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A novel multiple attribute decision making method based on q-rung dual hesitant uncertain linguistic sets and Muirhead mean

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This paper aims to propose a new multi-attribute decision making (MADM) method in complicated and fuzzy decision-making environment. To express both decision makers (DMs') quantitative and qualitative evaluation information comprehensively and consider their high hesitancy in giving their assessment values in MADM process, we combine q-rung dual hesitant fuzzy sets (q-RDHFSs) with uncertain linguistic variables and develop a new tool, called the q-rung dual hesitant uncertain linguistic sets (q-RDHULSs). First, the definition, operations and comparison method of q-RDHULSs are proposed. Second, given the interrelationship among multiple q-rung dual hesitant uncertain linguistic variables (q-RDHULVs) we introduce some aggregation operators (AOs) to fuse q-rung dual hesitant uncertain linguistic (q-RDHUL) information based on the Muirhead mean, i.e. the q-RDHUL Muirhead mean operator, the q-RDHUL weighted Muirhead mean operator, the q-RDHUL dual Muirhead mean operator, and the q-RDHUL weighted dual Muirhead mean operator. To cope with MADM problems with q-RDHUL information, we propose a new method based on the proposed AOs. Afterwards, we apply the proposed method to an enterprise informatization level evaluation problem to verify its effectiveness. In addition, we also explain why our proposed method is more powerful and flexible than others.

Key words: q-rung dual hesitant uncertain linguistic sets; Muirhead mean; q-rung dual hesitant uncertain linguistic Muirhead mean; multi-attribute decision making; informatization level evaluation

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1. Introduction

As practical decision-making problems are becoming more and more complicated, decision-makers (DMs) always experience the difficulties of appropriately expressing their evaluation information. Generally, it is usually impossible for DMs to express their evaluation information in the form of clear numbers. In order to effectually deal with vague information, Zadeh [1] creatively introduced a new methodology, called fuzzy sets (FSs). The uniqueness of FS is that it incorporates the concept of membership degree (MD), depicting the degree that an element belongs to a given fixed set. Given the striking advantages of FSs on dealing with ambiguity, FS theory based multi-attribute decision making (MADM) methods have attracted quite a few scientists' interests [2]. With the emergence of new and complex MADM problems, the shortcomings of FSs have been becoming more and more obvious, which received extensive attention from scholars globally. The FS uses only one value to represent the MD, however, in most practical MADM problems, DMs hesitate among several values when establishing the MD. To effectively deal with such cases as well as DMs' high hesitancy, Torra [3] introduced a concept of a hesitant fuzzy set (HFS), which allows the MD to be denoted by several single values instead of only one. Owing to its good ability of depicting fuzzy information and DMs' hesitancy, HFS has been regarded as one the most powerful and flexible tools in MADM procedure [4–12].

Nevertheless, HFSs are still insufficient in handling complicated decision-making systems. The main drawback of HFS is that it only describes the degree that an element belongs to a given set by MD, whereas ignores the grade that an element does not be included to a given set. Motivated by the intuitionistic fuzzy set (IFS) which describes fuzzy information from both positive and negative aspects, Zhu et al. [13] introduced the dual hesitant fuzzy set (DHFS), which permits not only the MD but also the non-membership degree (NMD) to be denoted by several values. Hence, compared with HFS, DHFS is better to model DMs' hesitancy and can describe fuzzy data more comprehensively. Afterwards, Wang et al. [14] and Yu and Li [15] proposed simple dual hesitant fuzzy weighted aggregation operators. To capture more information from aggregated DHFSs, Xing et al. [16] introduced the dual hesitant fuzzy point operators. Ju et al. [17] explored operational laws of dual hesitant fuzzy elements (DHFEs) based on Hamacher t-norm and t-conorm. Wang et al. [18] demonstrated the dual hesitant fuzzy operations on the basis of Archimedean t-conorm and t-norm (ATT) and proposed dual hesitant fuzzy power average operators based on the proposed operational rules. Ju et al. [19] generalized DHFSs to interval-valued DHFSs and studied their aggregation operators (AOs). Peng et al. [20] investigated interval-valued dual hesitant fuzzy AOs based on ATT. Zhao et al. [21] proposed AOs for DHFEs based on Einstein t-norm and t-conorm. Ju [22] introduced the Choquet integral to DHFSs to consider the relationship among attributes. Ren et al. [23]

proposed new comparison method for DHFEs and a new dual hesitant fuzzy MADM method based on VIKOR. Yang and Ju [24] introduced the dual hesitant fuzzy linguistic sets (DHFLSs) and their AOs. Su et al. [25] investigated distance and similarity measures for DHFSs and applied them in pattern recognition. Liu and Tang [26] introduced interval-valued dual hesitant fuzzy uncertain linguistic sets and proposed their generalized Shapely Choquet geometric operators. Tu et al. [27] presented the dual hesitant fuzzy Bonferroni mean operators to take into account the interrelationship among DHFEs. For the same purpose, Yu et al. [28] put forward the dual hesitant fuzzy Heronian mean operators. Tang et al. [29] studied dual hesitant fuzzy operational rules under Frank t-norm and t-norm. In addition, scholars also studied dual hesitant fuzzy rough sets [30–32], dual hesitant fuzzy soft sets [33–35], correlation coefficients of DHFSs [36, 37], etc.

The DHFSs must satisfy the constraint that the sum of MD and NMD is equal to or less than one. However, in complicated real MADM problems this constraint cannot be always met. For example, a group of DMs are hesitant between 0.5, 0.6, and 0.7 for determining MD and 0.7, 0.8 for providing NMD. Hence, DMs' evaluation can be denoted by $d = \{\{0.5, 0.6, 0.7\}, \{0.7, 0.8\}\}$. As $0.7 + 0.8 = 1.5 > 1$, then d cannot be handled by DHFSs. To enlarge the information space that DHFSs can describe, Xu et al. [38] gave the notion of q-rung dual hesitant fuzzy sets (q-RDHFSs). The q-RDHFSs are based on q-rung orthopair fuzzy sets (q-ROFSs), whose powerfulness in depicting fuzzy information has been proved by scientists [39–44]. Therefore, q-RDHFSs inherit the advantages of q-ROFSs, i.e. they can deal with the circumstances in which the sum and square sum of MD and NMD are larger than one. In Ref. [38], Xu et al. also proposed q-rung hesitant fuzzy Heronian mean operators to aggregate attribute values. However, the MADM method proposed by Xu et al. [38] still has limitations. Firstly, it only focuses on DMs' quantitative evaluation information, whereas neglects their qualitative assessments. More and more scholars have been aware of the importance and necessity of taking into consideration of both DMs' quantitative and qualitative evaluation ideas [45–47]. Secondly, the method proposed by Xu et al. [38] only captures the interrelationship among any two attributes. In practical MADM issues, interrelationship usually exists among multiple attributes. Hence, the MADM method introduced by Xu et al. [38] is still insufficient to deal with complex MADM problems in real life.

To overcome the above-mentioned shortcomings, we firstly propose a new tool, called q-rung dual hesitant uncertain linguistic sets (q-RDHULSs) to represent DMs' evaluation information both quantificationally and qualitatively. The q-RDHULS is a combination of q-RDHFS with uncertain linguistic variable (ULV). In other word, q-RDHULS utilizes q-RDHFS to denote the MD and NMD of an alternative to a ULV. The q-RDHULS is parallel to dual hesitant fuzzy uncertain linguistic sets (DHFULSs) but is more powerful as its constraint is laxer, making it more suitable and powerful to deal with complicated real

MADM. To capture the interrelationship among multiple attributes, we further propose a series of q-rung dual hesitant uncertain linguistic (q-RDHUL) Muirhead mean (MM) operators. The main superiority of MM [48] is its ability to proceed the interrelationship among multiple attributes, and recently it has successfully attracted many scholars' research interests and been extensively studied [49–53]. Finally, based on the proposed q-rung dual hesitant uncertain linguistic MM operators, we investigate a novel MADM method.

To present our works clearly, we organize the rest of this paper as follows. Section 2 recalls existing concepts and proposes the q-RDHULSs. Section 3 introduces q-rung dual hesitant uncertain linguistic AOs based on MM. Section 4 introduces a new MADM method within q-RDHUL context. Section 5 applies the proposed method to evaluate enterprise informatization level and analyzes the effectiveness and advantages of the new method. Conclusion remarks are provided in Section 6.

2. Basic concepts

In this section, we recall basic notions which will be used in the following section.

2.1. q-RDHFSs and q-RDHULSs

Xu et al. [38] generalized q-ROFSs to q-RDHFSs, which permit the MD and NMD to be denoted by several values rather than single ones.

Definition 1 [38] *Let X be a given ordinary set, a q-rung dual hesitant fuzzy set (q-RDHFS) A defined on X is expressed as*

$$A = \{\langle x, g_A(x), t_A(x)\rangle \mid x \in X\}, \tag{1}$$

where $g_A(x)$ and $t_A(x)$ are two sets of values in the interval [0, 1], representing the possible MD and NMD of $x \in X$ to the set A. In addition, $g_A(x)$ and $t_A(x)$ satisfy the condition that $\delta^q + \pi^q \leqslant 1$ $(q \geqslant 1)$, where $\delta \in g_A(x)$ and $\pi \in t_A(x)$ for all $x \in X$. For easy description, the ordered pair $d(x) = (g_A(x), t_A(x))$ is called a q-rung dual hesitant fuzzy element (q-RDHFE), which can be denoted as $d = (g, t)$ for simplicity, with the condition $\delta \in g$, $\pi \in t$, $0 \leqslant \delta, \pi \leqslant 1$ and $0 \leqslant \delta^q + \pi^q \leqslant 1$ $(q \geqslant 1)$.

The linguistic term set (LTS) and linguistic variables (LVs) proposed by Prof. Zadeh are effective to represent quantitative information. Recently, Xu [54] extended the classical LTS to continuous LTS and proposed the concept of ULVs. Compared with LVs, ULVs can describe DMs' quantitative information more appropriately and comprehensively. However, the main shortcoming of ULV is that when DMs' use ULVs to express their evaluation information, the MD and NMD of an element to a ULV are ignored. In the other words, the proposed ULVs express DMs' quantitative but neglect their quantitative decision information. Hence, Liu and Jin [55] utilized the IFS to represent the MD and NMD of an element to a given ULV and proposed the concept of intuitionistic uncertain linguistic set. Similarly, this paper proposes the concept of q-RDHULS by combining q-RDHFS with ULVs. In q-RDHULS, q-RDHFS is utilized to represent the MD and NMD of an ULV.

Definition 2 *Let X be a given fixed set $\widetilde{S}$ be a continuous linguistic term set, a q-rung dual hesitant uncertain linguistic set (q-RDHULS) A defined on X is expressed as*

$$A = \left\{\left\langle x, \left(\left[s_{\theta(x)}, s_{\xi(x)}\right], (g_A(x), t_A(x))\right)\right\rangle | x \in X\right\}, \tag{2}$$

where $\left[s_\theta, s_\xi\right] \in S$ is a ULV, and $g_A(x), t_A(x)\colon X \to [0,1]$, with the condition that $0 \leqslant (g_A(x))^q + (t_A(x))^q \leqslant 1$. The values of $g_A(x)$ and $t_A(x)$ denote the MD and NMD of the element x to the ULV $\left[s_\theta, s_\xi\right]$, respectively. For convenience, we call the pair $\alpha(x) = \left\langle\left[s_{\theta(x)}, s_{\xi(x)}\right], (g_A(x), t_A(x))\right\rangle$ a q-rung dual hesitant uncertain linguistic variable (q-RDHULV), which can be denoted as $\alpha = \left\langle\left[s_\theta, s_\xi\right], (g, t)\right\rangle$ for simplify.

From Definition 2, we can find out that when $q = 1$, then q-RDHULS reduces to the dual hesitant uncertain linguistic set. When $q = 2$, then q-RDHULS reduces to the dual hesitant Pythagorean uncertain linguistic set.

In the following, we introduce operations of q-RDHULVs.

Definition 3 *Let $\alpha_1 = \left\langle\left[s_{\theta_1}, s_{\xi_1}\right], (g_1, t_1)\right\rangle$, $\alpha_2 = \left\langle\left[s_{\theta_2}, s_{\xi_2}\right], (g_2, t_2)\right\rangle$ and $\alpha = \left\langle\left[s_\theta, s_\xi\right], (g, t)\right\rangle$ be any three q-RDHULVs and λ be a positive real number, then*

(1) $\alpha_1 \oplus \alpha_2 = \left\langle\left[s_{\theta_1+\theta_2}, s_{\xi_1+\xi_2}\right], \cup_{\substack{\delta_1\in g_1,\ \delta_2\in g_2,\\ \pi_1\in t_1,\ \pi_2\in t_2}} \left\{\left\{\left(\delta_1^q + \delta_2^q - \delta_1^q\delta_2^q\right)^{1/q}\right\}, \{\pi_1\pi_2\}\right\}\right\rangle$;

(2) $\alpha_1 \otimes \alpha_2 = \left\langle\left[s_{\theta_1\times\theta_2}, s_{\xi_1\times\xi_2}\right], \cup_{\substack{\delta_1\in g_1,\ \delta_2\in g_2,\\ \pi_1\in t_1,\ \pi_2\in t_2}} \left\{\{\delta_1\delta_2\}, \left\{\pi_1^q + \pi_2^q - \pi_1^q\pi_2^q\right\}\right\}\right\rangle$;

(3) $\lambda\alpha = \left\langle\left[s_{\lambda\cdot\theta}, s_{\lambda\cdot\xi}\right], \cup_{\delta\in g,\ \pi\in t} \left\{\left\{\left(1-(1-\delta^q)^\lambda\right)^{1/q}\right\}, \left\{\pi^\lambda\right\}\right\}\right\rangle$;

(4) $\alpha^\lambda = \left\langle\left[s_{\theta^\lambda}, s_{\xi^\lambda}\right], \cup_{\delta\in g,\ \pi\in t} \left\{\left\{\delta^\lambda\right\}, \left\{\left(1-(1-\pi^q)^\lambda\right)^{1/q}\right\}\right\}\right\rangle$.

According to Definition 3, the following theorem can be easily obtained.

Theorem 1 *Let α_1, α_2 and α be any three q-RDHULVs, then*

(1) $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$;
(2) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$;
(3) $\lambda(\alpha_1 \oplus \alpha_2) = \lambda\alpha_1 \oplus \lambda\alpha_2, \quad \lambda > 0$;
(4) $(\alpha_1 \otimes \alpha_2)^\lambda = \alpha_1^\lambda \otimes \alpha_2^\lambda, \quad \lambda > 0$;
(5) $\lambda_1\alpha \oplus \lambda_2\alpha = (\lambda_1 + \lambda_2)\alpha, \quad \lambda_1, \lambda_2 > 0$;
(6) $\alpha^{\lambda_1} \otimes \alpha^{\lambda_2} = \alpha^{(\lambda_1+\lambda_2)}, \quad \lambda_1, \lambda_2 > 0.$

To rank any two q-RDHULVs, we introduce a comparison method for q-RDHULVs.

Definition 4 *Let $\alpha = \left\langle \left[s_\theta, s_\xi\right], (g, t)\right\rangle$ be a q-RDHULV, then the score function of α is given as*

$$S(\alpha) = \frac{1}{8}(\theta + \xi) \times \left(1 + \left(\frac{1}{\#g}\sum_{\delta \in g}\delta\right)^q - \left(\frac{1}{\#t}\sum_{\pi \in t}\pi\right)^q\right), \tag{3}$$

and the accuracy function of α is expressed as

$$H(\alpha) = \frac{1}{4}(\theta + \xi) \times \left(\left(\frac{1}{\#g}\sum_{\delta \in g}\delta\right)^q - \left(\frac{1}{\#t}\sum_{\pi \in t}\pi\right)^q\right), \tag{4}$$

where $\#g$ and $\#t$ denote the numbers of values in g and t respectively. For any two q-RDHULVs $\alpha_1 = \left\langle \left[s_{\theta_1}, s_{\xi_1}\right], (g_1, t_1)\right\rangle$ and $\alpha_2 = \left\langle \left[s_{\theta_2}, s_{\xi_2}\right], (g_2, t_2)\right\rangle$, then

(1) *If $S(\alpha_1) > S(\alpha_2)$, then $\alpha_1 > \alpha_2$;*
(2) *If $S(\alpha_1) > S(\alpha_2)$, then*
If $H(\alpha_1) > H(\alpha_2)$, then $\alpha_1 > \alpha_2$;
If $H(\alpha_1) = H(\alpha_2)$, then $\alpha_1 = \alpha_2$.

2.2. Muirhead mean operator and its dual form

Definition 5 [48] *Let b_j $(j = 1, 2, \ldots, n)$ a set of positive real numbers and $L = (l_1, l_2, \ldots, l_n) \in R^n$ be a collection of parameters. If*

$$MM^L(b_1, b_2, \ldots, b_n) = \left(\frac{1}{n!}\sum_{\sigma \in R_n}\prod_{j=1}^{n} b_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}, \tag{5}$$

then MM^L is called the Muirhead mean (MM), where R_n represents all possible permutations of $(1, 2, \ldots, n)$ and $\sigma(j)$ $(j = 1, 2, \ldots, n)$ is any one of R_n.

The dual form of MM operator is presented as follows.

Definition 6 [53] *Let b_j $(j = 1, 2, \ldots, n)$ a set of positive real numbers and $L = (l_1, l_2, \ldots, l_n) \in R^n$ be a collection of parameters. Then the dual Muirhead mean (DMM) operator is defined as*

$$DMM^L(b_1, b_2, \ldots, b_n) = \frac{1}{\sum\limits_{j=1}^{n} l_j} \left(\prod_{\sigma \in R_n} \sum_{j=1}^{n} \left(l_j b_{\sigma(j)} \right) \right)^{\frac{1}{n!}}, \tag{6}$$

where R_n represents all possible permutations of $(1, 2, \ldots, n)$ and $\sigma(j)$ $(j = 1, 2, \ldots, n)$ is any one of R_n.

3. Some q-rung dual hesitant uncertain linguistic aggregation operators

In this section, we propose some AOs for fusing q-RDHUL information based on MM and DMM.

3.1. The q-rung dual hesitant uncertain linguistic Muirhead mean (q-RDHULMM) operator

Definition 7 *Let $\alpha_j = \left\langle \left[s_{\theta_j}, s_{\xi_j} \right], \left(g_j, t_j \right) \right\rangle$ $(j = 1, 2, \ldots, n)$ be a collection of q-RDHULVs and $L = (l_1, l_2, \ldots, l_n) \in R^n$ be a collection of parameters. If*

$$q\text{-}RDHULMM^L(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{1}{n!} \bigoplus_{\sigma \in R_n} \bigotimes_{j=1}^{n} \alpha_{\sigma(j)}^{l_j} \right)^{\frac{1}{\sum_{j=1}^{n} l_j}}, \tag{7}$$

then q-RDHULMM^L is called the q-rung dual hesitant uncertain linguistic Muirhead mean (q-RDHULMM) operator, where R_n represents all possible permutations of $(1, 2, \ldots, n)$ and $\sigma(j)$ $(j = 1, 2, \ldots, n)$ is any one of R_n.

Theorem 2 *Let $\alpha_j = \left\langle \left[s_{\theta_j}, s_{\xi_j} \right], \left(g_j, t_j \right) \right\rangle$ $(j = 1, 2, \ldots, n)$ be a collection of q-RDHULVs and $L = (l_1, l_2, \ldots, l_n) \in R^n$ be a collection of parameters. The aggregated value by the q-RDHULMM operator is also a q-RDHULV and*

$$
\begin{aligned}
&q\text{-}RDHULMM^{L}(\alpha_1,\alpha_2,\ldots,\alpha_n)=\\
&=\left\langle\left[s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\theta_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}},\, s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\xi_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}}\right],\right.\\
&\qquad\bigcup_{\substack{\delta_{\sigma(j)}\in g_{\sigma(j)},\\ \pi_{\sigma(j)}\in t_{\sigma(j)}}}\left\{\left\{\left(\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{ql_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\right\},\right.\\
&\qquad\left.\left.\left\{\left(1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi_{\sigma(j)}^{q}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\right)^{1/q}\right\}\right\}\right\rangle.
\end{aligned}
\tag{8}
$$

Proof. We firstly prove that Eq. (8) holds, and afterwards we prove the aggregated value is a q-RDHULV. According to the operations of q-RDHULVs, we have

$$
\alpha_{\sigma(j)}^{l_j}=\left\langle\left[s_{\theta_{\sigma(j)}^{l_j}},\, s_{\xi_{\sigma(j)}^{l_j}}\right],\bigcup_{\substack{\delta_{\sigma(j)}\in g_{\sigma(j)},\\ \pi_{\sigma(j)}\in t_{\sigma(j)}}}\left\{\left\{\delta_{\sigma(j)}^{l_j}\right\},\left\{\left(1-\left(1-\pi_{\sigma(j)}^{q}\right)^{l_j}\right)^{1/q}\right\}\right\}\right\rangle,
$$

and

$$
\begin{aligned}
&\bigotimes_{j=1}^{n}\alpha_{\sigma(j)}^{l_j}=\\
&=\left\langle\left[s_{\prod_{j=1}^{n}\theta_{\sigma(j)}^{l_j}},\, s_{\prod_{j=1}^{n}\xi_{\sigma(j)}^{l_j}}\right],\bigcup_{\substack{\delta_{\sigma(j)}\in g_{\sigma(j)},\\ \pi_{\sigma(j)}\in t_{\sigma(j)}}}\left\{\left\{\prod_{j=1}^{n}\delta_{\sigma(j)}^{l_j}\right\},\left\{\left(1-\prod_{j=1}^{n}\left(1-\pi_{\sigma(j)}^{q}\right)^{l_j}\right)^{1/q}\right\}\right\}\right\rangle.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\bigoplus_{\sigma\in R_n}\bigotimes_{j=1}^{n}\alpha_{\sigma(j)}^{l_j}=&\left\langle\left[s_{\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\theta_{\sigma(j)}^{l_j}},\, s_{\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\xi_{\sigma(j)}^{l_j}}\right],\bigcup_{\substack{\delta_{\sigma(j)}\in g_{\sigma(j)},\\ \pi_{\sigma(j)}\in t_{\sigma(j)}}}\right.\\
&\left.\left\{\left\{\left(1-\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{ql_j}\right)\right)^{1/q}\right\},\left\{\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi_{\sigma(j)}^{q}\right)^{l_j}\right)^{1/q}\right\}\right\}\right\rangle,
\end{aligned}
$$

and

$$\frac{1}{n!}\bigoplus_{\sigma\in R_n}\bigotimes_{j=1}^{n}\alpha_{\sigma(j)}^{l_j}=\left\langle\left[s_{\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\theta_{\sigma(j)}^{l_j}},s_{\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\xi_{\sigma(j)}^{l_j}}\right],\bigcup_{\substack{\delta_{\sigma(j)}\in g_{\sigma(j)},\\ \pi_{\sigma(j)}\in t_{\sigma(j)}}}\right.$$
$$\left.\left\{\left\{\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{ql_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right\},\left\{\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi_{\sigma(j)}^{q}\right)^{l_j}\right)^{1/q}\right)^{\frac{1}{n!}}\right\}\right\}\right\rangle.$$

Thus,

$$\left(\frac{1}{n!}\bigoplus_{\sigma\in R_n}\bigotimes_{j=1}^{n}\alpha_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}=\left\langle\left[s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\theta_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}},s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\xi_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}}\right],\right.$$
$$\bigcup_{\substack{\delta_{\sigma(j)}\in g_{\sigma(j)},\\ \pi_{\sigma(j)}\in t_{\sigma(j)}}}\left\{\left\{\left(\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{ql_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\right\},\right.$$
$$\left.\left.\left\{\left(1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi_{\sigma(j)}^{q}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\right)^{1/q}\right\}\right\}\right\rangle$$

which proves the rightness of Eq. (8). In the followings, we shall prove that the aggregated value is a q-RDHULV. For easy description, let

$$\delta=\left(\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{ql_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}},$$

$$\pi=\left(1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi_{\sigma(j)}^{q}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\right)^{1/q}.$$

It is easy to prove that $0 \leqslant \delta, \pi \leqslant 1$. Since $\delta^q_{\sigma(j)} + \pi^q_{\sigma(j)} \leqslant 1$, then $\delta^q_{\sigma(j)} \leqslant 1 - \pi^q_{\sigma(j)}$, and we can obtain

$$\begin{aligned}\delta^q + \pi^q &= \left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta^{ql_j}_{\sigma(j)}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}} \\ &\quad + 1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi^q_{\sigma(j)}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}} \\ &\leqslant \left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi^q_{\sigma(j)}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}} \\ &\quad + 1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi^q_{\sigma(j)}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}} = 1,\end{aligned}$$

which proves that the aggregated value is also a q-RDHULVs.

In the followings, we discuss some properties of the proposed q-RDHULMM operator.

Property 1 (**Idempotency**) *Let α_j $(j = 1, 2, \ldots, n)$ be a collection of q-RDHULVs. If all q-RDHULVs are equal, i.e. $\alpha_j = \alpha$ holds for $j = 1, 2, \ldots, n$, and α only has one ULV, one MD and one NMD, then*

$$q\text{-}RDHULMM^L(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha. \tag{9}$$

Proof. Since $\alpha_j = \alpha = \left\langle \left[s_\theta, s_\xi\right], \{\{\delta\}, \{\pi\}\}\right\rangle$ holds for all j, then we have

$$\begin{aligned}q\text{-}RDHULMM^L(\alpha_1, \alpha_2, \ldots, \alpha_n) &= \left\langle \left[s_{\left(\frac{1}{n!}\sum_{\sigma\in R_n}\prod_{j=1}^{n}\theta^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}},\ s_{\left(\frac{1}{n!}\sum_{\sigma\in R_n}\prod_{j=1}^{n}\xi^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}} \right],\right. \\ &\bigcup_{\delta\in g,\ \pi\in t}\left\{\left\{\left(\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta^{ql_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}\right\},\right. \\ &\left.\left.\left\{\left(1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi^q\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}\right)^{1/q}\right\}\right\}\right\rangle.\end{aligned}$$

Further,

$$s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\theta^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}} = s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\theta^{\sum_{j=1}^{n} l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}} = s_{\left(\frac{1}{n!}n!\theta^{\sum_{j=1}^{n} l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}} = s_\theta .$$

Similarly,

$$s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\xi_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}} = s_\xi .$$

In addition,

$$\left(\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{ql_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}} =$$

$$= \left(\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta^{ql_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}} = \left(\left(1-\left(\left(1-\prod_{j=1}^{n}\delta^{ql_j}\right)^{n!}\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}},$$

and

$$\left(\left(\prod_{j=1}^{n}\delta^{ql_j}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}} = \left(\left(\delta^{q\sum_{j=1}^{n} l_j}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}} = \delta.$$

Similarly,

$$\left(1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi_{\sigma(j)}^{q}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}\right)^{1/q} = \pi.$$

Hence, q-RDHULMM^L $(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha$.

Property 2 (**Monotonicity**) *Let* $\alpha_j = \left\langle\left[s_{\theta_j}, s_{\xi_j}\right], \left(g_j, t_j\right)\right\rangle$ *and* $\beta_j = \left\langle\left[s_{\psi_j}, s_{\vartheta_j}\right], \left(k_j, m_j\right)\right\rangle$ $(j = 1, 2, \ldots, n)$ *be two collections of q-RDHULVs. If* $s_{\theta_j} \leqslant s_{\psi_j}$, $s_{\xi_j} \leqslant s_{\vartheta_j}$, $\delta \leqslant \eta$ *and* $\pi \geqslant \gamma$ *hold for* $j = 1, 2, \ldots, n$, *where* $\delta \in g_j$, $\eta \in k_j$, $\pi \in t_j$ *and* $\rho \in m_j$, *then*

$$q\text{-}RDHULMM^L\left(\alpha_1, \alpha_2, \ldots, \alpha_n\right) \leqslant q\text{-}RDHULMM^L\left(\beta_1, \beta_2, \ldots, \beta_n\right). \qquad (10)$$

Proof. For easy description, we assume

$$
\begin{aligned}
&q\text{-}RDHULMM^{L}\left(\alpha_1,\alpha_2,\ldots,\alpha_n\right)=\left\langle\left[s_\theta,\ s_\xi\right],\ (g,t)\right\rangle\\
&=\left\langle\left[s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\theta_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}},\ s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\xi_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}}\right],\right.\\
&\bigcup_{\substack{\delta_{\sigma(j)}\in g_{\sigma(j)},\\ \pi_{\sigma(j)}\in t_{\sigma(j)}}}\left\{\left\{\left(\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{ql_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\right\},\right.\\
&\left.\left.\left\{\left(1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi_{\sigma(j)}^{q}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\right)^{1/q}\right\}\right\}\right\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
&q\text{-}RDHULMM^{L}\left(\beta_1,\beta_2,\ldots,\beta_n\right)=\left\langle\left[s_\psi,\ s_\vartheta\right],\ (k,m)\right\rangle\\
&=\left\langle\left[s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\psi_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}},\ s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\vartheta_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}}\right],\right.\\
&\bigcup_{\substack{\eta_{\sigma(j)}\in k_{\sigma(j)},\\ \gamma_{\sigma(j)}\in m_{\sigma(j)}}}\left\{\left\{\left(\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\eta_{\sigma(j)}^{ql_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\right\},\right.\\
&\left.\left.\left\{\left(1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\gamma_{\sigma(j)}^{q}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\right)^{1/q}\right\}\right\}\right\rangle.
\end{aligned}
$$

Since $s_{\theta_j} \leqslant s_{\psi_j}$ and $s_{\xi_j} \leqslant s_{\vartheta_j}$ hold for $j = 1, 2, \ldots, n$, it is easy to prove that

$$s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\theta_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}} \leqslant s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\psi_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}},$$

$$s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\xi_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}} \leqslant s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\vartheta_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}},$$

i.e. $s_\theta \leqslant s_\psi$ and $s_\xi \leqslant s_\vartheta$.

In addition, since $\delta \leqslant \eta$ holds for $j = 1, 2, \ldots, n$, where $\delta \in g_j$, $\eta \in k_j$, we have

$$\prod_{j=1}^{n}\delta_{\sigma(j)}^{ql_j} \leqslant \prod_{j=1}^{n}\eta_{\sigma(j)}^{ql_j} \Rightarrow 1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{ql_j} \geqslant 1-\prod_{j=1}^{n}\eta_{\sigma(j)}^{ql_j}$$

$$\Rightarrow \prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{ql_j}\right) \geqslant \prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\eta_{\sigma(j)}^{ql_j}\right)$$

$$\Rightarrow \left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{ql_j}\right)\right)^{\frac{1}{n!}} \geqslant \left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\eta_{\sigma(j)}^{ql_j}\right)\right)^{\frac{1}{n!}}$$

$$\Rightarrow 1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{ql_j}\right)\right)^{\frac{1}{n!}} \leqslant 1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\eta_{\sigma(j)}^{ql_j}\right)\right)^{\frac{1}{n!}}$$

$$\Rightarrow \left(\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{ql_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}} \leqslant \left(\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\eta_{\sigma(j)}^{ql_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}.$$

Similarly,

$$\left\{\left(1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi_{\sigma(j)}^{q}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\right)^{1/q}\right\} \leqslant$$

$$\leqslant \left\{\left(1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\gamma_{\sigma(j)}^{q}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n}l_j}}\right)^{1/q}\right\}.$$

Hence, for any element $\delta \in g$ there exits an element η ($\eta \in k$), satisfying $\delta \leqslant \eta$. For any element $\pi \in t$, there exits an element γ ($\gamma \in m$), satisfying $\delta \leqslant \gamma$. Therefore, according to Definition 4, we can obtain

$$q\text{-}RDHULMM^L(\alpha_1, \alpha_2, \ldots, \alpha_n) \leqslant q\text{-}RDHULMM^L(\beta_1, \beta_2, \ldots, \beta_n).$$

Property 3 (**Boundedness**) *Let* $\alpha_j = \left\langle \left[s_{\theta_j}, s_{\xi_j}\right], \left(g_j, t_j\right) \right\rangle$ $(j = 1, 2, \ldots, n)$ *be a collection of q-RDHULVs. If*

$$\alpha^+ = \left\langle \left[s_{\max_{j=1}^{n}(\theta_j)}, \ s_{\max_{j=1}^{n}(\xi_j)} \right], \bigcup_{\delta_j \in g_j, \pi_j \in t_j} \left\{ \left\{ \max_{j=1}^{n} \left(\delta_j\right) \right\}, \left\{ \min_{j=1}^{n} \left(\pi_j\right) \right\} \right\} \right\rangle, \tag{11}$$

and

$$\alpha^- = \left\langle \left[s_{\min_{j=1}^{n}(\theta_j)}, \ s_{\min_{j=1}^{n}(\xi_j)} \right], \bigcup_{\delta_j \in g_j, \pi_j \in t_j} \left\{ \left\{ \min_{j=1}^{n} \left(\delta_j\right) \right\}, \left\{ \max_{j=1}^{n} \left(\pi_j\right) \right\} \right\} \right\rangle. \tag{12}$$

Then

$$\alpha^- \leqslant q\text{-}RDHULMM^L(\alpha_1, \alpha_2, \ldots, \alpha_n) \leqslant \alpha^+. \tag{13}$$

Proof. According to property 2, we can easily obtain that

$$\begin{aligned} q\text{-}RDHULMM^L(\alpha^-, \alpha^-, \ldots, \alpha^-) &\leqslant q\text{-}RDHULMM^L(\alpha_1, \alpha_2, \ldots, \alpha_n) \\ &\leqslant q\text{-}RDHULMM^L(\alpha^+, \alpha^+, \ldots, \alpha^+). \end{aligned}$$

In addition, both α^- and α^+ only have one ULV, one MD and one NMD. Therefore,

$$q\text{-}RDHULMM^L(\alpha^-, \alpha^-, \ldots, \alpha^-) = \alpha^-,$$
$$q\text{-}RDHULMM^L(\alpha^+, \alpha^+, \ldots, \alpha^+) = \alpha^+.$$

Hence,

$$\alpha^- \leqslant q\text{-}RDHULMM^L(\alpha_1, \alpha_2, \ldots, \alpha_n) \leqslant \alpha^+.$$

In the following part, we shall present special cases of the q-RDHULMM operator. As we know, the MM operators have some special cases. In other words, some existing AOs are special cases of MM. The q-RDHULMM operator can be regard as a generalized of MM to q-RDHULSs so that it also has some special cases with respect to the parameter vector L and parameter q.

(1) If $L = (1, 0, \ldots, 0)$, then we have

$$
\begin{aligned}
&q\text{-}RDHULMM^{(1,0,\ldots,0)}(\alpha_1, \alpha_2, \ldots, \alpha_n) \\
&= \frac{1}{n} \bigoplus_{j=1}^{n} \alpha_j = \\
&= \left\langle \left[s_{\frac{1}{n}\sum\limits_{j=1}^{n}\theta_j}, s_{\frac{1}{n}\sum\limits_{j=1}^{n}\xi_j} \right], \bigcup_{\substack{\delta_j \in g_j,\\ \pi_j \in t_j}} \left\{ \left\{ \left(1 - \prod_{j=1}^{n} \left(1 - \delta_j^q \right)^{1/n} \right)^{1/q} \right\}, \left\{ \prod_{j=1}^{n} \pi_j^{1/n} \right\} \right\} \right\rangle,
\end{aligned}
\tag{14}
$$

and so that the q-RDHULMM operator reduces to the q-rung dual hesitant uncertain linguistic average (q-RDHULA) operator.

(2) If $L = (1, 1, 0, 0, \ldots, 0)$, then we have

$$
\begin{aligned}
&q\text{-}RDHULMM^{(1,1,0,0,\ldots,0)}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \\
&= \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1; i \neq j}^{n} \left(\alpha_i \otimes \alpha_j \right) \right)^{1/2} = \\
&= \left\langle \left[s_{\left(\frac{1}{n(n-1)} \sum\limits_{i,j=1; i \neq j}^{n} \theta_i \theta_j \right)^{1/2}}, s_{\left(\frac{1}{n(n-1)} \sum\limits_{i,j=1; i \neq j}^{n} \xi_i \xi_j \right)^{1/2}} \right], \right. \\
&\bigcup_{\substack{\delta_i \in g_i, \delta_j \in g_j,\\ \pi_i \in t_i, \pi_j \in t_j}} \left\{ \left\{ \left(1 - \left(\prod_{i,j=1; i \neq j}^{n} \left(1 - \left(\delta_i \delta_j \right)^q \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2q}} \right\}, \right. \\
&\left. \left. \left\{ \left(1 - \left(1 - \left(\prod_{i,j=1; i \neq j}^{n} \left(\pi_i^q + \pi_j^q - \pi_i^q \pi_j^q \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right)^{\frac{1}{q}} \right\} \right\} \right\rangle,
\end{aligned}
\tag{15}
$$

and so that the q-RDHULMM operator reduces to the q-rung dual hesitant uncertain linguistic Bonferroni mean operator.

(3) If $L = (\overbrace{1, 1, \ldots, 1}^{k}, \overbrace{0, 0, \ldots, 0}^{n-k})$, then we have

$$q\text{-}RDHULMM^{(\overbrace{1,1,\ldots,1}^{k},\overbrace{0,0,\ldots,0}^{n-k})}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \\ = \left(\frac{\bigoplus\limits_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \bigotimes\limits_{j=1}^{k} \alpha_{i_j}}{C_n^k} \right)^{1/k} = \\ = \left\langle \left[s_{\left(\frac{\sum\limits_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \prod\limits_{j=1}^{k} \theta_{i_j}}{C_n^k}\right)^{1/k}}, s_{\left(\frac{\sum\limits_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \prod\limits_{j=1}^{k} \xi_{i_j}}{C_n^k}\right)^{1/k}} \right], \right. \\ \left. \bigcup_{\substack{\delta_{i_j} \in g_{i_j}, \\ \pi_{i_j} \in t_{i_j}}} \left\{ \begin{array}{l} \left\{ \left(1 - \left(1 - \left(1 - \prod\limits_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \left(\prod\limits_{j=1}^{k} \delta_{i_j} \right)^q \right) \right) \right)^{\frac{1}{C_n^k}} \right)^{1/qk} \right\}, \\ \left\{ \left(1 - \left(1 - \prod\limits_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \left(1 - \prod\limits_{j=1}^{k} \left(1 - \pi_{i_j} \right) \right)^{\frac{1}{C_n^k}} \right)^{1/k} \right)^{1/q} \right\} \end{array} \right\} \right\rangle, \tag{16}$$

and so that the q-RDHULMM operator reduces to the q-rung dual hesitant uncertain linguistic Maclaurin symmetric mean operator.

(4) If $L = (1, 1, \ldots, 1)$, then we have

$$q\text{-}RDHULMM^{(1,1,\ldots,1)}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \bigotimes_{j=1}^{n} \alpha_j^{1/n} = \\ = \left\langle \left[s_{\prod\limits_{j=1}^{n} \theta_j^{1/n}}, s_{\prod\limits_{j=1}^{n} \xi_j^{1/n}} \right], \bigcup_{\substack{\delta_j \in g_j, \\ \pi_j \in t_j}} \left\{ \left\{ \prod_{j=1}^{n} \delta_j^{1/n} \right\}, \left\{ \left(1 - \prod_{j=1}^{n} \left(1 - \pi_j^q \right)^{1/n} \right)^{1/q} \right\} \right\} \right\rangle, \tag{17}$$

and so that the q-RDHULMM operator reduces to the q-rung dual hesitant uncertain linguistic geometric (q-RDHULG) operator.

(5) If $L = (1/n, 1/n, \ldots, 1/n)$, then the q-RDHULMM operator also becomes the q-RDHULG operator shown as Eq. (17).

(6) If $q = 1$, then we have

$$
\begin{aligned}
&q\text{-}RDHULMM^{L}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \\
&= \left\langle \left[s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\theta_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}},\ s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\xi_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}} \right], \right. \\
&\qquad \bigcup_{\substack{\delta_{\sigma(j)}\in g_{\sigma(j)},\\ \pi_{\sigma(j)}\in t_{\sigma(j)}}} \left\{ \left\{ \left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}} \right\}, \right. \\
&\qquad\qquad \left. \left. \left\{ 1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi_{\sigma(j)}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}} \right\} \right\} \right\rangle,
\end{aligned}
\tag{18}
$$

and so that the q-RDHULMM operator reduces to the dual hesitant uncertain linguistic Muirhead mean operator.

(7) If $q = 2$, then we have

$$
\begin{aligned}
&q\text{-}RDHULMM^{L}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \\
&= \left\langle \left[s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\theta_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}},\ s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\xi_{\sigma(j)}^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}} \right], \right. \\
&\qquad \bigcup_{\substack{\delta_{\sigma(j)}\in g_{\sigma(j)},\\ \pi_{\sigma(j)}\in t_{\sigma(j)}}} \left\{ \left\{ \left(\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\delta_{\sigma(j)}^{2l_j}\right)\right)^{\frac{1}{n!}}\right)^{1/2}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}} \right\}, \right. \\
&\qquad\qquad \left. \left. \left\{ \left(1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\pi_{\sigma(j)}^{2}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}\right)^{1/2} \right\} \right\} \right\rangle,
\end{aligned}
\tag{19}
$$

and so that the q-RDHULMM operator reduces to the dual Pythagorean hesitant uncertain linguistic Muirhead mean operator.

3.2. The q-rung dual hesitant uncertain linguistic weighted Muirhead mean (q-RDHULWMM) operator

Definition 8 *Let* $\alpha_j = \left\langle \left[s_{\theta_j}, s_{\xi_j}\right], \left(g_j, t_j\right)\right\rangle$ $(j = 1, 2, \ldots, n)$ *be a collection of q-RDHULVs and* $L = (l_1, l_2, \ldots, l_n) \in R^n$ *be a collection of parameters. Let* $w = (w_1, w_2, \ldots, w_n)^T$ *be the weight vector of* α_j $(j = 1, 2, \ldots, n)$*, satisfying* $0 \leqslant w_j \leqslant 1$ *and* $\sum\limits_{j=1}^{n} w_j = 1$*. If*

$$q\text{-}RDHULWMM^L(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{1}{n!} \bigoplus_{\sigma\in R_n} \bigotimes_{j=1}^{n} \left(nw_{\sigma(j)}\alpha_{\sigma(j)}\right)^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}, \quad (20)$$

then $q\text{-}RDHULWMM^L$ *is the q-rung dual hesitant uncertain linguistic weighted Muirhead mean (q-RDHULWMM) operator, where* R_n *represents all possible permutations of* $(1, 2, \ldots, n)$ *and* $\sigma(j)$ $(j = 1, 2, \ldots, n)$ *is any one of* R_n.

Theorem 3 *Let* $\alpha_j = \left\langle \left[s_{\theta_j}, s_{\xi_j}\right], \left(g_j, t_j\right)\right\rangle$ $(j = 1, 2, \ldots, n)$ *be a collection of q-RDHULVs and* $L = (l_1, l_2, \ldots, l_n) \in R^n$ *be a collection of parameters. The aggregated value by the q-RDHULWMM operator is still a q-RDHULV and*

$$q\text{-}RDHULWMM^L(\alpha_1, \alpha_2, \ldots, \alpha_n) =$$

$$= \left\langle \left[s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\left(nw_{\sigma(j)}\theta_{\sigma(j)}\right)^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}}, s_{\left(\frac{1}{n!}\sum\limits_{\sigma\in R_n}\prod\limits_{j=1}^{n}\left(nw_{\sigma(j)}\xi_{\sigma(j)}\right)^{l_j}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}} \right], \right.$$

$$\bigcup_{\substack{\delta_{\sigma(j)}\in g_{\sigma(j)},\\ \pi_{\sigma(j)}\in t_{\sigma(j)}}} \left\{\left\{\left(\left(1 - \left(\prod_{\sigma\in R_n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(1 - \delta_{\sigma(j)}^{q}\right)^{nw_{\sigma(j)}}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}\right\}, \quad (21)$$

$$\left.\left\{\left(1 - \left(1 - \left(\prod_{\sigma\in R_n}\left(1 - \prod_{j=1}^{n}\left(1 - \left(\pi_{\sigma(j)}^{nw_{\sigma(j)}}\right)^{q}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}\right)^{1/q}\right\}\right\}\right\rangle.$$

The proof of Theorem 3 is similar to that of Theorem 2. In addition, it is easy to prove that the q-RDHULWMM operator has the properties of monotonicity and boundedness.

3.3. The q-rung dual hesitant uncertain linguistic dual Muirhead mean (q-RDHULDMM) operator

Definition 9 *Let* $\alpha_j = \left\langle \left[s_{\theta_j}, s_{\xi_j} \right], \left(g_j, t_j \right) \right\rangle$ $(j = 1, 2, \ldots, n)$ *be a collection of q-RDHULVs and* $L = (l_1, l_2, \ldots, l_n) \in R^n$ *be a collection of parameters. If*

$$q\text{-}RDHULDMM^L \left(\alpha_1, \alpha_2, \ldots, \alpha_n \right) = \frac{1}{\sum\limits_{j=1}^{n} l_j} \left(\bigotimes_{\sigma \in R_n} \bigoplus_{j=1}^{n} \left(l_j \alpha_{\sigma(j)} \right) \right)^{\frac{1}{n!}}, \tag{22}$$

then q-$RDHULDMM^L$ *is the q-rung dual hesitant uncertain linguistic dual Muirhead mean (q-RDHULDMM) operator, where* R_n *represents all possible permutations of* $(1, 2, \ldots, n)$ *and* $\sigma(j)$ $(j = 1, 2, \ldots, n)$ *is any one of* R_n.

Theorem 4 *Let* $\alpha_j = \left\langle \left[s_{\theta_j}, s_{\xi_j} \right], \left(g_j, t_j \right) \right\rangle$ $(j = 1, 2, \ldots, n)$ *be a collection of q-RDHULVs and* $L = (l_1, l_2, \ldots, l_n) \in R^n$ *be a collection of parameters. The aggregated value by the q-RDHULDMM operator is still a q-RDHULV and*

$$\begin{aligned} & q\text{-}RDHULDMM^L \left(\alpha_1, \alpha_2, \ldots, \alpha_n \right) = \\ & = \left\langle \left[s_{\frac{1}{\sum_{j=1}^{n} l_j} \left(\prod_{\sigma \in R_n} \sum_{j=1}^{n} \left(l_j \theta_{\sigma(j)} \right) \right)^{\frac{1}{n!}}}, s_{\frac{1}{\sum_{j=1}^{n} h_j} \left(\prod_{\sigma \in R_n} \sum_{j=1}^{n} \left(l_j \xi_{\sigma(j)} \right) \right)^{\frac{1}{n!}}} \right], \right. \\ & \bigcup_{\substack{\delta_{\sigma(j)} \in g_{\sigma(j)}, \\ \pi_{\sigma(j)} \in t_{\sigma(j)}}} \left\{ \left\{ \left(1 - \left(1 - \left(\prod_{\sigma \in R_n} \left(1 - \prod_{j=1}^{n} \left(1 - \delta_{\sigma(j)}^q \right)^{l_j} \right) \right)^{\frac{1}{n!}} \right)^{\frac{1}{\sum_{j=1}^{n} l_j}} \right)^{1/q} \right\}, \right. \\ & \left. \left. \left\{ \left(\left(1 - \left(\prod_{\sigma \in R_n} \left(1 - \prod_{j=1}^{n} \pi_{\sigma(j)}^{q l_j} \right) \right)^{\frac{1}{n!}} \right)^{1/q} \right)^{\frac{1}{\sum_{j=1}^{n} l_j}} \right\} \right\} \right\rangle. \end{aligned} \tag{23}$$

The proof of Theorem 4 is similar to that of Theorem 2. In addition, it is easy to prove that the q-RDHULDMM operator has the properties of idempotency, monotonicity and boundedness. In the following, we discuss special cases of the q-RDHULDMM operator with respect to the parameter vector L and parameter q.

(1) If $L = (1, 0, \ldots, 0)$ in the q-RDHULDMM operator, then it will become to the q-RDHULG operator, shown as Eq. (17).

(2) If $L = (1, 1, 0, \ldots, 0)$ in the q-RDHULDMM operator, then we have

$$q\text{-}RDHULDMM^{(1,1,0,\ldots,0)}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{2}\left(\bigotimes_{i,j=1;i\neq j}^{n}\left(\alpha_i \oplus \alpha_j\right)\right)^{\frac{1}{n(n-1)}} =$$

$$= \left\langle \left[s_{\frac{1}{2}\prod\limits_{i,j=1;i\neq j}^{n}(\theta_i+\theta_j)^{\frac{1}{n(n-1)}}},\ s_{\frac{1}{2}\prod\limits_{i,j=1;i\neq j}^{n}(\xi_i+\xi_j)^{\frac{1}{n(n-1)}}} \right], \right.$$

$$\bigcup_{\substack{\delta_i\in g_i,\delta_j\in g_j,\\ \pi_i\in t_i,\pi_j\in t_j}} \left\{ \left\{ \left(1 - \left(1 - \left(\prod_{i,j=1;i\neq j}^{n} \left(\delta_i^q + \delta_j^q - \delta_i^q \delta_j^q \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2}} \right)^{\frac{1}{q}} \right\}, \right. \tag{24}$$

$$\left. \left. \left\{ \left(1 - \left(\prod_{i,j=1;i\neq j}^{n} \left(1 - \left(\pi_i \pi_j \right)^q \right) \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{2q}} \right\} \right\} \right\rangle,$$

which is the q-rung dual hesitant uncertain linguistic geometric Bonferroni mean operator.

(3) If $L = (\overbrace{1, 1, \ldots, 1}^{k}, \overbrace{0, 0, \ldots, 0}^{n-k})$ in the q-RDHULDMM operator, then we have

$$q\text{-}RDHULDMM^{(\overbrace{1, 1, \ldots, 1,}^{k}\ \overbrace{0, 0, \ldots, 0}^{n-k})}(\alpha_1, \alpha_2, \ldots, \alpha_n) =$$

$$= \frac{1}{k}\left(\bigotimes_{\substack{1\leqslant i_1<\ldots\\ \ldots<i_k\leqslant n}}\left(\bigoplus_{j=1}^{k}\alpha_{i_j}\right)^{1/C_n^k}\right) = \left\langle \left[s_{\frac{1}{k}\left(\prod\limits_{\substack{1\leqslant i_1<\ldots\\ \ldots<i_k\leqslant n}}\left(\sum\limits_{j=1}^{k}\theta_{i_j}\right)^{\frac{1}{C_n^k}}\right)},\ s_{\frac{1}{k}\left(\prod\limits_{\substack{1\leqslant i_1<\ldots\\ \ldots<i_k\leqslant n}}\left(\sum\limits_{j=1}^{k}\xi_{i_j}\right)^{\frac{1}{C_n^k}}\right)} \right], \right.$$

$$\bigcup_{\substack{\delta_{i_j}\in g_{i_j},\\ \pi_{i_j}\in t_{i_j}}} \left\{ \left\{ \left(1 - \left(1 - \prod_{1\leqslant i_1<\ldots<i_k\leqslant n} \left(1 - \prod_{j=1}^{k} \left(1 - \delta_{i_j} \right) \right)^{\frac{1}{C_n^k}} \right)^{1/k} \right)^{1/q} \right\}, \right. \tag{25}$$

$$\left. \left. \left\{ \left(1 - \left(1 - \left(1 - \prod_{1\leqslant i_1<\ldots<i_k\leqslant n} \left(1 - \left(\prod_{j=1}^{k} \pi_{i_j} \right)^q \right) \right) \right)^{\frac{1}{C_n^k}} \right)^{1/qk} \right\} \right\} \right\rangle,$$

which is the q-rung dual hesitant uncertain linguistic dual Maclaurin symmetric mean operator.

(4) If $L = (1, 1, \ldots, 1)$ or $L = (1/n,\ 1/n,\ \ldots,\ 1/n)$ in the q-RDHULDMM operator, then it will become the q-RDHULA, shown as Eq. (14).

(5) If $q = 1$ in the in the q-RDHULDMM operator, then we have

$$q\text{-}RDHULDMM^L(\alpha_1, \alpha_2, \ldots, \alpha_n) = \\ = \left\langle \left[s_{\frac{1}{\sum_{j=1}^n l_j}\left(\prod_{\sigma\in R_n}\sum_{j=1}^n (l_j\theta_{\sigma(j)})\right)^{\frac{1}{n!}}}, s_{\frac{1}{\sum_{j=1}^n l_j}\left(\prod_{\sigma\in R_n}\sum_{j=1}^n (l_j\xi_{\sigma(j)})\right)^{\frac{1}{n!}}} \right], \right. \\ \bigcup_{\substack{\delta_{\sigma(j)}\in g_{\sigma(j)},\\ \pi_{\sigma(j)}\in t_{\sigma(j)}}} \left\{ \left\{ 1 - \left(1 - \left(\prod_{\sigma\in R_n}\left(1 - \prod_{j=1}^n \left(1-\delta_{\sigma(j)}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n l_j}} \right\}, \right. \\ \left. \left. \left\{ \left(1 - \left(\prod_{\sigma\in R_n}\left(1 - \prod_{j=1}^n \pi_{\sigma(j)}^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n l_j}} \right\} \right\} \right\rangle, \tag{26}$$

which is the dual hesitant uncertain linguistic dual Muirhead mean operator.

(6) If $q = 2$ in the in the q-RDHULDMM operator, then we have

$$q\text{-}RDHULDMM^L(\alpha_1, \alpha_2, \ldots, \alpha_n) = \\ = \left\langle \left[s_{\frac{1}{\sum_{j=1}^n l_j}\left(\prod_{\sigma\in R_n}\sum_{j=1}^n (l_j\theta_{\sigma(j)})\right)^{\frac{1}{n!}}}, s_{\frac{1}{\sum_{j=1}^n l_j}\left(\prod_{\sigma\in R_n}\sum_{j=1}^n (l_j\xi_{\sigma(j)})\right)^{\frac{1}{n!}}} \right], \right. \\ \bigcup_{\substack{\delta_{\sigma(j)}\in g_{\sigma(j)},\\ \pi_{\sigma(j)}\in t_{\sigma(j)}}} \left\{ \left\{ 1 - \left(1 - \left(\prod_{\sigma\in R_n}\left(1 - \prod_{j=1}^n \left(1-\delta_{\sigma(j)}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^n l_j}} \right\}, \right. \\ \left. \left. \left\{ \left(\left(1 - \left(\prod_{\sigma\in R_n}\left(1 - \prod_{j=1}^n \pi_{\sigma(j)}^{2l_j}\right)\right)^{\frac{1}{n!}}\right)^{1/2}\right)^{\frac{1}{\sum_{j=1}^n l_j}} \right\} \right\} \right\rangle, \tag{27}$$

which is the dual Pythagorean hesitant uncertain linguistic dual Muirhead mean operator.

3.4. The q-rung dual hesitant uncertain linguistic weighted dual Muirhead mean (q-RDHULDMM) operator

Definition 10 *Let* $\alpha_j = \left\langle \left[s_{\theta_j}, s_{\xi_j}\right], \left(g_j, t_j\right)\right\rangle$ $(j = 1,2,\ldots,n)$ *be a collection of q-RDHULVs and* $L = (l_1, l_2, \ldots, l_n) \in R^n$ *be a collection of parameters. Let* $w = (w_1, w_2, \ldots, w_n)^T$ *be the weight vector of* α_j $(j = 1,2,\ldots,n)$*, satisfying* $0 \leqslant w_j \leqslant 1$ *and* $\sum_{j=1}^{n} w_j = 1$*. If*

$$q\text{-}RDHULWDMM^L(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{\sum\limits_{j=1}^{n} l_j}\left(\bigotimes_{\sigma\in R_n}\bigoplus_{j=1}^{n}\left(l_j \alpha_{\sigma(j)}^{n w_{\sigma(j)}}\right)\right)^{\frac{1}{n!}}, \tag{28}$$

then q-RDHULWMM^L *is the q-rung dual hesitant uncertain linguistic weighted dual Muirhead mean (q-RDHULWDMM) operator, where* R_n *represents all possible permutations of* $(1,2,\ldots,n)$ *and* $\sigma(j)$ $(j = 1,2,\ldots,n)$ *is any one of* R_n*.*

Theorem 5 *Let* $\alpha_j = \left\langle \left[s_{\theta_j}, s_{\xi_j}\right], \left(g_j, t_j\right)\right\rangle$ $(j = 1,2,\ldots,n)$ *be a collection of q-RDHULVs and* $L = (l_1, l_2, \ldots, l_n) \in R^n$ *be a collection of parameters. The aggregated value by the q-RDHULWDMM operator is still a q-RDHULV and*

$$\begin{aligned} & q\text{-}RDHULWDMM^L(\alpha_1, \alpha_2, \ldots, \alpha_n) = \\ & = \left\langle \left[s_{\frac{1}{\sum_{j=1}^{n} l_j}\left(\prod\limits_{\sigma\in R_n}\sum\limits_{j=1}^{n}\left(l_j \theta_{\sigma(j)}^{n w_{\sigma(j)}}\right)\right)^{\frac{1}{n!}}},\ s_{\frac{1}{\sum_{j=1}^{n} l_j}\left(\prod\limits_{\sigma\in R_n}\sum\limits_{j=1}^{n}\left(l_j \xi_{\sigma(j)}^{n w_{\sigma(j)}}\right)\right)^{\frac{1}{n!}}} \right], \right. \\ & \bigcup_{\substack{\delta_{\sigma(j)}\in g_{\sigma(j)},\\ \pi_{\sigma(j)}\in t_{\sigma(j)}}} \left\{\left\{\left(1-\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\left(\delta_{\sigma(j)}^{n w_{\sigma(j)}}\right)^q\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}\right)^{1/q}\right\}, \right. \\ & \left.\left.\left\{\left(\left(1-\left(\prod_{\sigma\in R_n}\left(1-\prod_{j=1}^{n}\left(1-\left(1-\pi_{\sigma(j)}^q\right)^{n w_{\sigma(j)}}\right)^{l_j}\right)\right)^{\frac{1}{n!}}\right)^{1/q}\right)^{\frac{1}{\sum_{j=1}^{n} l_j}}\right\}\right\}\right\rangle. \end{aligned} \tag{29}$$

The proof of Theorem 5 is similar to that of Theorem 2. In addition, the proposed q-RDHULWDMM operator has the properties of monotonicity and boundedness.

4. A new MADM method with q-rung dual hesitant uncertain linguistic information

In the above sections, we have analyzed the powerfulness of the q-RDHULSs and the proposed q-RDHUL AOs. In this section, we consider MADM problem in q-RDHUL decision-making environment. There are m alternatives to be evaluated, which can be denoted as $X = \{X_1, X_2, \ldots, X_m\}$. DMs are invited to be a decision-making committee to evaluate the candidates from n aspects, which can be presented by $C = \{C_1, C_2, \ldots, C_n\}$. The importance degree of attribute C_j is w_j, satisfying $0 \leqslant w_j \leqslant 1$ and $\sum_{j=1}^{n} w_j = 1$. In other word, $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of attributes. Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ be a LTS and based on which DMs use a q-RDHULV $\alpha_{ij} = \left\langle \left[s_{\theta_{ij}}, s_{\xi_{ij}}\right], \left(g_{ij}, t_{ij}\right)\right\rangle$ to express their evaluation value for attribute C_j over alternative X_i. Hence, a q-RDHUL decision matrix $D = \left(\alpha_{ij}\right)_{m\times n}$ is determined. In the followings, we solve this MADM problem based on the proposed AOs.

Step 1 Standardize the original q-RDHUL decision matrix according to the following equation

$$D = \left(\alpha_{ij}\right)_{m\times n} = \begin{cases} \left\langle \left[s_{\theta_{ij}}, s_{\xi_{ij}}\right], \left(g_{ij}, t_{ij}\right)\right\rangle & C_j \in I_1, \\ \left\langle \left[s_{\theta_{ij}}, s_{\xi_{ij}}\right], \left(t_{ij}, g_{ij}\right)\right\rangle & C_j \in I_2, \end{cases} \tag{30}$$

where I_1 and I_2 denote benefit type and cost type of attribute.

Step 2 Utilize the q-RDHULWMM operator

$$\alpha_i = q\text{-RDHULWMM}^L \left(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}\right), \tag{31}$$

or the q-RDHULWDMM operator

$$\alpha_i = q\text{-RDHULWDMM}^L \left(\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{in}\right), \tag{32}$$

to aggregate attribute values of alternative X_i $(i = 1, 2, \ldots, m)$. Hence, a collective of overall evaluation values are derived.

Step 3 Calculate the scores of the comprehensive evaluation values according to Definition 4.

Step 4 Rank alternatives according to their corresponding scores and select the optimal one.

5. A practical application of proposed method in enterprise informatization level evaluation

Enterprise information construction refers to improving the production and operation efficiency of enterprises through the deployment of computer technology, reducing operational risks and costs, thereby improving the overall management level and the ability of continuous operation. The main purpose of enterprise informatization is to use advanced information technology and modern management methods to enhance and optimize the business process and management level of the enterprise. In the fierce domestic and international competition, many Chinese companies are accelerating the pace of informatization construction. Before the construction of information technology, enterprises need to comprehensively evaluate the current level of informatization from multiple aspects. In essence, the evaluation of enterprise informatization level is a MADM problem. A group wants to evaluate the information level of its four subsidiaries, which can be denoted as $X = \{X_1, X_2, X_3, X_4\}$. In order to make an accurate evaluation, the company has invited a number of senior experts in the field of enterprise information construction to evaluate the four possible alternatives. Taking into account the actual business and business conditions of the company, the decision-making experts evaluate the informatization level of the four subsidiaries from the following four aspects, i.e. C_1 (enterprise scale level), C_2 (proportion of investment for informatization), C_3 (institutional standards construction), and C_4 (attention from leader). The weight vector of these attributes is $w = (0.2, 0.1, 0.3, 0.4)^T$. Let $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$ be an LTS and DMs are invited to use a q-RDHULV $\alpha_{ij} = \left\langle \left[s_{\theta_{ij}}, s_{\xi_{ij}}\right], \left(g_{ij}, t_{ij}\right)\right\rangle$ $(i, j = 1, 2, 3, 4)$ to express their evaluation information. Hence, a q-RDHUL decision matrix can be obtained, which is shown as Table 1. In the followings, we utilize the proposed method to evaluate the overall performance of the four alternatives.

5.1. The decision making process

In the following, we utilize the proposed method to deal with the MADM problem.

Step 1 Given that all attributes are benefit type, the original q-rung dual hesitant uncertain linguistic does not need to be normalized.

Step 2 Utilize the q-RDHULWMM operator to aggregate attribute values of each alternative. Hence, for each alternative an overall evaluation value is obtained. Without loss of generality, let $L = (1, 1, 1, 1)$ and $q = 2$. As the comprehensive evaluation values of each alternatives are too complex, we omit them here to save space.

Table 1: The q-rung dual hesitant uncertain linguistic decision matrix provided by experts

	C_1	C_2
X_1	$\langle[s_3, s_4], \{\{0.3, 0.5\}, \{0.2, 0.4\}\}\rangle$	$\langle[s_5, s_6], \{\{0.5, 0.6\}, \{0.3, 0.4\}\}\rangle$
X_2	$\langle[s_1, s_3], \{\{0.2, 0.4, 0.5\}, \{0.1, 0.3\}\}\rangle$	$\langle[s_4, s_6], \{\{0.5, 0.6, 0.7\}, \{0.1, 0.2, 0.3\}\}\rangle$
X_3	$\langle[s_2, s_4], \{\{0.3, 0.4\}, \{0.1, 0.2, 0.3\}\}\rangle$	$\langle[s_5, s_7], \{\{0.3, 0.5\}, \{0.1, 0.2\}\}\rangle$
X_4	$\langle[s_2, s_4], \{\{0.3, 0.4\}, \{0.1, 0.2, 0.3\}\}\rangle$	$\langle[s_6, s_7], \{\{0.2, 0.6\}, \{0.1, 0.3, 0.4\}\}\rangle$
	C_3	C_4
X_1	$\langle[s_1, s_3], \{\{0.2, 0.3, 0.5\}, \{0.2, 0.3\}\}\rangle$	$\langle[s_5, s_7], \{\{0.6, 0.7\}, \{0.1, 0.2\}\}\rangle$
X_2	$\langle[s_2, s_3], \{\{0.1, 0.5\}, \{0.1, 0.5\}\}\rangle$	$\langle[s_6, s_7], \{\{0.1, 0.6\}, \{0.1, 0.3\}\}\rangle$
X_3	$\langle[s_2, s_3], \{\{0.1, 0.3, 0.4\}, \{0.1, 0.4\}\}\rangle$	$\langle[s_4, s_7], \{\{0.2, 0.5\}, \{0.3, 0.5\}\}\rangle$
X_4	$\langle[s_2, s_4], \{\{0.2, 0.4, 0.6\}, \{0.1, 0.3\}\}\rangle$	$\langle[s_5, s_6], \{\{0.3, 0.6\}, \{0.3, 0.4\}\}\rangle$

Step 3 Calculate the scores of the comprehensive evaluation values, and we can obtain

$$S(\alpha_1) = 0.2774, \quad S(\alpha_2) = 0.2398, \quad S(\alpha_3) = 0.2686, \quad S(\alpha_4) = 0.2782.$$

Step 4 Rank alternatives according to their scores and we have $X_4 \succ X_1 \succ X_3 \succ X_2$, and X_4 is the best alternative. In other word, the subsidiary X_4 has the highest informatization level.

In Step 2, if we utilize the q-RDHULWDMM operator to aggregate attributes of each alternative, then the scores of alternatives are ($q = 2$ and $L = (1, 1, 1, 1)$)

$$S(\alpha_1) = 0.0715, \quad S(\alpha_2) = 0.0669, \quad S(\alpha_3) = 0.0728, \quad S(\alpha_4) = 0.0729.$$

Hence, the ranking orders is $X_4 \succ X_3 \succ X_1 \succ X_2$, and the optimal alternative is X_4, which also illustrates that X_4 has highest informatization level.

5.2. The validity of our proposed method

In this subsection, we prove the validity and effectiveness of our proposed method by solving real MADM problems. We compare the decision results obtained by our method with those derived by some existing methods and conduct analysis in detail.

Example 1 (From Ref. [24]) A company wants to select an investment project among four possible alternatives, denoted by A_i ($i = 1, 2, 3, 4$). A group of DMs are organized as a committee to evaluate the four potential alternatives under three attributes C_j ($j = 1, 2, 3$), i.e. the risk analysis (C_1), the growth analysis

(C_2), and the environmental impact analysis (C_3). The weight vector of attributes is $w = (0.35,\ 0.25,\ 0.40)^T$. Let $S = \{s_0,\ s_1,\ s_2,\ s_3,\ s_4,\ s_5,\ s_6\}$ be a LTS and for attribute C_j $(j = 1, 2, 3)$ of alternative A_i $(i = 1, 2, 3, 4)$, DMs utilize a dual hesitant fuzzy linguistic element (DHFLE) $\alpha_{ij} = \left\langle s_{\theta_{ij}}, \left(g_{ij}, t_{ij}\right)\right\rangle$ to express their evaluation values. Therefore, a dual hesitant fuzzy linguistic decision matrix can be obtained, which is listed in Table 2.

Table 2: The dual hesitant fuzzy linguistic decision matrix of Example 1

	C_1	C_2	C_3
A_1	$\langle s_3, \{\{0.4, 0.5, 0.6\}, \{0.3, 0.4\}\}\rangle$	$\langle s_4, \{\{0.3, 0.5\}, \{0.2, 0.3\}\}\rangle$	$\langle s_4, \{\{0.3, 0.5, 0.6\}, \{0.1, 0.2\}\}\rangle$
A_2	$\langle s_2, \{\{0.4, 0.5\}, \{0.3, 0.4\}\}\rangle$	$\langle s_5, \{\{0.4, 0.5\}, \{0.4, 0.5\}\}\rangle$	$\langle s_3, \{\{0.2, 0.5, 0.6\}, \{0.2, 0.4\}\}\rangle$
A_3	$\langle s_4, \{\{0.5, 0.7\}, \{0.2, 0.3\}\}\rangle$	$\langle s_3, \{\{0.2, 0.4, 0.5\}, \{0.3, 0.4\}\}\rangle$	$\langle s_3, \{\{0.4, 0.5, 0.7\}, \{0.2, 0.3\}\}\rangle$
A_4	$\langle s_4, \{\{0.4, 0.6, 0.8\}, \{0.1, 0.2\}\}\rangle$	$\langle s_2, \{\{0.5, 0.6\}, \{0.2, 0.4\}\}\rangle$	$\langle s_5, \{\{0.6, 0.7\}, \{0.1, 0.3\}\}\rangle$

It is worth to point out that in Example 1, DMs utilize DHFLEs to represent the attribute values. Actually, a DHFLE is a special case of the proposed q-RDHULV and we can transform a DHFLE into a q-RDHULV. For example, let $\alpha = \langle s_3, \{\{0.4, 0.5, 0.6\}, \{0.3, 0.4\}\}\rangle$ be a DHFLE and we transform it into a q-RDHULV, i.e. $\alpha = \langle [s_3, s_3], \{\{0.4, 0.5, 0.6\}, \{0.3, 0.4\}\}\rangle$. It is obvious that the information of α has not been changed during the transformation. Hence, we can transform the original dual hesitant fuzzy linguistic decision matrix of Example 1 into a q-RDHUL decision matrix, which is shown as Table 3. We utilize the method proposed by Yang and Ju [24] based on the dual hesitant fuzzy linguistic weighted geometric (DHFLWG) operator and our proposed method based on the q-RDHULWDMM operator to solve Example 1 and present their results in Table 4.

Table 3: The corresponding q-rung dual hesitant uncertain linguistic decision matrix of Example 1

	C_1	C_2	C_3
A_1	$\langle [s_3, s_3], \{\{0.4, 0.5, 0.6\}, \{0.3, 0.4\}\}\rangle$	$\langle [s_4, s_4], \{\{0.3, 0.5\}, \{0.2, 0.3\}\}\rangle$	$\langle [s_4, s_4], \{\{0.3, 0.5, 0.6\}, \{0.1, 0.2\}\}\rangle$
A_2	$\langle [s_2, s_2], \{\{0.4, 0.5\}, \{0.3, 0.4\}\}\rangle$	$\langle [s_5, s_5], \{\{0.4, 0.5\}, \{0.4, 0.5\}\}\rangle$	$\langle [s_3, s_3], \{\{0.2, 0.5, 0.6\}, \{0.2, 0.4\}\}\rangle$
A_3	$\langle [s_4, s_4], \{\{0.5, 0.7\}, \{0.2, 0.3\}\}\rangle$	$\langle [s_3, s_3], \{\{0.2, 0.4, 0.5\}, \{0.3, 0.4\}\}\rangle$	$\langle [s_3, s_3], \{\{0.4, 0.5, 0.7\}, \{0.2, 0.3\}\}\rangle$
A_4	$\langle [s_4, s_4], \{\{0.4, 0.6, 0.8\}, \{0.1, 0.2\}\}\rangle$	$\langle [s_2, s_2], \{\{0.5, 0.6\}, \{0.2, 0.4\}\}\rangle$	$\langle [s_5, s_5], \{\{0.6, 0.7\}, \{0.1, 0.3\}\}\rangle$

Table 4: The decision results of Example 1 by different methods

Method	Score values $S(\alpha_i)$ $(i = 1, 2, 3, 4)$	Ranking order
The method introduced by Yang and Ju [24] based on the DHFLWG operator	$S(\alpha_1) = 0.1196$ $S(\alpha_2) = 0.0304$ $S(\alpha_3) = 0.0905$ $S(\alpha_4) = 0.1951$	$A_4 \succ A_1 \succ A_3 \succ A_2$
The method presented in this paper based on the q-RDHULWDMM operator ($q = 1$ and $L = (1, 0, 0)$)	$S(\alpha_1) = 0.6028$ $S(\alpha_2) = 0.5145$ $S(\alpha_3) = 0.7056$ $S(\alpha_4) = 0.7202$	$A_4 \succ A_3 \succ A_1 \succ A_2$

From Table 4, we can find out that although the ranking order derived by our method is slightly different from that obtained by Yang and Ju's [24] method, but the best and worst alternatives are all A_4 and A_2, respectively, which proves the effectiveness and correctness of our proposed method.

5.3. The effects of the parameters on the results

In this subsection, we investigate the influence of the parameters on the result. First, we study the effect of the parameter q on the scores and ranking orders. We utilize different values of q in the q-RDHULWMM and q-RDHULWDMM operators when aggregating attribute values and present the scores and ranking results in Tables 5 and 6. Without loss of generality, we assume $L = (1,\ 1,\ 1,\ 1)$.

Table 5: The decision results by using the q-RDHULWMM operator with different values of q

q	Score values $S(\alpha_i)$ $(i = 1, 2, 3, 4)$	Ranking order
1	$S(\alpha_1) = 0.2826$ $S(\alpha_2) = 0.2455$ $S(\alpha_3) = 0.2749$ $S(\alpha_4) = 0.2899$	$X_4 \succ X_1 \succ X_3 \succ X_2$
2	$S(\alpha_1) = 0.2774$ $S(\alpha_2) = 0.2398$ $S(\alpha_3) = 0.2686$ $S(\alpha_4) = 0.2782$	$X_4 \succ X_1 \succ X_3 \succ X_2$
3	$S(\alpha_1) = 0.2701$ $S(\alpha_2) = 0.2322$ $S(\alpha_3) = 0.2602$ $S(\alpha_4) = 0.2677$	$X_1 \succ X_4 \succ X_3 \succ X_2$
5	$S(\alpha_1) = 0.2567$ $S(\alpha_2) = 0.2189$ $S(\alpha_3) = 0.2455$ $S(\alpha_4) = 0.2543$	$X_1 \succ X_4 \succ X_3 \succ X_2$
7	$S(\alpha_1) = 0.2469$ $S(\alpha_2) = 0.2095$ $S(\alpha_3) = 0.2351$ $S(\alpha_4) = 0.2478$	$X_4 \succ X_1 \succ X_3 \succ X_2$

As seen from Table 5, if we assign different values to the parameter q in the q-RDHULWMM operator, different scores of alternatives are derived, which further leads to different ranking orders. Moreover, we notice that the increase of

Table 6: The decision results by using the q-RDHULWDMM with different values of q

q	Score values $S(\alpha_i)$ $(i = 1, 2, 3, 4)$	Ranking order
1	$S(\alpha_1) = 0.0683$ $S(\alpha_2) = 0.0631$ $S(\alpha_3) = 0.0687$ $S(\alpha_4) = 0.0697$	$X_4 \succ X_3 \succ X_1 \succ X_2$
2	$S(\alpha_1) = 0.0715$ $S(\alpha_2) = 0.0669$ $S(\alpha_3) = 0.0728$ $S(\alpha_4) = 0.0729$	$X_4 \succ X_3 \succ X_1 \succ X_2$
3	$S(\alpha_1) = 0.0744$ $S(\alpha_2) = 0.0695$ $S(\alpha_3) = 0.0758$ $S(\alpha_4) = 0.0761$	$X_4 \succ X_3 \succ X_1 \succ X_2$
5	$S(\alpha_1) = 0.0783$ $S(\alpha_2) = 0.0719$ $S(\alpha_3) = 0.0786$ $S(\alpha_4) = 0.0805$	$X_4 \succ X_3 \succ X_1 \succ X_2$
7	$S(\alpha_1) = 0.0802$ $S(\alpha_2) = 0.0723$ $S(\alpha_3) = 0.0792$ $S(\alpha_4) = 0.0826$	$X_4 \succ X_1 \succ X_3 \succ X_2$

the value of q results in the decrease of the scores. In Table 6, we can also find the phenomenon that different scores and corresponding ranking orders of alternatives are obtained with different values of q in the q-RDHULWDMM operator. What is opposite is that the increase of the value of q results in the increase of score values. This reveals that the parameter q in q-RDHULWMM and q-RDHULWDMM operators has inverse influence on the score values. Hence, how to select an appropriate value of q is a fundamental problem before determining the best alternative. Basically, we argue that the value of q should be taken as the smallest integer such that $\delta^q + \pi^q \leqslant 1$, for any $\delta \in g$ and $\pi \in t$. For instance, if a DM provides $\langle [s_3, s_6], \{\{0.3, 0.5, 0.7\}, \{0.2, 0.4, 0.9\}\}\rangle$ as his/her evaluation value, as $0.7^3 + 0.9^3 = 1.072 > 1$ and $0.7^4 + 0.9^4 = 0.8962 < 1$, then q can be taken as 4. In the followings, we further study the influence of the parameter vector L on the results. Similarly, we assign different parameter vector to L in the q-RDHULWMM and q-RDHULWDMM operators and present the score values and corresponding ranking orders in Tables 7 and 8.

As we can see from Tables 7 and 8, different score values are derived with different values of L in the q-RDHULWMM and q-RDHULWDMM operators. For easy description, we denote the number of related parameters in parameter vector L as n_L $(n_L = 1, 2, 3, 4)$. As noticed in Table 7, when $n_L = 1$ the scores of alternatives are extremely small, which is even smaller than the smallest score of any attribute value. This is inconsistent with our intuition and the reality. When $n_L = 2, 3, 4$, the scores of alternatives of alternatives are comparatively reasonable. In Table 8, we can find the similar phenomenon. What is inverse is that when $n_L = 1$, the scores of alternatives are extremely bigger, which is even bigger than the biggest score of attribute values. The main reason is that when $n_L = 1$ in the q-RDHULWMM (q-RDHULWDMM) operator, the interrelationship among

Table 7: The decision results by using different parameter vector L in the q-RDHULWMM operator

L	Score values $S(\alpha_i)$ $(i = 1, 2, 3, 4)$	Ranking order
$L = (1, 0, 0, 0)$	$S(\alpha_1) = 0.0561$ $S(\alpha_2) = 0.0382$ $S(\alpha_3) = 0.0554$ $S(\alpha_4) = 0.0719$	$X_4 > X_1 > X_3 > X_2$
$L = (1, 1, 0, 0)$	$S(\alpha_1) = 0.1623$ $S(\alpha_2) = 0.1265$ $S(\alpha_3) = 0.1562$ $S(\alpha_4) = 0.1766$	$X_4 > X_1 > X_3 > X_2$
$L = (1, 1, 1, 0)$	$S(\alpha_1) = 0.2345$ $S(\alpha_2) = 0.1958$ $S(\alpha_3) = 0.2269$ $S(\alpha_4) = 0.2446$	$X_4 > X_1 > X_3 > X_2$
$L = (1, 1, 1, 1)$	$S(\alpha_1) = 0.2826$ $S(\alpha_2) = 0.2455$ $S(\alpha_3) = 0.2749$ $S(\alpha_4) = 0.2899$	$X_4 > X_1 > X_3 > X_2$
$L = (2, 2, 2, 2)$	$S(\alpha_1) = 0.2826$ $S(\alpha_2) = 0.2455$ $S(\alpha_3) = 0.2749$ $S(\alpha_4) = 0.2899$	$X_4 > X_1 > X_3 > X_2$
$L = (2, 0, 0, 0)$	$S(\alpha_1) = 0.0561$ $S(\alpha_2) = 0.0382$ $S(\alpha_3) = 0.0554$ $S(\alpha_4) = 0.0719$	$X_4 > X_1 > X_3 > X_2$
$L = (5, 0, 0, 0)$	$S(\alpha_1) = 0.0561$ $S(\alpha_2) = 0.0382$ $S(\alpha_3) = 0.0554$ $S(\alpha_4) = 0.0719$	$X_4 > X_1 > X_3 > X_2$

Table 8: The decision results by using different parameter vector L in the q-RDHULWDMM operator

L	Score values $S(\alpha_i)$ $(i = 1, 2, 3, 4)$	Ranking order
$L = (1, 0, 0, 0)$	$S(\alpha_1) = 0.5637$ $S(\alpha_2) = 0.4953$ $S(\alpha_3) = 0.5445$ $S(\alpha_4) = 0.5717$	$X_4 > X_1 > X_3 > X_2$
$L = (1, 1, 0, 0)$	$S(\alpha_1) = 0.2130$ $S(\alpha_2) = 0.1916$ $S(\alpha_3) = 0.2096$ $S(\alpha_4) = 0.2161$	$X_4 > X_1 > X_3 > X_2$
$L = (1, 1, 1, 0)$	$S(\alpha_1) = 0.1114$ $S(\alpha_2) = 0.1018$ $S(\alpha_3) = 0.1110$ $S(\alpha_4) = 0.1133$	$X_4 > X_1 > X_3 > X_2$
$L = (1, 1, 1, 1)$	$S(\alpha_1) = 0.0683$ $S(\alpha_2) = 0.0631$ $S(\alpha_3) = 0.0687$ $S(\alpha_4) = 0.0697$	$X_4 > X_3 > X_1 > X_2$
$L = (2, 2, 2, 2)$	$S(\alpha_1) = 0.0683$ $S(\alpha_2) = 0.0631$ $S(\alpha_3) = 0.0687$ $S(\alpha_4) = 0.0697$	$X_4 > X_3 > X_1 > X_2$
$L = (2, 0, 0, 0)$	$S(\alpha_1) = 0.5637$ $S(\alpha_2) = 0.4953$ $S(\alpha_3) = 0.5445$ $S(\alpha_4) = 0.5717$	$X_4 > X_1 > X_3 > X_2$
$L = (5, 0, 0, 0)$	$S(\alpha_1) = 0.5637$ $S(\alpha_2) = 0.4953$ $S(\alpha_3) = 0.5445$ $S(\alpha_4) = 0.5717$	$X_4 > X_1 > X_3 > X_2$

attributes is not taken into account. Another interesting phenomenon we can find out is that the increase of n_L in the q-RDHULWMM operator leads to the increase of the score functions $S(\alpha_i)$ $(i = 1, 2, 3, 4)$ and inversely, the increase of n_L in the q-RDHULWDMM operator leads to the decrease of the score functions.

5.4. Superiorities of the proposed method

In the followings, we compare the proposed method based on the q-RDHULWMM operator with that proposed by Wang et al. [56] based on the q-rung orthopair uncertain linguistic weighted Muirhead mean (q-ROULWMM) operator, that proposed by Yang and Ju [24] based on the dual hesitant linguistic weighted average (DHLWA) operator, and that proposed by Xu et al. [38] based on the q-rung hesitant fuzzy weighted Heronian mean (q-RDHFWHM) operator. We utilize some of these methods to solve some MADM problems, compare their results and discuss the advantages and superiorities of our method in detail.

5.4.1. Its ability of capturing DMs' hesitancy and uncertainty

Our proposed method is based on q-RDHULSs, which allow the NDs and NMDs of ULVs to be denoted by two sets of values. The MADM method proposed by Wang et al. [56] is based on the q-ROULSs wherein the MD and NMD of an ULV are denoted by a single value. Hence, q-RDHULSs are more powerful and flexible than q-ROULSs, as they can appropriately express DMs' high hesitancy in giving their evaluation values. In other word, the q-ROULSs are a special case the proposed q-RDHULSs where there are only one MD and one NMD. Thus, our proposed method can also effectively deal with MADM problems in which DMs' evaluation values are in the form of q-ROULSs. To better demonstrate this advantage of our method, we give the following example.

Example 2 (From Ref. [57]) To make a great profit, a company wants to invest some money into an alternative. After primary evaluation, there are four potential alternatives to be selected, which are denoted by A_i $(i = 1, 2, 3, 4)$. To determine the most suitable alternative, the company invites a set of DMs to evaluation the performance of the possible alternatives under four attributes, which are denoted by C_i $(i = 1, 2, 3, 4)$. The weight vector of attributes is $w = (0.32, 0.26, 0.18, 0.27)^T$. Let $S = \{s_0, s_1, s_2, s_3, s_4, s_5, s_6\}$, and DMs are required to utilize q-rung orthopair uncertain linguistic variables (q-ROULVs) to express their evaluation values, which constructs a q-rung orthopair uncertain linguistic decision matrix (see Table 9). We utilize the method presented by Wang

Table 9: The q-rung orthopair uncertain linguistic decision matrix of Example 2

	C_1	C_2	C_3	C_4
A_1	$\langle [s_5, s_5], (0.7, 0.1)\rangle$	$\langle [s_3, s_3], (0.7, 0.3)\rangle$	$\langle [s_4, s_5], (0.6, 0.1)\rangle$	$\langle [s_4, s_5], (0.7, 0.2)\rangle$
A_2	$\langle [s_4, s_5], (0.6, 0.2)\rangle$	$\langle [s_5, s_5], (0.6, 0.3)\rangle$	$\langle [s_2, s_3], (0.8, 0.1)\rangle$	$\langle [s_3, s_4], (0.6, 0.4)\rangle$
A_3	$\langle [s_3, s_4], (0.7, 0.2)\rangle$	$\langle [s_4, s_5], (0.6, 0.2)\rangle$	$\langle [s_1, s_2], (0.7, 0.1)\rangle$	$\langle [s_2, s_3], (0.7, 0.1)\rangle$
A_4	$\langle [s_3, s_3], (0.5, 0.2)\rangle$	$\langle [s_2, s_3], (0.7, 0.1)\rangle$	$\langle [s_3, s_4], (0.6, 0.3)\rangle$	$\langle [s_4, s_4], (0.5, 0.3)\rangle$

et al. [56] based on the q-rung orthopair uncertain linguistic weighted Muirhead mean (q-ROULWMM) operator and our proposed method to solve Example 2 and present the results in Table 10.

Table 10: The decision results of Example 2 by different methods

Method	Score values $S(\alpha_i)$ $(i = 1, 2, 3, 4)$	Ranking order
Wang et al.'s [56] method based on the q-ROULWMM operator (when $q = 2$ and $P = (1, 1, 1, 1)$)	$S(\alpha_1) = 0.4108$ $S(\alpha_2) = 0.4007$ $S(\alpha_3) = 0.3916$ $S(\alpha_4) = 0.3215$	$A_1 \succ A_2 \succ A_3 \succ A_4$
Our proposed method based on the q-RDHULWMM operator (when $q = 2$ and $L = (1, 1, 1, 1)$)	$S(\alpha_1) = 0.4500$ $S(\alpha_2) = 0.4102$ $S(\alpha_3) = 0.4002$ $S(\alpha_4) = 0.3564$	$A_1 \succ A_2 \succ A_3 \succ A_4$

The decision results in Table 10 also demonstrate the correctness of our proposed method. However, Example 2 assumes that DMs are not hesitant when providing their evaluation values. In most actual MADM problems, DMs often hesitate among a set of values when giving the MDs and NMDs of ULVs and generally they hope to provide several values for MD and NMD of an ULV instead of single values. In addition, in some real situations DMs have different opinions on MD and NMD of an ULV. For instance, as per attribute C_1 of A_1 some DMs may give 0.5 and 0.1 as for the MD and NMD of the ULV$[s_5, s_5]$. Some DMs would like to provide 0.6 and 0.2 as the MD and NMD, and the others would like to give 0.7 and 0.3 for the MD and NMD. Hence, to comprehensive represent the attribute value of C_1 OF alternative A_1, we can use a q-RDHULV, i.e. $\langle [s_5, s_5], \{\{0.5, 0.6, 0.7\}, \{0.1, 0.2, 0.3\}\}\rangle$. Afterward, we utilize Wang et al.'s [56] method and our proposed method to solve Example 2 and present the results in Table 11.

Table 11: The new decision results of Example 2 by different methods

Method	Score values $S(\alpha_i)$ $(i = 1, 2, 3, 4)$	Ranking order
Wang et al.'s [56] method based on the q-ROULWMM operator	Cannot be computed	No
Our proposed method based on the q-RDHULWMM operator (when $q = 2$ and $L = (1, 1, 1, 1)$)	$S(\alpha_1) = 0.4223$ $S(\alpha_2) = 0.4102$ $S(\alpha_3) = 0.4002$ $S(\alpha_4) = 0.3564$	$A_1 \succ A_2 \succ A_3 \succ A_4$

From Table 11 we can easily find that the method proposed by Wang et al. [56] cannot be applied to solve the revised Example 2, whereas is suitable to solve this example. This is because Wang et al.'s [56] method is based on q-ROULSs in which the MD and NMD of an ULV can only be represented by two single values. In other word, Wang et al.'s [56] method cannot effectively deal with DMs inherent hesitancy in expressing their evaluation information. However, our proposed method is based on q-RDHULSs, which allow the MD and NMD of ULVs to be represented by two sets of values. In other word, our proposed method has good ability of reflecting and capturing DMs' high hesitancy in the evaluation process. This feature makes our method suitable to deal with complicated practical MADM problems.

5.4.2. Larger information space that it can depict

Our proposed method is based on q-RDHULSs, whose constraint is that the sum of q-th power of MD and q-th power of NMD should be less than or equal to one. This characteristic provides great freedom for DMs to comprehensively express their evaluation values and results ion less information distortion. To better demonstrate this advantage, we give the following example.

Example 3 In Example 1, DMs use DHFLEs to express their preference information. The constraint of DHFLE is that the sum of MD and NMD of a linguistic variable should be less than or equal to one. However, in real MADM problems such restriction cannot always been satisfied. Basically, we should not set rigorous constraint for DMs before they provide their evaluation information. To accurately and comprehensively capture DMs' ideas, we should give them more freedom. For instance, for the attribute C_1 of A_1 DMs provide $\langle s_3, \{\{0.4, 0.5, 0.8\}, \{0.3, 0.9\}\}\rangle$ as their assessment, then as $0.8 + 0.9 = 1.7 > 1$, the method proposed by Yang and Ju [24] is not suitable to deal with this case. Nevertheless, as our proposed method has a laxer constraint, our method can be applied to solve this problem and the results are listed in Table 12. (As $0.8^4 + 0.9^4 = 1.0657 > 1$ and $0.8^5 + 0.8^5 = 0.9182 < 1$, then q can be taken as 5).

Table 12: The decision results of Example 3 by different MADM method

Method	Score values $S(\alpha_i)$ $(i = 1, 2, 3, 4)$	Ranking order
The proposed method introduced by Yang and Ju [24]	Cannot be calculated	No
Our proposed method in this paper (when $q = 5$ and $L = (1, 0, 0)$)	$S(\alpha_1) = 0.4844$ $S(\alpha_2) = 0.3963$ $S(\alpha_3) = 0.5695$ $S(\alpha_4) = 0.5651$	$A_3 \succ A_4 \succ A_1 \succ A_2$

In practical MADM problems, we cannot require DMs to meet a rigorous constraint, otherwise some important decision-making information will lose in the evaluating process. We should give DMs enough freedom to express their preference and so that we can get comprehensive evaluation values of possible alternatives. As our proposed method has a laxer constraint, which makes it effective to describe larger information space. Hence, our method provides enough freedom for DMs to fully and comprehensively give their preference information. This characteristic makes it suitable to handle actual MADM problems.

5.4.3. The ability of considering both DMs' quantitative and qualitative evaluation information

In Ref. [38], Xu et al. proposed the concept of q-RDHFSs, studied their aggregation operators and investigated their applications in MADM. This paper proposes a new tool to express DMs' evaluation information, called q-RDHFULS, which is combination of Xu et al.'s [38] q-RDHFSs with ULVs. Basically, the MADM method proposed by Xu et al. [38] can only reflect DMs' preference information quantificationally. In real MADM problems, the shortcoming of Xu et al.'s [38] method is conspicuous. Due to the increasing complexity of MADM problems, DMs would like to express their assessments from both quantitative and qualitative aspects. The overlook of either DMs' quantitative or qualitative decision information will lead to unreasonable results. In our method, DMs utilize q-RDHFULVs to denote their assessments, which can fully depict DMs' preference information both quantificationally and qualitatively. Hence, our method is more sufficient and suitable to deal with practical MADM problems.

5.4.4. The flexibility of dealing with the interrelationship among any numbers of attributes

In most real MADM problems, attributes are usually correlated and ignorance of such kind of interrelationship among attributes will lead to unreasonable decision results. For instance, in the enterprise informatization level evaluation problem, if leaders pay more attention to the informatization construction, the enterprise will invest more money into the informatization procedure, and the enterprise scale level will also improve. In other word, there exists significant interrelationship among the attributes C_1, C_2 and C_3. In Example 2, there is also significant correlation among the attributes C_1, C_2 and C_3. However, the MADM method proposed by Yang and Ju [24] is based on the simple weighted geometric operator, which ignores the interrelationship among attribute values. The method presented by Xu et al. [38] can only consider the interrelationship between any two attributes. Our proposed method is based on MM so that it is powerful to reflect the interrelationship among any numbers of attributes (Details can be found in subsection 3.2). Hence, our proposed method is more suitable to deal with MADM problems. To better demonstrate the advantages superiorities of our method. We present the characteristics of some MADM methods in Table 13.

Table 13: The characteristics of different methods

Method	Whether considers DMs' quantitative evaluation information	Whether considers DMs' qualitative evaluation information	Whether comprehensively captures DMs' hesitancy in providing their evaluations
Yang and Ju's [24] method based on the DHFLWG operator	Yes	Yes	Yes
Wang et al.' method [56] based on the q-ROULWMM operator	Yes	Yes	No
Xu et al.'s method [38] based on the q-RDHFWHM operator	Yes	No	Yes
The proposed method based on the q-RDHULWMM operator	Yes	Yes	Yes
Method	Whether permits the sum of MD and NMD to be greater than one	Whether captures the interrelationship between any two attributes	Whether captures the interrelationship among multiple attributes
Yang and Ju's [24] method based on the DHFLWG operator	No	No	No
Wang et al.' method [56] based on the q-ROULWMM operator	Yes	Yes	Yes
Xu et al.'s method [38] based on the q-RDHFWHM operator	Yes	Yes	No
The proposed method based on the q-RDHULWMM operator	Yes	Yes	Yes

6. Conclusions

This paper studies new method for MADM problems. We firstly introduced the q-RDHFULSs, which are effective to denote DMs' evaluations qualitatively and quantificationally. Moreover, they can efficiently and comprehensively capture DMs' high hesitancy in providing their evaluation values. With respect to MADM issues wherein attribute values are in the form of q-RDHFULVs, we introduced some AOs to effectively integrate DMs' evaluation values. Based on these AOs, a novel approach to q-rung dual hesitant uncertain linguistic MADM problems are illustrated stepwise. We utilized the proposed method to solve an enterprise informatization level evaluation instance to show its validity and effectiveness.

We also detailedly discussed and analyzed the advantages and superiorities of our proposed method over some existing MADM methods through comparative analysis. Given the good performance of the proposed MADM method, in further works we will study the applications of the method in some other practical MADM problems, such as low carbon supplier selection, medical diagnosis, the best airline selection, etc.

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