

# Performance evaluation of unreliable system with infinite number of servers

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**Abstract.** In the paper, we investigate queuing system  $M/G/\infty$  with non-homogeneous customers. By non-homogeneity we mean that each customer is characterized by some arbitrarily distributed random volume. The arriving customers appear according to a stationary Poisson process. Service time of a customer is proportional to its volume. The system is unreliable, which means that all its servers can break simultaneously and then the repair period goes on for random time having an arbitrary distribution. During this period, customers present in the system and arriving to it are not served. Their service continues immediately after repair period termination. Time intervals of the system in good repair mode have an exponential distribution. For such system, we determine steady-state sojourn time and total volume of customers present in it distributions. We also estimate the loss probability for the similar system with limited total volume. An analysis of some special cases and some numerical examples are attached as well.

**Key words:** queuing system with non-homogeneous customers, unreliable queuing system, total volume, loss probability, Laplace–Stieltjes transform.

## 1. Introduction

In classical queuing theory, we usually investigate systems with finite or infinite number of identical servers that work continuously without any breaks. It means that all servers in the system are reliable and we do not need stop their work (repair them) during the process of service. For such systems, we usually obtain characteristics of the number of customers present in the system, their waiting and sojourn time (for the systems with finite number of servers and infinite queues) at any time instant  $t$  or at least in steady state [1]. From the theoretical point of view, the class of systems with infinite number of servers is very important because such systems may be treated as the limitary case of systems with big number of servers ( $n \rightarrow \infty$ ) and obtained results in this case may be used as approximations of characteristics for analogous systems with finite and numerous number of servers. Formulae describing analyzed characteristics are usually much less mathematically complicated and more convenient from the numerical point of view. Moreover, models with infinite number of servers have some practical applications, especially when we model customers storage process in the multi-phase queueing systems [2] or analyze the process of reading data from files located in some database. Indeed, the model of  $M/G/\infty$  queueing system can be understood as the limitary case of the Erlang-type  $M/G/n/0$  one [1, 2]. It is obvious that, in this model, all customers are served without any waiting and their sojourn time is the time of their ser-

vice. If we additionally assume that servers can break (which is a very practical assumption because servers often need some repair or break), situation changes because sojourn time of a customer is then the sum of his service time and summary repair time (sum of all times of repairs that were done during its service). The analysis of mentioned above models becomes much more interesting when we additionally assume that all customers arriving to the system have some random volume. It means that we investigate models of queueing systems with non-homogeneous customers. These models are used in modeling of real computer or telecommunication systems in which we store some information. During the investigations, we additionally obtain characteristics of the customers total volume (which is the sum of the volumes of all customers present in the system) at any time instant  $t$  or in steady state. These characteristics show the summary size of all customers present in the system and may be used during the process of designing of real computer (or telecommunication) systems that have limited buffers space (memory). It is clear that even in the case when we have infinite number of servers (e.g.  $M/G/\infty$  model) but total customers volume is limited (by value  $V$ ), we face the problem of the customers losses. In this case, loss probability  $P_{loss}$  strictly depends on  $V$  value that can be used to choose such level of buffer space to minimize the loss probability (when we have the exact formulae for its calculation). In the beginning, such models were analyzed with the usage of results known from the classical queueing theory [3, 4] but it became obvious that classical models cannot exactly describe queueing models with non-homogeneous customers and some generalizations are needed. For example, total volume characteristics depend also on the character of dependency between customer volume and his service time that was shown in many

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Manuscript submitted 2019-05-05, revised -, initially accepted for publication 2020-02-10, published in April 2020

later papers [5–10]. In previous investigations, this fact was not taken into account. As a result, we have mistakes in formulae for characteristics of the total volume and loss probability. It is worth noticing that in the theory of queueing systems with non-homogeneous customers we also investigate models with unlimited total volume (which seems to be impractical) because results of such models analysis may be used to approximate loss characteristics in analogous models with limited total volume [2, 11, 12]. This fact is very important because, for many models, we are not able to obtain exact formulae and then we have to use their approximations. The analysis of various problems connected with investigation of queueing systems with non-homogeneous customers and limited or unlimited total volume can be found in many papers and books that have appeared in recent years [2, 5–11, 13–19].

In this paper, we investigate the modification of the classical queueing model of the  $M/G/\infty$ -type in which:

- 1) customers arriving to the system have some random volume (size) characterized by arbitrary distribution function;
- 2) all servers can break simultaneously in some exponentially distributed random moment of time and then they are repaired for some random time having arbitrary distribution;
- 3) service time of the customer is proportional to his volume (so they are dependent);
- 4) total customers volume is unlimited.

Our main purpose is to obtain the characteristics of customers total volume and their sojourn time for the system under consideration and show possible practical applications of obtained formulae.

The rest of the paper is organized as follows. In the next Section 2, we introduce necessary notation and functions describing behavior of the system under consideration. Then, in Section 3, we obtain main results connected with steady-state characteristics of the customers total volume and their sojourn time. Section 4 presents an interesting practical case of the investigated model which can be understood as the model of database table that serves customers requests for reading information from the table, while customers writing new information stop the process of reading (readers and writers problem). In Section 5, we present results for the case of exponentially distributed customers volume. In Section 6, for the analyzed model, we additionally obtain characteristics of steady-state customers number distribution. Finally, in Section 7 we show the possibility of obtaining the approximations of loss characteristics for analogous systems with finite buffers together with some numerical examples obtained with the help of *Mathematica* environment [20]. The last Section 8 contains conclusions and final remarks.

## 2. The model and basic notation

Consider a service system  $M/G/\infty$  with identical servers. Let  $\alpha$  be the parameter of customers entrance flow. All servers of the system can break simultaneously at some random moments of time. Time intervals of the system in good repair mode (working without breaks) have an exponential distribution with pa-

rameter  $\alpha > 0$ . After system breakage, the repair period goes on for some random time  $\chi$ . Denote by  $G(t) = P\{\chi < t\}$  its distribution function (DF). Let  $g(q) = \int_0^{\infty} e^{-qt} dG(t)$  be the Laplace-

Stieltjes transform (LST) of the function  $G(t)$  and  $g_k$  be the  $k$ th moment of RV  $\chi$ . All customers present in the system at its breaking moment continue their service after repair period termination and all customers that arrive during this period begin their service after its termination. For the considered system, the steady-state distribution of number of customers present in it was determined in [21] in the case of exponentially distributed service time.

Assume additionally that each customer is characterized by some random volume  $\zeta$  which does not depend on other customers volumes and his arriving time. Denote by  $L(x) = P\{\zeta < x\}$  its DF and by  $\varphi(s) = \int_0^{\infty} e^{-sx} dL(x)$  – its LST. The

$k$ th moment of RV  $\zeta$  we shall denote by  $\varphi_k$ ,  $k = 1, 2, \dots$ . Assume that service time  $\xi$  of a customer is proportional to his volume:  $\xi = c\zeta$ ,  $c > 0$ . Then, we obtain for DF and LST of RV  $\xi$ :  $B(t) = P\{\xi < t\} = L(t/c)$ ,  $\beta(q) = \int_0^{\infty} e^{-qt} dB(t) = \varphi(cq)$ .

Denote by  $\beta_k$ ,  $k = 1, 2, \dots$ , the  $k$ th moment of service time  $\xi$ . It is clear that  $\beta_k = c^k \varphi_k$ . We shall denote by  $\eta$  the steady-state number of customers present in the system. Let  $\sigma$  be the whole sum of the volumes of customers present in the system in steady state, i.e. RV  $\sigma$  is the total volume of these customers. Let  $D(x) = P\{\sigma < x\}$  be DF of RV  $\sigma$  and  $\delta(s) = \int_0^{\infty} e^{-sx} dD(x)$  be its LST. Our main aim is the determination of the function  $\delta(s)$ .

## 3. General solution

Let  $\gamma$  be a customer sojourn time, i.e. the length of time interval from the moment of his arrival to the system to his service termination. Denote by  $V(t) = P\{\gamma < t\}$  DF of RV  $\gamma$  and by  $v(q) = \int_0^{\infty} e^{-qt} dV(t)$  its LST. Note that RVs  $\zeta$  and  $\gamma$  are generally dependent, because service time (being a part of sojourn time) of the customer depends on his volume. Let  $R(x, t) = P\{\zeta < x, \gamma < t\}$  be the joint DF of RVs  $\zeta$  and  $\gamma$ ,  $r(s, q) = \int_0^{\infty} \int_0^{\infty} e^{-sx-qt} dR(x, t)$  be double LST of DF  $R(x, t)$ . First, we determine  $r(s, q)$ .

Denote by  $\nu$  the mode indicator of customer's arrival epoch: assume that  $\nu = 0$ , if the customer arrives to the system when it is in the good repair mode, and  $\nu = 1$ , if the customer arrives to the system when it is in the repair mode. Let  $x$  be the customer volume ( $\zeta = x$ ). Determine the following conditional DF:

$$V(t | \nu = 0, \zeta = x) = P\{\gamma < t | \nu = 0, \zeta = x\}. \quad (1)$$

Evidently (from the total probability theorem), we have:

$$V(t|v=0, \zeta=x) = \sum_{k=0}^{\infty} \frac{(\alpha cx)^k}{k!} e^{-\alpha cx} G_*^{(k)}(t-cx), \quad (2)$$

where  $G_*^{(k)}(t)$  is the  $k$ th Stieltjes convolution of DF  $G(t)$ .  $G_*^{(k)}(t-cx)$  means that the total sojourn time of the customer is the sum of his service time  $cx$  and  $k$  independent intervals having the same DF  $G(t)$  (under condition that  $\zeta=x$ ,  $v=0$  and during service the system was broken  $k$  times).

Denote by  $R(x, t|v=i)$  the next conditional DF:

$$R(x, t|v=i) = P\{\zeta < x, \gamma < t|v=i\}, \quad i=0, 1.$$

Taking into consideration the probability sense, we obtain from the relation (1):

$$\begin{aligned} dR(x, t|v=0) &= P\{\zeta \in [x; x+dx), \gamma \in [t; t+dt)|v=0\} = \\ &= dV(t|v=0, \zeta=x)dL(x). \end{aligned}$$

Then, based on formula (2), double LST (with respect to  $x$  and  $t$ ) of the function  $R(x, t|v=0)$  takes the form:

$$\begin{aligned} r(s, q|v=0) &= \int_0^{\infty} \int_0^{\infty} e^{-sx-qt} dR(x, t|v=0) = \\ &= \int_0^{\infty} \int_0^{\infty} e^{-sx-qt} \sum_{k=0}^{\infty} \frac{(\alpha cx)^k}{k!} e^{-\alpha cx} d_t [G_*^{(k)}(t-cx)] dL(x) = \\ &= \int_0^{\infty} e^{-sx-qt} \sum_{k=0}^{\infty} \frac{(\alpha cx)^k}{k!} e^{-\alpha cx} e^{-qcx} (g(q))^k dL(x) = \\ &= \varphi(s+cq+c\alpha(1-g(q))). \end{aligned}$$

It is clear that if a customer arrives to the system when it is in repair mode, his sojourn time includes time interval  $\psi$  from his arriving epoch to the repair period termination. Assume that  $g_1 = E\chi = \int_0^{\infty} t dG(t) < \infty$ . Then, DF  $K(t)$  of RV  $\psi$  has the form:

$$K(t) = (g_1)^{-1} \int_0^t [1-G(u)] du$$

and its LST  $\kappa(q)$  is determined by the relation (see e.g. [2]):

$$\kappa(q) = \frac{1-g(q)}{qg_1}.$$

Therefore (on the base of LST properties) we obtain:

$$r(s, q|v=1) = \frac{1-g(q)}{qg_1} r(s, q|v=0).$$

It is evident that  $P\{v=0\} = \frac{1}{1+\alpha g_1}$  and  $P\{v=1\} = \frac{\alpha g_1}{1+\alpha g_1}$ , whereas we obtain:

$$\begin{aligned} r(s, q) &= r(s, q|v=0)P\{v=0\} + r(s, q|v=1)P\{v=1\} = \\ &= \frac{r(s, q|v=0)}{1+\alpha g_1} \left\{ 1 + \frac{\alpha[1-g(q)]}{q} \right\} = \\ &= \frac{\varphi(s+cq+c\alpha(1-g(q)))}{1+\alpha g_1} \left\{ 1 + \frac{\alpha[1-g(q)]}{q} \right\}. \quad (3) \end{aligned}$$

From the relation (3) we can obtain LST  $v(q)$  of customer's sojourn time:

$$\begin{aligned} v(q) = r(0, q) &= \frac{\varphi(cq+c\alpha(1-g(q)))}{1+\alpha g_1} \left\{ 1 + \frac{\alpha[1-g(q)]}{q} \right\} = \\ &= \frac{\beta(q+\alpha(1-g(q)))}{1+\alpha g_1} \left\{ 1 + \frac{\alpha[1-g(q)]}{q} \right\}. \quad (4) \end{aligned}$$

We can calculate the moments of RV  $\gamma$  (if exist) using the relation (4). For example, the first and second moments take the form:

$$v_1 = E\gamma = -v'(q)|_{q=0} = \beta_1(1+\alpha g_1) + \frac{\alpha g_2}{2(1+\alpha g_1)}, \quad (5)$$

$$\begin{aligned} v_2 = E\gamma^2 &= v''(q)|_{q=0} = \\ &= 2\alpha\beta_1 g_2 + \beta_2(1+\alpha g_1)^2 + \frac{\alpha g_3}{3(1+\alpha g_1)}. \quad (6) \end{aligned}$$

It follows from [2] that LST  $\delta(s)$  of DF of steady-state total volume  $\sigma$ , in our case, takes the following form:

$$\begin{aligned} \delta(s) &= \exp\{-a[v_1 + r'_q(s, q)|_{q=0}]\} = \\ &= \exp\left\{-a\left[c(1+\alpha g_1)(\varphi'(s) + \varphi_1) + \frac{\alpha g_2(1-\varphi(s))}{2(1+\alpha g_1)}\right]\right\}. \quad (7) \end{aligned}$$

The first and second moments of total customers volume take the form:

$$\begin{aligned} \delta_1 = E\sigma &= -\delta'(s)|_{s=0} = \\ &= a\left[c(1+\alpha g_1)\varphi_2 + \frac{\alpha\varphi_1 g_2}{2(1+\alpha g_1)}\right], \quad (8) \end{aligned}$$

$$\begin{aligned} \delta_2 = E\sigma^2 &= \delta''(s)|_{s=0} = \\ &= \delta_1^2 + a\left[c(1+\alpha g_1)\varphi_3 + \frac{\alpha\varphi_2 g_2}{2(1+\alpha g_1)}\right]. \quad (9) \end{aligned}$$

#### 4. System $M/G/\infty$ under control of $M/G/1/\infty$ system

Consider the system  $M/G/\infty$  with customers of random volume as a model of a database table serving customers that read information from the table. All these customers are served without waiting (as in the system under consideration). Besides

reading customers, customers that modify tables information (writing customers) can arrive. If such customer arrives during presence of reading ones in the system, the service of them is immediately interrupted and will be continued after modification period termination. It means that reading customers are blocked as long as writing customers appear. This model may be treated as an interesting one for *readers and writers problem*. Assume that writing customers form a Poisson entrance flow with parameter  $\alpha$  and denote by  $K(t)$  DF of their service time. The process of table modification is described by  $M/G/1/\infty$  queueing model. The behavior of this system does not depend on the reading process. The analysis of this situation leads to our previous model (Section 3), if we assume that  $G(t) = \Pi(t)$ , where  $\Pi(t)$  is DF of busy period of the control  $M/G/1/\infty$  system. Let  $\kappa(q) = \int_0^\infty e^{-qt} dK(t)$  be LST of DF

$K(t)$  and  $\pi(q) = \int_0^\infty e^{-qt} d\Pi(t)$  be LST of DF  $\Pi(t)$ ,  $\kappa_i$  and  $\pi_i$  be the  $i$ th moments of service time and busy period of the control system, respectively,  $i = 1, 2, \dots$ .

Then, in this case, the functions  $r(s, q)$  and  $v(q)$  can be obtained from the relations (3) and (4), where  $g(q) = \pi(q)$ . It is known [2] that the function  $\pi(q)$  is a unique solution of the functional equation  $\pi(q) = \kappa(q + \alpha - \alpha\pi(q))$  (under assumption  $\alpha\kappa_1 < 1$ ). So, LST  $\delta(s)$  and first two moments of the sojourn time  $\gamma$  and total volume  $\sigma$  can be determined by the relations (5)–(9), where  $g_i = \pi_i$ ,  $i = 1, 2, 3$ . It is known (see e.g. [2]) that

$$\pi_1 = \frac{\kappa_1}{1 - \alpha\kappa_1}, \quad \pi_2 = \frac{\kappa_2}{(1 - \alpha\kappa_1)^3},$$

$$\pi_3 = \frac{\kappa_3}{(1 - \alpha\kappa_1)^4} + \frac{3\alpha\kappa_2^2}{(1 - \alpha\kappa_1)^5}.$$

### 5. The case of exponentially distributed customer volume

If customer volume has an exponential distribution with parameter  $f$ , we have  $\varphi(s) = \frac{f}{s+f}$ . Then, the function  $r(s, q)$  takes the form:

$$r(s, q) = \frac{f(q + \alpha - \alpha g(q))}{q(1 + \alpha g_1)[s + f + c(q + \alpha - \alpha g(q))]}.$$

The function  $v(q)$  is determined as follows:

$$v(q) = \frac{\mu(q + \alpha - \alpha g(q))}{q(1 + \alpha g_1)(q + \mu + \alpha - \alpha g(q))},$$

where  $\mu = f/c$  is the parameter of service time. The first and second moments of sojourn time are determined as

$$v_1 = \frac{1 + \alpha g_1}{\mu} + \frac{\alpha g_2}{2(1 + \alpha g_1)},$$

$$v_2 = \frac{2\alpha g_2}{\mu} + \frac{2(1 + \alpha g_1)^2}{\mu^2} + \frac{\alpha g_3}{3(1 + \alpha g_1)}.$$

Assume additionally that the repair period is also exponentially distributed with parameter  $b$ . Then, we have  $g(q) = \frac{b}{b+q}$ ,  $g_1 = 1/b$ ,  $g_2 = 2/b^2$  and  $g_3 = 6/b^3$ . Hence, the function  $v(q)$  takes the form:

$$v(q) = \frac{\mu b(q + \alpha + b)}{(\alpha + b)[q^2 + (\mu + \alpha + b)q + \mu b]}.$$

Therefore, DF  $V(t) = P\{\gamma < t\}$  can be obtained by Laplace transform  $v(q)/q$  inversion. Finally, we have:

$$V(t) = 1 - \frac{\mu b}{(\alpha + b)(q_2 - q_1)} \left( \frac{\alpha + b - q_1}{q_1} e^{-q_1 t} - \frac{\alpha + b - q_2}{q_2} e^{-q_2 t} \right),$$

where

$$q_1 = \frac{\mu + \alpha + b - \sqrt{(\mu + \alpha + b)^2 - 4\mu b}}{2},$$

$$q_2 = \frac{\mu + \alpha + b + \sqrt{(\mu + \alpha + b)^2 - 4\mu b}}{2}.$$

The first and second moments of RV  $\gamma$  have the form:

$$v_1 = \frac{1}{b} \left( \frac{\alpha + b}{\mu} + \frac{\alpha}{\alpha + b} \right),$$

$$v_2 = \frac{2}{b^2} \left[ \frac{2\alpha}{\mu} + \frac{(\alpha + b)^2}{\mu^2} + \frac{\alpha}{\alpha + b} \right].$$

In a similar way, we obtain relation for  $\delta(s)$  in the case of exponential distribution of customer volume:

$$\delta(s) = \exp \left\{ -\frac{as}{s+f} \left[ \frac{c(1 + \alpha g_1)(s + 2f)}{f(s+f)} + \frac{\alpha g_2}{2(1 + \alpha g_1)} \right] \right\}.$$

For the first and second moments of RV  $\sigma$  we have in this case:

$$\delta_1 = \frac{a}{f} \left[ \frac{2c(1 + \alpha g_1)}{f} + \frac{\alpha g_2}{2(1 + \alpha g_1)} \right],$$

$$\delta_2 = \delta_1^2 + \frac{a}{f^2} \left[ \frac{6c(1 + \alpha g_1)}{f} + \frac{\alpha g_2}{1 + \alpha g_1} \right].$$

In the case when repair period is also exponentially distributed with parameter  $b$ , we easily obtain:

$$\delta(s) = \exp \left\{ -\frac{as}{s+f} \left[ \frac{c(b+\alpha)(s+2f)}{bf(s+f)} + \frac{\alpha}{b(b+\alpha)} \right] \right\},$$

$$\delta_1 = \frac{a}{f} \left[ \frac{2c(b+\alpha)}{bf} + \frac{\alpha}{b(b+\alpha)} \right],$$

$$\delta_2 = \delta_1^2 + \frac{a}{f^2} \left[ \frac{6c(b+\alpha)}{bf} + \frac{2\alpha}{b(b+\alpha)} \right].$$

### 6. Customers number distribution

Using the relation (7), we can also determine the steady-state generating function  $P(z) = \sum_{k=0}^{\infty} P\{\eta = k\}z^k$  of number of customers present in the system. Indeed, RV  $\eta$  can be treated as the total volume of customers with the same volume  $\zeta \equiv 1$  and  $c = 1$ . Then, we have  $\varphi(s) = e^{-s}$ ,  $\varphi_1 = 1$  and the function  $\delta(s)$  takes the form:

$$\delta(s) = \exp \left\{ -a(1 - e^{-s}) \left[ 1 + \alpha g_1 + \frac{\alpha g_2}{2(1 + \alpha g_1)} \right] \right\}.$$

It is clear that  $\delta(s) = P(e^{-s})$ , whereas we obtain:

$$P(z) = \exp \left\{ -a(1 - z) \left[ 1 + \alpha g_1 + \frac{\alpha g_2}{2(1 + \alpha g_1)} \right] \right\} = e^{-ah} e^{azh},$$

where  $h = 1 + \alpha g_1 + \frac{\alpha g_2}{2(1 + \alpha g_1)}$ . Since  $e^{azh} = \sum_{k=0}^{\infty} \frac{(ah)^k}{k!} z^k$ , we can conclude that the number of customers present in the system under consideration has Poisson distribution with parameter  $ah$ , i.e.:

$$P\{\eta = k\} = \frac{(ah)^k}{k!} e^{-ah}, \quad k = 0, 1, \dots \quad (10)$$

### 7. Estimation of loss characteristics for the system with limited total volume

It is clear that, using the relation (7), we can obtain the explicit form of DF  $D(x)$  very rarely. But we can approximate it by DF  $D^*(x)$  having the form:

$$D^*(x) = p_0 + (1 - p_0) \frac{\gamma(p, gx)}{\Gamma(p)}, \quad (11)$$

where  $\gamma(p, gx)$  is incomplete gamma-function, i.e.  $\gamma(p, gx) = \int_0^{gx} t^{p-1} e^{-t} dt$  and  $\Gamma(p) = \gamma(p, \infty)$  is gamma-function,  $p_0 = P\{\eta = 0\} = P\{\sigma = 0\} = e^{-ah}$ , as it follows from (10). Let

$\delta_1^*$  and  $\delta_2^*$  be the first and second moment of RV with DF  $D^*(x)$ , respectively. It follows from (11) that

$$\delta_1^* = \frac{(1 - p_0)p}{g},$$

$$\delta_2^* = \frac{(1 - p_0)p(p + 1)}{g^2}.$$

Parameters  $p$  and  $g$  can be chosen so that the first and second moments of approximate distribution function have to be equal to corresponding moments of the DF  $D(x)$ . Finally, we obtain:

$$p = \frac{\delta_1^2}{(1 - p_0)\delta_2 - \delta_1^2},$$

$$g = \frac{(1 - p_0)\delta_1}{(1 - p_0)\delta_2 - \delta_1^2},$$

where the moments  $\delta_1$  and  $\delta_2$  are determined by relations (8) and (9), respectively.

Let us consider a system  $M/G/(\infty, V)$  which differs from the system analyzed in Sec. 2–6 in total volume limitation. Let  $V = const$  is the system buffer space, such that  $\sigma(t) < V$  for all  $t > 0$ . Let a customer having the volume  $x$  arrive to the system at epoch  $t$ . Then, he will be accepted for service, if  $\sigma(t^-) + x \leq V$ . Otherwise the customer will be lost. Therefore, for this system, we have to introduce some measures of losses. Such traditional characteristic is the loss probability  $P_{loss}$ , i.e. the probability that arriving customer will be lost (it has a sense of a part of lost customers). In this case, it is determined as follows:

$$P_{loss} = 1 - \int_0^V D_V(V - x) dL(x).$$

Let  $D_V(x)$  be DF of steady-state total volume  $\sigma$  in the system with limited buffer space. We evidently have (see e.g. [2]):

$$P_{loss} \leq P_{loss}^* = 1 - \int_0^V D(V - x) dL(x), \quad (12)$$

since  $D(x) \leq D_V(x)$  for all  $x \geq 0$ .

Other measure of losses is the probability of unit of volume loss  $Q_{loss}$ , having the sense of a part of lost volume (information). For the system with limited volume, it can be determined as:

$$Q_{loss} = 1 - \frac{1}{\varphi_1} \int_0^V x D_V(V - x) dL(x).$$

It can be easily shown that

$$Q_{loss} \leq Q_{loss}^* = 1 - \frac{1}{\varphi_1} \int_0^V x D(V - x) dL(x). \quad (13)$$

It can be also proved [2] that  $Q_{loss} \geq P_{loss}$ .

The values  $P_{loss}^*$  and  $Q_{loss}^*$  can be the estimators of loss characteristics  $P_{loss}$  and  $Q_{loss}$ , respectively, under assumption that customers losses are rather rare events. For their approximate calculation, we can replace the function  $D(x)$  by  $D^*(x)$  in relations (12) and (13). Of course, in such case, it is possible that relations (12)–(13) will not be satisfied, as it is connected with approximation mistake. Now we present some numerical examples in which we obtain estimators of loss characteristics for the system with limited total volume and compare them to results obtained by simulation. Approximations  $P_{loss}^*$  and  $Q_{loss}^*$  were calculated with the help of *Mathematica* environment [20] and simulation results  $P_{loss}^{SIM}$  and  $Q_{loss}^{SIM}$  were obtained with the help of previously written *Python3* scripts [22] and the usage of discrete event simulation method (DES) [23].

**7.1. Customer volume and repair period exponentially distributed.** Assume that the arrival flow parameter  $a = 1$ , parameter of presence system in good repair mode  $\alpha = 2$ , coefficient  $c = 1$  and customer volume is exponentially distributed with parameter  $f = 2$ . Suppose additionally that repair period is also exponentially distributed with parameter  $b = 1$ . Then, on the base of calculations from Section 5 and using (12), (13) formulae, we can find loss characteristics approximations of analyzed system. They are presented in Table 1.

Table 1  
Loss characteristics  $a = 1, \alpha = 2, c = 1, f = 2, b = 1$

| $V$ | $P_{loss}^{SIM}$ | $Q_{loss}^{SIM}$ | $P_{loss}^*$ | $Q_{loss}^*$ |
|-----|------------------|------------------|--------------|--------------|
| 2   | 0.218136         | 0.409677         | 0.483899     | 0.625923     |
| 4   | 0.076142         | 0.155185         | 0.143673     | 0.207518     |
| 6   | 0.024476         | 0.051370         | 0.038280     | 0.057290     |
| 8   | 0.007105         | 0.014971         | 0.009775     | 0.014854     |
| 10  | 0.001666         | 0.003756         | 0.002440     | 0.003739     |
| 12  | 0.000399         | 0.000933         | 0.000600     | 0.000925     |
| 14  | 0.000076         | 0.000180         | 0.000146     | 0.000226     |
| 16  | 0.000015         | 0.000036         | 0.000035     | 0.000055     |

Now we change only one parameter. Suppose that  $b = 1.5$ . Then, of course, loss characteristics are decreasing, what is presented in Table 2.

Now we change one more parameter. Assume that  $a = 0.75$ . In such case loss characteristics are also decreasing (Table 3).

**7.2. Customer volume exponentially distributed and repair period uniformly distributed.** One more time we assume that the arrival flow parameter  $a = 1$ , parameter of presence system in good repair mode  $\alpha = 2$ , coefficient  $c = 1$  and customer volume is exponentially distributed with parameter  $f = 2$ . But now we suppose that the repair period is uniformly distributed on the interval  $[0, 2]$ . Then we have  $g_1 = 1$  and  $g_2 = 4/3$  and, on the base of calculations from Section 5, we can obtain loss characteristics approximations that are presented in Table 4.

Table 2  
Loss characteristics  $a = 1, \alpha = 2, c = 1, f = 2, b = 1.5$

| $V$ | $P_{loss}^{SIM}$ | $Q_{loss}^{SIM}$ | $P_{loss}^*$ | $Q_{loss}^*$ |
|-----|------------------|------------------|--------------|--------------|
| 2   | 0.170649         | 0.347311         | 0.356096     | 0.503283     |
| 4   | 0.046844         | 0.104930         | 0.086337     | 0.133603     |
| 6   | 0.011270         | 0.026692         | 0.020003     | 0.031568     |
| 8   | 0.002267         | 0.005576         | 0.004577     | 0.007251     |
| 10  | 0.000410         | 0.001104         | 0.001042     | 0.001654     |
| 12  | 0.000087         | 0.000231         | 0.000236     | 0.000376     |
| 14  | 0.000011         | 0.000019         | 0.000053     | 0.000085     |
| 16  | 0.000001         | 0.000005         | 0.000012     | 0.000019     |

Table 3  
Loss characteristics  $a = 0.75, \alpha = 2, c = 1, f = 2, b = 1.5$

| $V$ | $P_{loss}^{SIM}$  | $Q_{loss}^{SIM}$ | $P_{loss}^*$ | $Q_{loss}^*$ |
|-----|-------------------|------------------|--------------|--------------|
| 2   | 0.136910          | 0.298190         | 0.263957     | 0.406753     |
| 4   | 0.030872          | 0.073847         | 0.053377     | 0.088405     |
| 6   | 0.005597          | 0.014215         | 0.010800     | 0.017964     |
| 8   | 0.000912          | 0.002536         | 0.002203     | 0.003656     |
| 10  | 0.000129          | 0.000391         | 0.000452     | 0.000748     |
| 12  | 0.000019          | 0.000049         | 0.000093     | 0.000154     |
| 14  | 0.000003          | 0.000006         | 0.000019     | 0.000032     |
| 16  | $5 \cdot 10^{-7}$ | 0.000002         | 0.000004     | 0.000007     |

Table 4  
Loss characteristics  $a = 1, \alpha = 2, c = 1, f = 2, g_1 = 1$

| $V$ | $P_{loss}^{SIM}$ | $Q_{loss}^{SIM}$ | $P_{loss}^*$ | $Q_{loss}^*$ |
|-----|------------------|------------------|--------------|--------------|
| 2   | 0.201224         | 0.392348         | 0.453320     | 0.597143     |
| 4   | 0.065004         | 0.139362         | 0.129946     | 0.189749     |
| 6   | 0.017467         | 0.040206         | 0.034086     | 0.051309     |
| 8   | 0.004362         | 0.010480         | 0.008651     | 0.013173     |
| 10  | 0.000801         | 0.002012         | 0.002157     | 0.003306     |
| 12  | 0.000148         | 0.000405         | 0.000532     | 0.000819     |
| 14  | 0.000019         | 0.000054         | 0.000130     | 0.000201     |
| 16  | 0.000005         | 0.000020         | 0.000032     | 0.000049     |

If we decrease parameter of the uniformly distributed repair period, then loss characteristics are decreasing. For example, if repair period is uniformly distributed on the interval  $[0, 4/3]$  (then  $g_1 = 2/3$  and  $g_2 = 16/27$ ), loss characteristics are as shown in Table 5.

## Performance evaluation of unreliable system with infinite number of servers

Table 5

Loss characteristics  $a = 1$ ,  $\alpha = 2$ ,  $c = 1$ ,  $f = 2$ ,  $g_1 = 2/3$ 

| $V$ | $P_{loss}^{SIM}$ | $Q_{loss}^{SIM}$ | $P_{loss}^*$ | $Q_{loss}^*$ |
|-----|------------------|------------------|--------------|--------------|
| 2   | 0.160724         | 0.335958         | 0.338173     | 0.484500     |
| 4   | 0.040880         | 0.095678         | 0.080402     | 0.125233     |
| 6   | 0.008796         | 0.022070         | 0.018509     | 0.029236     |
| 8   | 0.001576         | 0.004201         | 0.004234     | 0.006708     |
| 10  | 0.000252         | 0.000661         | 0.000966     | 0.001532     |
| 12  | 0.000041         | 0.000123         | 0.000220     | 0.000349     |
| 14  | 0.000009         | 0.000028         | 0.000050     | 0.000080     |
| 16  | 0.000001         | 0.000003         | 0.000011     | 0.000018     |

If we change additionally parameter  $a$  (e.g.  $a = 0.75$ ) then loss characteristics also decrease, which is presented in Table 6.

Table 6

Loss characteristics  $a = 0.75$ ,  $\alpha = 2$ ,  $c = 1$ ,  $f = 2$ ,  $g_1 = 2/3$ 

| $V$ | $P_{loss}^{SIM}$  | $Q_{loss}^{SIM}$  | $P_{loss}^*$ | $Q_{loss}^*$ |
|-----|-------------------|-------------------|--------------|--------------|
| 2   | 0.128631          | 0.287237          | 0.250857     | 0.391729     |
| 4   | 0.026580          | 0.066338          | 0.050030     | 0.083729     |
| 6   | 0.004615          | 0.012614          | 0.010106     | 0.016820     |
| 8   | 0.000617          | 0.001842          | 0.002068     | 0.003427     |
| 10  | 0.000102          | 0.000301          | 0.000426     | 0.000704     |
| 12  | 0.000006          | 0.000011          | 0.000088     | 0.000146     |
| 14  | 0.000003          | 0.000004          | 0.000018     | 0.000030     |
| 16  | $1 \cdot 10^{-7}$ | $2 \cdot 10^{-7}$ | 0.000004     | 0.000006     |

### 7.3. Customer volume uniformly distributed and repair period exponentially distributed.

Assume that  $a = 1$ ,  $\alpha = 2$ ,  $c = 1$  but this time customer volume is uniformly distributed on the interval  $[0, 1]$ . Then we have  $\varphi_1 = 1/2$ ,  $\varphi_2 = 1/3$  and  $\varphi_3 = 1/4$ . Suppose that repair period is exponentially distributed with parameter  $b = 1$ . From the formulae (8), (9) we obtain the next loss characteristics approximations. We present them in Table 7.

Table 7

Loss characteristics  $a = 1$ ,  $\alpha = 2$ ,  $c = 1$ ,  $\varphi_1 = 1/2$ ,  $b = 1$ 

| $V$ | $P_{loss}^{SIM}$ | $Q_{loss}^{SIM}$ | $P_{loss}^*$ | $Q_{loss}^*$ |
|-----|------------------|------------------|--------------|--------------|
| 2   | 0.234872         | 0.305140         | 0.520424     | 0.768946     |
| 4   | 0.048110         | 0.063562         | 0.118915     | 0.296182     |
| 6   | 0.008071         | 0.010651         | 0.020292     | 0.076842     |
| 8   | 0.001231         | 0.001646         | 0.003064     | 0.016242     |
| 10  | 0.000188         | 0.000252         | 0.000437     | 0.003052     |
| 12  | 0.000016         | 0.000024         | 0.000061     | 0.000533     |
| 14  | 0.000001         | 0.000001         | 0.000008     | 0.000089     |

Now we change only parameter  $b$  ( $b = 1.5$ ). Then loss characteristics are decreasing. We present this fact in Table 8.

Table 8

Loss characteristics  $a = 1$ ,  $\alpha = 2$ ,  $c = 1$ ,  $\varphi_1 = 1/2$ ,  $b = 1.5$ 

| $V$ | $P_{loss}^{SIM}$  | $Q_{loss}^{SIM}$  | $P_{loss}^*$ | $Q_{loss}^*$ |
|-----|-------------------|-------------------|--------------|--------------|
| 2   | 0.164901          | 0.219152          | 0.405249     | 0.685058     |
| 4   | 0.019035          | 0.026077          | 0.077075     | 0.227167     |
| 6   | 0.001628          | 0.002263          | 0.011851     | 0.053214     |
| 8   | 0.000140          | 0.000199          | 0.001687     | 0.010505     |
| 10  | 0.000011          | 0.000016          | 0.000233     | 0.001884     |
| 12  | $4 \cdot 10^{-7}$ | $7 \cdot 10^{-7}$ | 0.000032     | 0.000318     |
| 14  | $2 \cdot 10^{-7}$ | $3 \cdot 10^{-7}$ | 0.000004     | 0.000052     |

Now we change also parameter  $a$ . Assume that  $a = 0.75$ . For this case, loss characteristics are presented in Table 9.

Table 9

Loss characteristics  $a = 0.75$ ,  $\alpha = 2$ ,  $c = 1$ ,  $\varphi_1 = 1/2$ ,  $b = 1.5$ 

| $V$ | $P_{loss}^{SIM}$  | $Q_{loss}^{SIM}$  | $P_{loss}^*$ | $Q_{loss}^*$ |
|-----|-------------------|-------------------|--------------|--------------|
| 2   | 0.117111          | 0.157804          | 0.329611     | 0.621760     |
| 4   | 0.008558          | 0.011843          | 0.055620     | 0.186190     |
| 6   | 0.000471          | 0.000641          | 0.008052     | 0.040870     |
| 8   | 0.000012          | 0.000019          | 0.001113     | 0.007739     |
| 10  | $4 \cdot 10^{-7}$ | $5 \cdot 10^{-7}$ | 0.000152     | 0.001350     |

### 7.4. Customer volume and repair period uniformly distributed.

Once again we assume that the arrival flow parameter  $a = 1$ , parameter of presence system in good repair mode  $\alpha = 2$ , coefficient  $c = 1$  and customer volume is uniformly distributed on the interval  $[0, 1]$ . But now we suppose that that repair period is uniformly distributed on the interval  $[0, 2]$ . Then we have  $b_1 = 1$  and  $b_2 = 4/3$  and, using calculations from Section 4, we obtain the following loss characteristics approximations (Table 10).

If we decrease parameter of the uniformly distributed repair period, loss characteristics are also decreasing. If repair period is uniformly distributed on the interval  $[0, 4/3]$ , loss characteristics present like in Table 11.

If we decrease once again entrance flow parameter into  $a = 0.75$  then loss characteristics also decrease, what is presented in Table 12.

If we analyze obtain numerical results, we can notice some interesting facts:

- Approximation of loss characteristics is much better for small values of  $P_{loss}$  and  $Q_{loss}$ , so we can use them only in such cases. For bigger values of loss characteristics we have use simulation methods.

Table 10

Loss characteristics  $a = 1$ ,  $\alpha = 2$ ,  $c = 1$ ,  $\varphi_1 = 1/2$ ,  $g_1 = 1$ 

| $V$ | $P_{loss}^{SIM}$  | $Q_{loss}^{SIM}$  | $P_{loss}^*$ | $Q_{loss}^*$ |
|-----|-------------------|-------------------|--------------|--------------|
| 2   | 0.212566          | 0.280554          | 0.485251     | 0.744637     |
| 4   | 0.031640          | 0.042970          | 0.105184     | 0.274415     |
| 6   | 0.002931          | 0.004013          | 0.017441     | 0.069127     |
| 8   | 0.000250          | 0.000337          | 0.002592     | 0.014334     |
| 10  | 0.000018          | 0.000023          | 0.000367     | 0.002659     |
| 12  | $3 \cdot 10^{-7}$ | $4 \cdot 10^{-7}$ | 0.000051     | 0.000460     |

Table 11

Loss characteristics  $a = 1$ ,  $\alpha = 2$ ,  $c = 1$ ,  $\varphi_1 = 1/2$ ,  $g_1 = 2/3$ 

| $V$ | $P_{loss}^{SIM}$  | $Q_{loss}^{SIM}$  | $P_{loss}^*$ | $Q_{loss}^*$ |
|-----|-------------------|-------------------|--------------|--------------|
| 2   | 0.150319          | 0.201998          | 0.384964     | 0.668666     |
| 4   | 0.012168          | 0.017010          | 0.071193     | 0.216118     |
| 6   | 0.000653          | 0.000913          | 0.010806     | 0.049852     |
| 8   | 0.000021          | 0.000029          | 0.001529     | 0.009748     |
| 10  | 0.000002          | 0.000003          | 0.000211     | 0.001738     |
| 12  | $1 \cdot 10^{-7}$ | $2 \cdot 10^{-7}$ | 0.000029     | 0.000292     |

Table 12

Loss characteristics  $a = 0.75$ ,  $\alpha = 2$ ,  $c = 1$ ,  $\varphi_1 = 1/2$ ,  $g_1 = 2/3$ 

| $V$ | $P_{loss}^{SIM}$  | $Q_{loss}^{SIM}$  | $P_{loss}^*$ | $Q_{loss}^*$ |
|-----|-------------------|-------------------|--------------|--------------|
| 2   | 0.104941          | 0.142867          | 0.314908     | 0.608354     |
| 4   | 0.004722          | 0.006751          | 0.052164     | 0.178749     |
| 6   | 0.000141          | 0.000208          | 0.007500     | 0.038831     |
| 8   | 0.000003          | 0.000005          | 0.001035     | 0.007308     |
| 10  | $1 \cdot 10^{-7}$ | $3 \cdot 10^{-7}$ | 0.000141     | 0.001270     |

- Approximation of loss probability  $P_{loss}$  is better than approximation of  $Q_{loss}$  characteristic.
- Approximations in the case of exponentially distributed customer volume are better than in the case of uniformly distributed one (compare results from Table 1 – Table 6 and Table 7 – Table 12). In the case of uniformly distributed customer volume, approximation mistakes are relatively big.
- Loss characteristics (and their approximations) strictly depend on the forms of distribution functions of customer volume and repair period (not only on their first moments). In all calculations, we choose parameters of customer volume and repair period in such way that first moments were the same (e.g. compare Table 1, 4, 7 and 10 and analogously Table 2, 5, 8 and 11 or Table 3, 6, 9 and 12).
- Loss characteristics for  $M/G/(\infty, V)$ -type system with unreliable servers depend on: arrival rate  $a$ , parameter of pres-

ence system in good repair mode  $\alpha$ , coefficient  $c$  and the forms of distribution functions of customer volume and repair period.

## 8. Conclusions and final remarks

In the present paper, we investigated queueing system of the  $M/G/\infty$ -type with non-homogeneous customers, unlimited memory space and unreliable servers. In the beginning, after exact mathematical analysis, we obtained Laplace-Stieltjes transforms of customer sojourn time and customers total volume in steady state together with formulae defining first moments of these random variables. Later on, we presented some interesting special cases of analyzed model including the case of exponentially distributed customer volume and interesting model of database table that can be treated as possible representation of readers and writers problem. Then, we showed how to use obtained formulae to calculate the steady-state distribution of number of customers for the system under consideration. Finally, we discussed how and when one can use the obtained loss characteristics approximations and simulation to analogous system with limited memory space. We illustrated our discussion with some numerical calculations for some special cases of analyzed model, comparing them with simulation results. Both the theoretical and simulation results show that loss characteristics for the model with limited total volume strictly depend on the forms of customer volume and repair period distributions. Practical applications of analyzed model and obtained results are possible in computer systems designing. Indeed, we may use proper numerical characteristics to choose the size of needed memory volume so as to avoid excessive loss characteristics.

## REFERENCES

- [1] P.P. Bocharov, C. D'Apice, A.V. Pechinkin, and S. Salerno: *Queueing Theory*, VSP, Utrecht-Boston, 2004.
- [2] O. Tikhonenko, *Computer Systems Probability Analysis*, Akademicka Oficyna Wydawnicza EXIT, Warsaw, 2006 (In Polish).
- [3] M. Schwartz, *Computer-communication Network Design and Analysis*, Prentice-Hall, Englewood Cliffs, New York, 1977.
- [4] M. Schwartz, *Telecommunication Networks: Protocols, Modeling and Analysis*, Addison-Wesley Publishing Company, New York, 1987.
- [5] E. Morozov, R. Nekrasova, L. Potakhina, and O. Tikhonenko, "Asymptotic analysis of queueing systems with finite buffer space", In: Kwiecien, A., Gaj, P., Stera, P. (eds.) CN 2014, *CCIS* 431, Springer, Cham, 223–232 (2014).
- [6] O.M. Tikhonenko, "Destriicted capacity queueing systems: determination of their characteristics", *Autom. Remote Control* 58 (6), 969–972 (1997).
- [7] O.M. Tikhonenko and K.G. Klimovich, "Queueing systems for random-length arrivals with limited cumulative volume", *Problems of Information Transmission* 37 (1), 70–79 (2001).
- [8] O.M. Tikhonenko, "Generalized Erlang problem for service systems with finite total capacity", *Problems of Information Transmission* 41 (3), 243–253 (2005).

- [9] O.M. Tikhonenko, "Queueing systems with processor sharing and limited resources", *Autom. Remote Control*, 71 (5), 803–815 (2010).
- [10] M. Ziółkowski, " $M/\bar{G}/n/0$  Erlang queueing system with heterogeneous servers and non-homogeneous customers", *Bull. Pol. Ac.: Tech.* 66 (1), 59–66 (2018).
- [11] O. Tikhonenko and M. Ziółkowski, "Single server queueing system with external and internal customers", *Bull. Pol. Ac.: Tech.* 66 (4), 539–551 (2018).
- [12] O. Tikhonenko and M. Ziółkowski, "Queueing models of systems with non-homogeneous customers and their applications in computer science", In: Proceedings of the IEEE 15th International Scientific Conference on Informatics "Informatics 2019", Poprad 20-22 November 2019, 423–428 (2019).
- [13] A.M. Alexandrov and B.A. Katz, "Non-homogeneous demands flows service", *Izvestiya AN SSSR. Tekhnicheskaya Kibernetika* 2, 47–53 (1973)(In Russian).
- [14] B. Sengupta, "The spatial requirements of an  $M/G/1$  queue, or: How to design for buffer space", In: Baccelli F., Fayolle G. (eds.), *Modelling and Performance Evaluation Methodology. Lect. Notes in Contr. and Inf. Sci.* 60, Springer, Heidelberg, 547–562 (1984).
- [15] O. Tikhonenko, "Queueing systems with common buffer: a theoretical treatment", In: Kwiecien, A., Gaj, P., Stera, P. (eds.) CN 2011, *CCIS* 160, Springer, Heidelberg, 61–69 (2011).
- [16] M. Ziółkowski and O. Tikhonenko, "Multiserver queueing system with non-homogeneous customers and sectorized memory space", In: Kwiecien, A., Gaj, P., Sawicki, M., Suchacka, G. (eds.) CN 2018, *CCIS* 860, Springer, Cham, 272–285 (2018).
- [17] S.F. Yashkov and A.S. Yashkova, "Processor sharing: A survey of the mathematical theory", *Autom. Remote Control* 68 (9), 1662–1731 (2007).
- [18] O. Tikhonenko, M. Ziółkowski, and M. Kurkowski, " $M/G/n/(0,V)$  Erlang queueing system with non-homogeneous customers, non-identical servers and limited memory space", *Bull. Pol. Ac.: Tech.* 67 (3), 489–500 (2019).
- [19] O. Tikhonenko and M. Ziółkowski, "Queueing systems with non-homogeneous customers and infinite sectorized memory space", In: Kwiecien, A., Gaj, P., Sawicki, M. (eds.) CN 2019, *CCIS*, vol. 1039, Springer, Cham, 316–329 (2019).
- [20] M.L. Abell and J.P. Braselton, *The Mathematica Handbook*, Elsevier, 1992.
- [21] Gia-Shie Liu, "Three m-failure group maintenance models for  $M/M/N$  unreliable queueing service systems", *Comput. Ind. Eng.* 62 (4), 1011–1024 (2012).
- [22] M. Dawson, *Python Programming for the Absolute Beginner*, Cengage Learning, 2010.
- [23] S. Robinson, *Simulation: The Practice of Model Development and Use*, Palgrave Macmillan, 2014.