$$

Let us observe that $C\ C_0 > 0$. Hence, inequality (63) proves that $S = 0$ is attainable in finite time less or equal to $\frac{\sqrt{2V(0)}}{C\ C_0}$. Finally, from Lemma 2, it follows that the origin $(E,\ \dot{E}) = (0,\ 0)$ may be attained in a finite time. □

Theorems 1 and 2 imply the following main result.

Theorem 3. By the fulfilment of assumptions from Theorems 1 and 2, control schemes (34)–(35) and (55), (56) result in stable convergence in a finite time of the task errors $(e,\ \dot{e},\ \ddot{e})$ to the origin $(e,\ \dot{e},\ \ddot{e}) = (0,\ 0,\ 0)$.

Proof. Application of sub-controller (55), (56) implies fulfilment of equality $E(t) = \dot{E}(t) = 0$ after a finite time $0 \leq T_d < \infty$. As a result, for $t \geq T_d$, control law (34), (35), according to Theorem 1, is realized, which implies stable convergence in a finite time $0 \leq T_k < \infty$ of task tracking errors $(e,\ \dot{e},\ \ddot{e})$ to the origin. Consequently, stable convergence of task errors $(e,\ \dot{e},\ \ddot{e})$ to the origin may be realized in a finite time less or equal to $0 \leq T_k + T_d < \infty$. □

Let us note that for the non-holonomic platforms of type (2, 0), actuation matrix B becomes diagonal with positive components. Consequently, there exists B^{-1}. On account of the fact that M is positive definite and symmetric matrix, there exists A' such that

$$0 < A' \leq \lambda_{\min}(M^{-1}). \tag{64}$$

If this is the case, we can propose the following control law:

$$\dot{v} = B^{-1} u(t,\, q,\, z,\, v,\, E,\, \dot{E},\, s). \tag{65}$$

where

$$u(t, q, z, v, E, \dot{E}, S) = \begin{cases} -\dfrac{C}{A'}\dfrac{S}{\|S\|}(\chi'+C_0) & \text{for } S \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{66}$$

$\chi' = \chi_{|w_7=1,\, w_8=0}$. We are now in position to give the following theorem.

Theorem 4. If the assumptions of Theorem 2 regarding Λ_0, Λ_1, C and C_0 are satisfied then control scheme (65), (66) enables the dynamic tracking errors $(E,\, \dot{E})$ to stably converge to the origin $(E,\, \dot{E}) = (0,\, 0)$ in a finite time.

Proof. The proof of Theorem 4 is a small modification of the proof of Theorem 2 with Lyapunov function candidate $V' = \dfrac{1}{2}\langle S,\, S\rangle$. Therefore, it is omitted. □

It is worth to emphasize the fact that finite-time controllers utilizing the Jacobian transpose matrix were also analysed in our most recent work [29]. However, there are significant differences between control law from [29] and that proposed herein. First, the structure of controller from [29] is more complex as compared to that proposed herein. Namely, centralized control law from [29] requires the inverse of actuation matrix B multiplied right sided by J^T whereas our sub-controllers (34), (35), (55), (56) need only transpose of J and B, respectively. The requirement of computing B^{-1} in [29] is a restrictive assumption for many non-holonomic mechanisms which are (by their nature) underactuated dynamic systems with B being not square and/or even singular. Second, the proof of finite-time stability in [29] is based on the assumption of the full rank of actuation matrix B. However, sub-controller (55), (56) does not require the strong assumption of invertibility of B. If $B = B(q)$ is singular at $q' = q(t')$ for some time instant t' and

$$0 \neq S(t') \notin \ker(B^T(q')) \tag{67}$$

then for sufficiently large C, expression $-C\langle B^T S,\, B^T S\rangle \dfrac{\chi + C_0}{A\|S\|}$ in equation (60) can take arbitrarily large negative values thus implying the negative value of $\dot{V}$. Hence, controller (55), (56) makes it possible to generate trajectory $q = q(t)$ passing through singular manifold $\{q\colon \det(BB^T) = 0\}$ at configuration $q' = q(t')$. Let us observe that, condition (67) is weaker than that of the full rank of B given in [29]. Consequently, matrix B may contain singularities and control laws (34), (35), (55), (56) still result in finite time stability. Moreover, transposed both Jacobian and actuation matrix sub-controllers (34), (35), (55), (56) provide (local) optimal solution by applying the sliding mode approach. In such a context, there exist several papers [8–11], which use sliding variables in control algorithms. Nevertheless, control laws proposed in [8–11] are not optimal in any sense and provide (in most cases) discontinuous steering signals.

Due to real-time nature of the sub-controllers (34), (35), (55), (56), we shall try to estimate the number of arithmetic operations required to implement the control algorithms presented in this section. Operations required for the computation of sin, cos and p_d^e functions are not taken into account. Furthermore, matrices $J(q)$ and $B(q)$ are assumed to be given. Moreover, estimations are carried out at any time instant of the robot task accomplishment. From (34) and (55), it follows that terms u_{ref}, u require $O(l+n-k)$ operations. Computation of the right-hand side of eqn (56) equals $O((l+n-k)^2)$. Let us note that for the $(2,\ 0)$ platform, matrix B is diagonal. Hence, computation of $B^T u$ requires $O(l+n-k)$ operations. Finally, the computational complexity of the whole mobile manipulator controller (34), (35), (55), (56) is of the order $O((l+n-k)^2)$.

4. Numerical example

Based on an exemplary task to be accomplished by the mobile manipulator, this section demonstrates the performance of the proposed cooperating controllers (34), (35) and (55), (56). For this purpose, the mobile manipulator operating in a three-dimensional work space and shown in Fig. 1, has been utilized. Kinematic and dynamic data correspond to KUKA youBot mobile platform and holonomic manipulator. However, in the computations carried out herein, the platform is assumed to be of the non-holonomic (2, 0) type and the holonomic manipulator has only three revolute kinematic pairs ($n = 3$) (the last three links of the original KUKA holonomic arm form the single link of the manipulator utilized in the computations). Consequently, kinematic equations of the mobile manipulator from Fig. 1 take the form

$$f_e(q) = \begin{pmatrix} \mathrm{a}c\theta + l_1 c_{\theta 1} + \dfrac{l_2}{2} c_{\theta 12} + \dfrac{l_3}{2} c_{\theta 123} + x_{1,c} \\ \mathrm{a}s\theta - l_1 s_{\theta 1} - \dfrac{l_2}{2} s_{\theta 12} - \dfrac{l_3}{2} s_{\theta 123} + x_{2,c} \\ \mathrm{c} - l_2 s y_2 - l_3 s y_{23} + x_{3,c} \end{pmatrix}, \tag{68}$$

where $c_{\theta 1} = \cos(-\theta + y_1)$, $c_{\theta 12} = \cos(-\theta + y_1 + y_2) + \cos(\theta - y_1 + y_2)$, $c_{\theta 123} = \cos(-\theta + y_1 + y_2 + y_3) + \cos(\theta - y_1 + y_2 + y_3)$, $s_{\theta 1} = \sin(-\theta + y_1)$, $s_{\theta 12} = \sin(-\theta + y_1 + y_2) - \sin(\theta - y_1 + y_2)$, $s_{\theta 123} = \sin(-\theta + y_1 + y_2 + y_3) - \sin(\theta - y_1 + y_2 + y_3)$. Vector q equals $q = (x_{1,c},\ x_{2,c},\ \theta,\ \phi_1,\ \phi_2,\ y_1,\ y_2,\ y_3)^T$. In the computer simulation, SI units are used. Taking into account the above assumptions and the KUKA youBot documentation [35], the kinematic parameters of the mobile manipulator from Fig. 1 take the following numeric values: $(\mathrm{a}, 0, \mathrm{c})^T = (0.167,\ 0,\ 0.161)^T$, $x_{3,c} = 0.084$, $l_1 = 0.033$, $l_2 = 0.155$, $l_3 = 0.342$, $W = 0.158$ and $R = 0.05$, respectively. Matrix C is equal to

$$C = \begin{bmatrix} N(x) & 0 \\ 0 & \mathbb{I}_3 \end{bmatrix}, \tag{69}$$

where $N(x)$ is given by formula (5). The task is to track the end-effector desired trajectory $p_d^e = (2 + \cos(t),\ 3 + \sin(t),\ 0.35)^T$,

$t \geq T$ $(m = 3)$. Hence, auxiliary matrix $\mathrm{j}(q)$ equals

$$\mathrm{j} = \begin{bmatrix} 1 & 0 & \mathrm{j}_{13} & 0 & 0 & \mathrm{j}_{14} & \mathrm{j}_{15} & \mathrm{j}_{16} \\ 0 & 1 & \mathrm{j}_{23} & 0 & 0 & \mathrm{j}_{24} & \mathrm{j}_{25} & \mathrm{j}_{26} \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{j}_{35} & \mathrm{j}_{36} \\ -s\theta & c\theta & 0 & 0 & 0 & 0 & 0 & 0 \\ c\theta & s\theta & W & -R & 0 & 0 & 0 & 0 \\ c\theta & s\theta & -W & 0 & -R & 0 & 0 & 0 \end{bmatrix}, \quad (70)$$

where

$$\mathrm{j}_{13} = -\mathrm{a}s\theta + l_1 s_{\theta 1} + \frac{l_2}{2} s_{\theta 12} + \frac{l_3}{2} s_{\theta 123};$$

$$\mathrm{j}_{14} = -l_1 s_{\theta 1} - \frac{l_2}{2} s_{\theta 12} - \frac{l_3}{2} s_{\theta 123};$$

$$\mathrm{j}_{15} = \frac{l_2}{2}(-\sin(-\theta + y_1 + y_2) - \sin(\theta - y_1 + y_2)) + \frac{l_3}{2}(-\sin(-\theta + y_1 + y_2 + y_3) - \sin(\theta - y_1 + y_2 + y_3));$$

$$\mathrm{j}_{16} = \frac{l_3}{2}(-\sin(-\theta + y_1 + y_2 + y_3) - \sin(\theta - y_1 + y_2 + y_3));$$

$$\mathrm{j}_{23} = \mathrm{a}c\theta + l_1 c_{\theta 1} + \frac{l_2}{2} c_{\theta 12} + \frac{l_3}{2} c_{\theta 123};$$

$$\mathrm{j}_{24} = -l_1 s_{\theta 1} - \frac{l_2}{2} c_{\theta 12} - \frac{l_3}{2} c_{\theta 123};$$

$$\mathrm{j}_{25} = -\frac{l_2}{2}(\cos(-\theta + y_1 + y_2) - \cos(\theta - y_1 + y_2)) - \frac{l_3}{2}(\cos(-\theta + y_1 + y_2 + y_3) - \cos(\theta - y_1 + y_2 + y_3));$$

$$\mathrm{j}_{26} = \frac{l_3}{2}(\cos(-\theta + y_1 + y_2 + y_3) - \cos(\theta - y_1 + y_2 + y_3));$$

$$\mathrm{j}_{35} = -l_2 c y_2 - l_3 c y_{23}; \qquad \mathrm{j}_{36} = -l_3 c y_{23}.$$

The components of the nominal dynamic equations are equal to: platform mass $m_p = 19.803$; wheel mass $m_w = 1.4$; masses of the holonomic manipulator links equal $m_1 = 1.39$, $m_2 = 1.318$ and $m_3 = 2.496$, respectively. The remaining dynamic parameters of the robot are taken from KUKA youBot documentation [35]. In order to simplify the computations, D is assumed to be equal to zero, i.e., $D = 0$. Hence, $\beta_0 = \beta_1 = 0$ and consequently $w_6 = 0$. The methodology of estimations of other constants, i.e., $w_1, \ldots, w_5, w_7, \ldots, w_9$, which depend only on configuration q, has been given in our work [29]. Nevertheless, in order to simplify the computations, rough values for w_i, $i = 1, \ldots, 9$, a and A have been assumed. Consequently, these constants are chosen as follows $a = A = 0.2$; $w_1 = 1.5$; $w_2 = 0.001$; $w_3 = 3$; $w_3^k = 2$; $w_4 = 9$; $w_4^k = 3$; $w_5 = 12$; $w_6 = 0$; $w_7 = 6$, $w_8 = 0.011$, $w_9 = 6$. Initial reduced velocity, control and configuration equal $z(0) = (0,\ 0,\ 0,\ 0,\ 0)^T$, $v(0) = (0,\ 0,\ 0,\ 0,\ 0)^T$, $q(0) = \left(-0.4, 0, 0, 0, 0, 0, -\frac{\pi}{18}, \frac{\pi}{9}\right)^T$, respectively. Moreover, $q_{rest} = q(0)$. Actuation matrix B takes the following form:

$$B = \begin{bmatrix} \frac{2}{R} & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{R} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (71)$$

In order to attain the accuracy of the task errors e less or equal to 10^{-4}, the following numerical values are assumed for the controller gains: $c = C = 2$; $c_0 = C_0 = 1$; $\lambda_0 = 1$; $\lambda_1 = 11$; $\lambda_2 = 6$; $\Lambda_0 = 11$; $\Lambda_1 = 6$; $\alpha_1 = \frac{3}{5}$; $c_{\mathscr{F}} = 1$ and $K_{\mathscr{F}} = \mathrm{diag}(0.001, 0.001, 0.001, 0.05, 0.05, 0.0001, 0.05, 0.05)$, respectively. The results for this simulation are given in Figs 2–4, which indicate a good tracking performance of optimal sub-controllers (34), (35) and (55), (56) (see Figs 2, 3). Let us observe (Figs 2, 3) that for $t > 5$ mobile manipulator accomplishes (locally) optimal movement. The corresponding torques v are depicted in Fig. 4. As is seen from Fig. 4, control laws (34), (35) and (55), (56) generate absolutely continuous steering signals (torques).

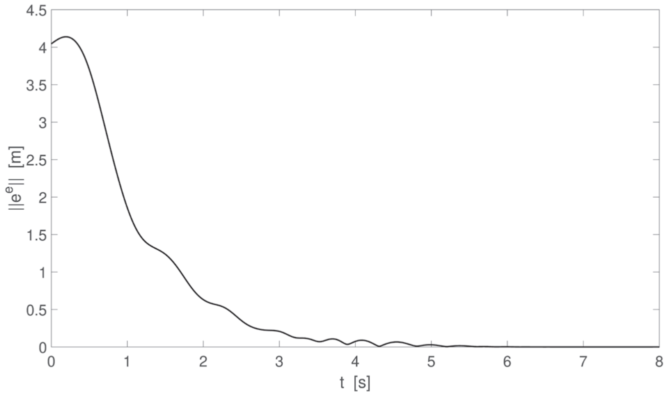

Fig. 2. Euclidean norm of task errors e^e for optimal sub-controllers (34), (35) and (55), (56)

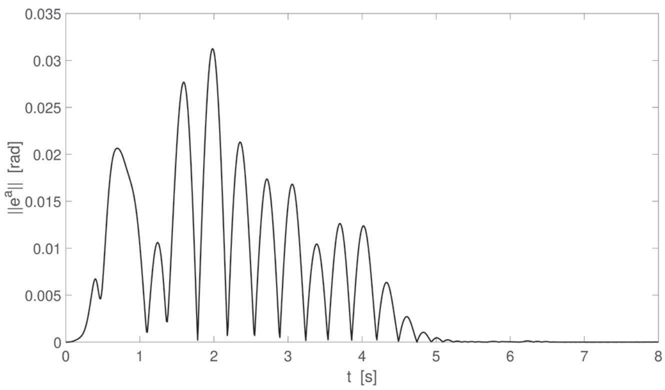

Fig. 3. Euclidean norm of auxiliary (user specified) errors e^a for optimal sub-controllers (34), (35) and (55), (56)

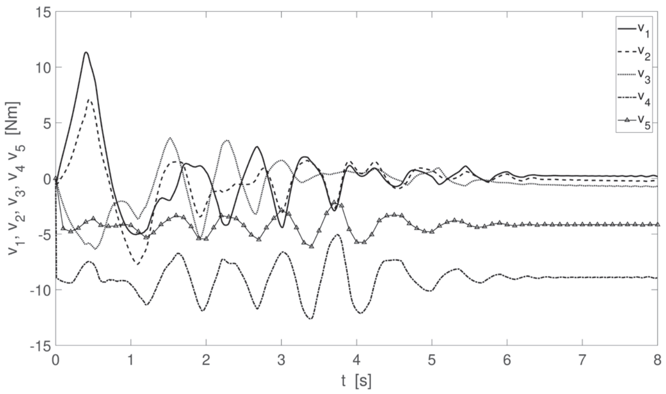

Fig. 4. Torques *v* for optimal sub-controllers (34), (35) and (55), (56)

5. Conclusions

A class of optimal hierarchical controllers designed to track desired trajectories expressed in Cartesian space, has been presented. The main advantage of the proposed control algorithm is the elimination of the inverse (or pseudo-inverse) of both extended Jacobian and actuation matrices. Moreover, the offered control scheme generates at least absolutely continuous steering signals. Based on the Lyapunov stability theory, the control strategies (34)–(35) and (55)–(56) are shown to be both finite time stable and (locally) optimal by fulfilment of reasonable assumptions regarding the matrices J and B. Let us note that presented control algorithms (34)–(35), (55)–(56) require some information extracted from both kinematic and dynamic equations. Namely, they only need upper norm estimates of some components of dynamic equations (and not the dynamic components directly) of the mobile manipulators. Nevertheless, the presented approach is able to handle both uncertainty in dynamics and external disturbances in the non-holonomic systems. Due to decentralized nature of our sub-controllers, it is also possible to extend the presented results to the case when only estimates of Jacobian matrix $J(q)$ and actuation one $B(q)$ are known with a given accuracy of estimation with respect to unknown J and B, respectively. This will be the subject of the future research. The proposed methodology to optimal trajectory tracking control problem may be directly applicable to many mobile manipulators operating in a six dimensional task space.

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