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On the region of attraction of dynamical systems: Application to Lorenz equations

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Many nonlinear dynamical systems can present a challenge for the stability analysis in particular the estimation of the region of attraction of an equilibrium point. The usual method is based on Lyapunov techniques. For the validity of the analysis it should be supposed that the initial conditions lie in the domain of attraction. In this paper, we investigate such problem for a class of dynamical systems where the origin is not necessarily an equilibrium point. In this case, a small compact neighborhood of the origin can be estimated as an attractor for the system. We give a method to estimate the basin of attraction based on the construction of a suitable Lyapunov function. Furthermore, an application to Lorenz system is given to verify the effectiveness of the proposed method.

Key words: nonlinear dynamical systems, Lyapunov function, Basin of attraction, Lorenz equations

1. Introduction

Given an ordinary differential equation, the problem of Lyapunov stability has attracted the attention of serval authors and has produced a large important results (see [1–3, 7–15]). The two major approaches for Lyapunov stability analysis [12, 17] are the linearization method and the direct method. Stability of a system can be investigated via the first linearization method, but in general and the most powerful technique is the second direct method. For this method one usually assumes the existence of the so called Lyapunov function which is a positive definite function with negative derivative along the trajectories of the system motivated by some earlier works (see [4, 9, 16, 18–20]). Another important problem is to estimate the region of attraction around the equilibrium, that is, the problem of finding a set which contains the origin such that the limit of every trajectory starting inside is the equilibrium point. Usually this problem is attacked by using a Lyapunov

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surface as an estimate for the region of attraction. Noting that the asymptotic stability is more important than stability, also the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable, thus the notion of ultimate boundedness of solutions is more suitable in several situations (see [5,6,10]). Frequently, chaos in many systems is a source of instability and a source of the generation of oscillation where chaotic systems commonly exist in various fields of application [21]. Quite often, one also desires that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner. In this paper, we study the problem of finding a suitable Lyapunov function to estimate the region of attraction for differential equations which don't admit necessarily the origin as an equilibrium point. We provide a result which estimates the domain of attractivity using Lyapunov's techniques. This yields us to study some classes of Lorenz equations where some numerical examples are given to show the effectiveness of the main result.

2. Problem formulation

Let consider the following differential equation: $\dot{x} = F(x)$. Unless otherwise stated, we assume throughout the paper that the function $F(.)$ encountered is sufficiently smooth. We often omit arguments of function to simplify notation, $\mathbb{R}^n$ is the n-dimensional Euclidean vector space; $\mathbb{R}^+$ is the set of all non-negative real numbers; $\|x\|$ is the Euclidean norm of a vector $x \in \mathbb{R}^n$. $B_r = \{x \in \mathbb{R}^n / \|x\| \leqslant r,\ r > 0\}$ denotes the ball centered at the origin and of radius $r > 0$. For all $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{R}^+$, we will denote by $x(t, t_0, x_0)$, or simply by $x(t)$, the unique solution at time t_0 starting from the point x_0. In this situation, since the considered equation is time-invariant, we can take $t_0 = 0$. We recall now some standard comparison functions which are used in stability theory to characterize the stability properties and uniform asymptotic stability (see [11, 16]): $\mathcal{K}$ is the class of functions $\mathbb{R}^+ \to \mathbb{R}^+$ which are zero at the origin, strictly increasing and continuous. $\mathcal{K}_\infty$ is the subset of $\mathcal{K}$ functions that are unbounded. $\mathcal{L}$ is the set of functions $\mathbb{R}^+ \to \mathbb{R}^+$ which are continuous, decreasing and converging to zero as their argument tends to $+\infty$. $\mathcal{KL}$ is the class of functions $\mathbb{R}^+ \to \mathbb{R}^+$ which are class $\mathcal{K}$ on the first argument and class $\mathcal{L}$ on the second argument.

2.1. Basin of attraction

We consider the following system:

$$\dot{x} = F(x), \tag{1}$$

where $x \in \mathbb{R}^n$ and $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous and locally Lipschitz, we denote by $x(t) = F_t(x)$, $t \geqslant 0$, the solution of (1), which starts from x at $t = 0$, it means that $F_0(x) = x$.

Definition 1 *A set M is called positively invariant set of (1) if any solution $x(t)$ that belongs to M at some time $t_0 > 0$ must belongs to it for all future time $(t \geqslant t_0 \geqslant 0)$;*

$$x(t_0) \in M \quad \Rightarrow \quad x(t) \in M, \ \ \forall t \geqslant t_0 \geqslant 0.$$

Lyapunov's direct method allows us to determine the stability of a system without explicitly integrating the differential equation. We would like to estimate the region of attraction of the system around the equilibrium, that is, we want to find a set $\mathcal{R}$, such that for every trajectory that starts in $\mathcal{R}$, the limit is the equilibrium point. We call such a set $\mathcal{R}$ an attraction region. The existence of such region is ensured by using the following geometrical statement. Let $\mathcal{R}$ the domain of attraction of (1), if the system (1) is asymptotically stable, then $\mathcal{R}$ is an non empty, invariant, open, and it's a connected set (see [14]). To approximate this region, we shall find a suitable function $V\colon \mathcal{R} \to \mathbb{R}$ continuously differentiable, such that $V(0) = 0$, $V(x) > 0$, $\forall x \in \mathcal{R} \setminus \{0\}$ and $\dot{V}(x) < 0$, $\forall x \in \mathcal{R} \setminus \{0\}$.

Now, in order to estimate the region of attraction in the case when the origin is not necessarily an equilibrium point, we shall introduce the following stability result of a small ball. Let $B_r = \{x \in \mathbb{R}^n \ / \ \ \| \ x \ \| \leqslant r, \ r > 0\}$ denotes the ball centered at the origin and of radius $r > 0$. The classic Lyapunov result for a nonlinear non-autonomous system can found in [1, 3, 13].

Definition 2 *The ball B_r is said to be globally uniformly asymptotically stable for* (1)*, if there exists a class $\mathcal{KL}$ function β such that the solution of* (1)*, from any initial state $x_0 \in \mathbb{R}^n$ and initial time $t_0 \in \mathbb{R}^+$ satisfies the following estimation:*

$$\|x(t)\| \leqslant \beta\left(\|x_0\|, t - t_0\right) + r, \ \ \textit{for all} \ \ t \geqslant t_0 \geqslant 0. \tag{2}$$

Note that, in the above definition, if we take $r = 0$, then one deals with the standard concept of the global asymptotic stability of the origin viewed as an equilibrium point. Moreover, we shall study the asymptotic behavior of a small ball centered at the origin for $0 \leqslant \|x(t)\| - r$, $\forall t \geqslant t_0 \geqslant 0$ so that the initial conditions are taken outside the ball B_r.

The following theorem, (see [1]) gives a result on asymptotic behavior of solutions when the origin is not necessarily an equilibrium point.

Theorem 1 *Consider system* (1) *and suppose that there exist a continuously differentiable function $V\colon \mathbb{R}^n \to \mathbb{R}$, $\mathcal{K}_\infty$ functions $\alpha_1(.)$, $\alpha_2(.)$, a class $\mathcal{K}$ $\alpha_3(.)$ and a small positive real number ϱ such that the following inequalities hold for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$*

$$\alpha_1(\|x\|) \leqslant V(x) \leqslant \alpha_2(\|x\|),$$

$$DV_x F(x) \leqslant -\alpha_3(\|x\|) + \varrho.$$

Then, the ball B_r is globally asymptotically stable with $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)$,

$$B_r = \{x \in \mathbb{R}^n / \|x\| \leqslant \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)\}.$$

In this case all solutions of (1) satisfy an inequality as in (2) with $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)$. Remark that, if there exists an open connected set $D_r \subset \mathbb{R}^n$ containing the origin such that $B_r \subset D_r$ and all the conditions of the above theorem hold for all $x \in \mathcal{D}_r \subset \mathbb{R}^n$, then the region of attraction will be $\mathcal{D}_r \setminus B_r$. Therefore, if B_r is asymptotically stable, then $\forall x \in D_r \setminus B_r$:

$$\lim_{t \to +\infty} d(F_t(x), B_r) = 0.$$

The region of attraction of the ball B_r, $r > 0$, denoted by $\mathcal{R}_r$, in this situation $\mathcal{R}_r = \mathcal{D}_r \setminus B_r$, is defined as the set of all points $x \in \mathcal{R}_r$, such that

$$\forall\, x \in \mathcal{R}_r\,, \quad \lim_{t \to +\infty} d(F_t(x), B_r) = 0,$$

which is defined by the property that every trajectory of the corresponding system starting from $x \in \mathcal{R}_r$ reaches the ball. Therefore, $\mathcal{R}_r$ is defined as:

$$\mathcal{R}_r = \{x \in \mathbb{R}^n \setminus B_r \ / \ \forall\, \varepsilon > 0, \ \exists T > 0, \ \forall\, t \geqslant T, \ d(F_t(x), B_r) < \varepsilon\}.$$

Here, we deal with the situation that every trajectory approaches the ball B_r without crossing the sphere $\mathcal{S}_r = \{x \in \mathbb{R}^n / \|x\| = r, \ r > 0\}$. We have the following properties concerning the last domain.

Proposition 1 *If B_r is asymptotically stable with respect the system (1), then $\mathcal{R}_r$ is a non empty, invariant, open and it is a connected set.*

Proof. Since B_r is asymptotically stable, then it is attractive and then,

$$\exists\, \rho > r, \ \ \forall\, x \in B_\rho \setminus B_r, \ \ \forall\, \varepsilon > 0, \ \exists\, T > 0, \ \forall\, t \geqslant T, \ \ d(F_t(x), B_r) < \varepsilon,$$

this implies that $x \in \mathcal{R}_r$ which gives that $\mathcal{R}_r \neq \emptyset$.

To show that $\mathcal{R}_r$ is invariant, we shall prove that,

$$x \in \mathcal{R}_r \ \Rightarrow \ F_s(x) \in \mathcal{R}_r, \ \ \forall s \geqslant 0.$$

Let $x \in \mathcal{R}_r$. For $s \in \mathbb{R}_+$, we have: $\forall\, \varepsilon > 0, \ \exists T > 0, \ \forall t \geqslant T, \ \ d(F_t(x), B_r) < \varepsilon$. Since $d(F_t(F_s(x)), B_r) = d(F_{t+s}(x), B_r)$ and $t + s \geqslant T$, then $d(F_{t+s}(x), B_r) < \varepsilon$. This implies that

$$F_s(x) \in \mathcal{R}_r, \quad \forall s \geqslant 0.$$

To show that $\mathcal{R}_r$ is open, we shall prove that for any point $x \in \mathcal{R}_r$, every point in a neighborhood of x belongs to $\mathcal{R}_r$. Let $x \in \mathcal{R}_r$. Since B_r is attractive, then there exists $\rho > r$ such that $\forall z \in \mathcal{B}_\rho \setminus B_r$, $\lim_{t \to +\infty} d(F_t(z), B_r) = 0$. Thus, for

$$\eta = r + \theta(\rho - r) > 0, \quad \theta \in]0, 1[,$$

there exists $T_1 > 0$, such that

$$\forall t \geqslant T_1, \quad d(F_t(x), B_r) < \frac{1}{2}\eta.$$

If $\mathcal{R}_r = B_\rho$ then $\mathcal{R}_r$ will be open. We take y outside B_ρ, y close to x, $y \in \mathcal{V}(x)$ a neighborhood of x such that by using the continuity of the solutions we get:

$$d(F_t(y), F_t(x)) < \frac{1}{2}\eta, \quad \text{for all } \ t \geqslant T_2,$$

for a certain $T_2 > 0$. Applying the triangular inequality, for all $t \geqslant \sup(T_1, T_2)$,

$$d(F_t(y), B_r) \leqslant d(F_t(y), F_t(x)) + d(F_t(x), B_r) < \eta.$$

This implies that, $F_t(y) \in \mathcal{B}_\rho \setminus B_r$ and hence, the solution starting at y approaches B_r as $t \to +\infty$. Thus, $y \in \mathcal{R}_r$, this implies that $\mathcal{V}(x) \subset \mathcal{R}_r$ and the set $\mathcal{R}_r$ is an open set. Finally, by invariance, $\mathcal{R}_r$ can not be the union of two open non-empty disjoint sets. This can be proven by using a contradiction. □

Let us now propose the main problem we are concerned with: Find a function V such that $\mathcal{R}_r$ will be the domain of attraction of (1). Let $V\colon \mathcal{R}_r \to \mathbb{R}$ be a continuously differentiable function, then we have $V(0) = 0$, $V(x) > 0$, $\forall\, x \in \mathcal{R}_r$ and $\dot{V}(x) < 0, \ \forall x \in \mathcal{R}_r$.

The problem of finding a Lyapunov function of a given non-linear ordinary differential equation that captures the non-linear behavior of F around a small ball centered at the origin is highly non-trivial and usually relies on engineering intuition. Note that the above statement is non-constructive, that is, it only ensures the existence of an attraction region, but does not provide it. In the sequel, we consider the constructive version of the theorem to estimate the region of attraction, that is, the problem of finding such an attraction region. The set Ω_c is given by

$$\Omega_c = \{x \in \mathbb{R}^n \setminus B_r \quad / \quad V(x) \leqslant c\}$$

with $c > 0$, which provides an interesting method to estimate the domain of attraction via an invariant set. A closed set $\Omega_c \subset \mathcal{R}_r$ is an estimation of region of attraction of an asymptotic stable ball centered at the origin B_r, $\ r > 0$, if there exists a Lyapunov function $V(x)$ such that $V(x)$ is positive definite on Ω_c, V is proper ($\lim_{\|x\| \to +\infty} = +\infty$) and $\dot{V}(x)$ is negative definite on Ω_c.

For $r > 0$, the problem is to find a suitable Lyapunov function $V(.)$ and a positive c, such that Ω_c is an invariant set which is contained inside R_r and the estimate of the domain of attractivity is maximized by solving the following two problems:

i) finding $\max c > 0$ such that, $\Omega_c \subset \mathcal{R}_r$.

ii) Ω_c is an invariant set, such that,

- $V(x) > 0, \quad \forall\, x \in \Omega_c$,
- $\dot{V}(x) < 0, \quad \forall\, x \in \Omega_c$.

Lyapunov function allows to gain some information about the global behavior of orbits. the problem is to expect that if V is strict in Ω_c for $c > 0$, then $\Omega_c \subset R_r$. The set Ω_c can be any compact positively invariant set. So we have to find three things: First, find where V is positive. Second, find where $\dot{V}$ is strictly negative and third find the largest level curve $V(x) = c$ of V such that Ω_c is inside the region when $\dot{V} < 0$.

Remark that when the system is globally asymptotically stable with respect B_r, the region of attraction is the region of the space $\mathbb{R}^n \setminus B_r$. So, local asymptotic stability of the origin can be studied when global Lyapunov function may not be possible to find. Of course global asymptotic stability is very desirable, but in many applications it is difficult to achieve. We are therefore interested in determining how far from a small ball the trajectory can be and still converge to it. In the next section, we will give an application to Lorenz equations in order to obtain a more accurate estimate of the basin of attraction which is contained inside a region bounded by a certain surface.

3. Lorenz system

The Lorenz system has played a fundamental role in the area of nonlinear science and chaotic dynamics. Although everyone believes the existence of the Lorenz attractor, no rigorous mathematical proof has been given so far. This problem has been listed as one of the fundamental mathematical problems. This problem is extensively discussed with the aid of numerical computation. It has been realized that it is extremely difficult to obtain the information of the chaotic attractor directly from the differential equation (3). Most of the results in the literature are computer simulations even based on computation of Lyapunov exponents of the system, one needs to assume the system being bounded in order to conclude that the system is chaotic. Therefore, the study of the globally attractive set of the Lorenz system is not only theoretically significant, but also practically important. The Lorenz equations is one of the most famous models of nonlinear dynamics, which is a nonlinear system that evolves in $\mathbb{R}^3$ whose equations are given by:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = cx - xz - y \\ \dot{z} = xy - bz \end{cases} \tag{3}$$

where the parameters a, b and c are assumed positive real numbers. We will first of all specify the physical origin of this system of equations. Lorenz arrived at these equations when modeling a two-dimensional fluid cell between two parallel plates which are at different temperatures. To simplify the problem, he expanded the unknown functions into Fourier series with respect to the spacial coordinates and set all coefficients except for three equal to zero: the variable x, which equals the convective flow; y, which equals the horizontal temperature distribution; and z, which equals the vertical temperature distribution. The resulting equation is (3), where a represents the ratio of fluid viscosity to thermal conductivity, c represents the difference in temperature between the top and bottom of the system and b is an aspect ratio.

3.1. Fixed points

For the remainder of this paper, the dot notation will be used to denote the derivative with respect to time, the system is then written as

$$\begin{cases} \dot{x} = a(y-x) \\ \dot{y} = cx - xz - y \\ \dot{z} = xy - bz \end{cases} \qquad \text{if } X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} a(y-x) \\ cx - xz - y \\ xy - bz \end{pmatrix}$$

in vector form the system becomes

$$\dot{X} = F(X).$$

The fixed points are given by $F(X) = 0$, solving this system reveals that the fixed points are:

$$\begin{cases} a(y-x) = 0 \\ cx - xz - y = 0 \\ xy - bz = 0 \end{cases} \iff \begin{cases} x = y \\ x(c-1-z) = 0 \\ x^2 = bz \end{cases}.$$

If

$$x = 0$$

then

$$p(0,0,0)$$

is a stationary point.

If

$$z = c-1, \quad x = y = \pm\sqrt{b(c-1)}$$

then

$$q_+ = \left(\sqrt{b(c-1)},\ \sqrt{b(c-1)},\ c-1\right),$$

$$q_- = \left(-\sqrt{b(c-1)},\ -\sqrt{b(c-1)},\ c-1\right)$$

are stationary points (for $c > 1$).

3.2. Stability analysis

The Jacobian matrix of the Lorenz system at critical point $p(0,0,0)$ is given by:

$$J_0 = \begin{bmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

with characteristic polynomial of the form:

$$\lambda^3 + (a+1+b)\lambda^2 + (b(a+1)a(1-c))\lambda + ab(1-c) = 0.$$

Then,

$$\begin{aligned} \lambda_1 &= -b, \\ \lambda_2 &= \frac{1}{2}\left(-1-a+\sqrt{4ac+a^2-2a+1}\right), \\ \lambda_3 &= \frac{1}{2}\left(-1-a-\sqrt{4ac+a^2-2a+1}\right). \end{aligned}$$

In the neighborhood of the origin:

- For $c < 1$; $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 < 0$ then, the system (3) is asymptotically stable.
- For $c = 1$; $\lambda_1 < 0$, $\lambda_2 = 0$, $\lambda_3 < 0$ then, we have a critical case.
- For $c > 1$; $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_3 < 0$ then, the system (3) is unstable.

Theorem 2 *The zero solution of the Lorenz system has the following cases:*

i) If $c < 1$, system (3) is asymptotically stable on the hole space.

ii) If $c = 1$, there exists $r_1 > 0$ such that system (3) is globally asymptotically stable on $\mathcal{R}_{r_1} = \mathbb{R}^3 \setminus B_{r_1}$.

iii) If $c > 1$, there exists $r_2 > 0$ such that system (3) is globally asymptotically stable on $\mathcal{R}_{r_2} = \mathbb{R}^3 \setminus B_{r_2}$.

Proof.
Case i) for $c < 1$, let consider the quadratic form defined by:

$$V(x, y, z) = \frac{1}{2} X^T P X = \frac{1}{2} \left(c x^2 + a y^2 + a z^2 \right),$$

where

$$P = \begin{bmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

is a real symmetric positive definite matrix. $V(x, y, z) > 0$ for all $x \in \mathbb{R}^3 \setminus \{0\}$. $V(x, y, z) \to \infty$ as $\|x\| \to \infty$ and

$$\frac{1}{2} \lambda_{\min}(P) \|(x, y, z)\|^2 \leqslant V(x, y, z) \leqslant \frac{1}{2} \lambda_{\max}(P) \|(x, y, z)\|^2 .$$

The derivative of V along the trajectories of the system (3) is given by:

$$\begin{aligned} \dot{V}(x, y, z) &= \begin{bmatrix} cx & ay & az \end{bmatrix} \begin{bmatrix} a(y - x) \\ cx - xz - y \\ xy - bz \end{bmatrix} \\ &= acxy - acx^2 + acxy - ay^2 - axyz + axyz - abz^2 \\ &= -acx^2 - ay^2 - abz^2 + 2acxy \\ &= -a(x\ y\ z)^T \begin{bmatrix} c & -c & 0 \\ -c & 1 & 0 \\ 0 & 0 & b \end{bmatrix} (x\ y\ z) \\ &= -a X^T B X . \end{aligned}$$

B is a real symmetric positive definite matrix ($\Delta_1 = c > 0$, $\Delta_2 = c - c^2 > 0$, $\Delta_3 = b(c - c^2) > 0$). Then, $\dot{V}(x, y, z) < 0$ for all $(x, y, z) \in \mathbb{R}^3 \setminus (0, 0, 0)$. We conclude that the origin is globally asymptotically stable with $\mathcal{R}_0 = \mathbb{R}^3 \setminus (0, 0, 0)$ is the domain of attraction.
Case ii) for $c = 1$, let consider the Lyapunov function defined by:

$$V(x, y, z) = \frac{1}{2} X^T P X = \frac{1}{2} (x^2 + a y^2 + a(z - 1)^2).$$

It's clear that V is a positive definite function, $V(x, y, z) \to \infty$ as $\|x\| \to \infty$. One has

$$\frac{1}{2} \lambda_{\min}(P) \|(x, y, z)\|^2 \leqslant V(x, y, z) \leqslant \frac{1}{2} \lambda_{\max}(P) \|(x, y, z)\|^2 .$$

The derivative of V along the trajectories of the system (3) is given by:

$$\begin{aligned}\dot{V}(x,y,z) &= \begin{bmatrix} x & ay & a(z-1)\end{bmatrix}\begin{bmatrix} a(y-x) \\ x-xz-y \\ xy-bz\end{bmatrix}\\ &= axy - ax^2 + axy - ay^2 - axyz + axyz - abz^2 - axy + abz\\ &= -ax^2 - ay^2 - abz^2 + axy + abz\\ &= -a(x\ y)^T \begin{bmatrix} 1 & -\dfrac{1}{2} \\ -\dfrac{1}{2} & 1\end{bmatrix}(x\ y) - abz^2 + abz.\end{aligned}$$

We have,

$$\widetilde{P}(x,y) = \begin{bmatrix} 1 & -\dfrac{1}{2} \\ -\dfrac{1}{2} & 1\end{bmatrix} \leqslant \lambda_{\max}(\widetilde{P})\|(x,y)\|^2 = \frac{3}{2}\|(x,y)\|^2.$$

Introduce the function: $f(z) = -abz^2 + abz$. We have $f'(z) = -2abz + ab$. When $f'(z) = 0$, this yields $z_0 = \dfrac{1}{2}$, then $f''(z_0) = -2ab < 0$. Thus $\sup_{z\in\mathbb{R}} f(z) = f(z_0) = \dfrac{ab}{4}$. So, we obtain

$$\begin{aligned}\dot{V}(X) &= -a(x\ y)^t \begin{bmatrix} c & -\dfrac{-1}{2} \\ -\dfrac{-1}{2} & 1\end{bmatrix}(x\ y) + f(z)\\ &\leqslant -a\frac{1}{2}\left(\sqrt{2}\sqrt{c^2+1} + c + 1\right)\|X\|^2 + f(z_0)\\ &= -a\frac{1}{2}\left(\sqrt{2}\sqrt{c^2+1} + c + 1\right)\|X\|^2 + \frac{ab}{4}.\end{aligned}$$

Thus, we obtain the following estimation:

$$\dot{V}(X) \leqslant -\alpha_3(\|X\|) + \rho,$$

where

$$\alpha_3(\|X\|) = a\frac{1}{2}\left(\sqrt{2}\sqrt{c^2+1} + c + 1\right)\|X\|^2.$$

Now if we take,

$$\alpha_1(\|X\|) = \frac{1}{2}\lambda_{\min}(P)\|X\|^2 \quad \text{and} \quad \alpha_2(\|X\|) = \frac{1}{2}\lambda_{\max}(P)\|X\|^2,$$

then by Theorem 1, it follows that B_{r_1} is globally asymptotically stable with $r_1 = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\rho)$, $\rho = \dfrac{ab}{4} > 0$ and $\mathcal{R}_{r_1} = \mathbb{R}^3 \setminus B_{r_1}$ estimates the region of attraction.

Case iii) when $c > 1$. In the neighborhood of the origin the system (3) is unstable. In this case, we need to determine a set and a real number r satisfying the conditions of theorem (1) to characterize the asymptotic behavior of the solutions. Let us define the following Lyapunov function candidate for the system (3):

$$V(x, y, z) = \frac{1}{2}\left(cx^2 + ay^2 + a(z - c + 1)^2\right).$$

It is clear that V is positive definite on $\mathbb{R}^3$. Therefore, the derivative of V along the trajectories of the system (3) is given by:

$$\begin{aligned}
\dot{V}(x, y, z) &= \begin{bmatrix} cx & ay & a(z - c + 1) \end{bmatrix} \begin{bmatrix} a(y - x) \\ cx - xz - y \\ xy - bz \end{bmatrix} \\
&= acxy - acx^2 + acxy - ay^2 - axyz + axyz - abz^2 \\
&\quad - acxy + abcz + axy - abz \\
&= -acx^2 - ay^2 - abz^2 + acxy + abcz + axy - abz \\
&= -acx^2 - ay^2 - abz^2 + acxy + axy + abcz - abz \\
&= -a(x\ y\ z)^t \begin{bmatrix} c & -\frac{1}{2}(c{+}1) & 0 \\ -\frac{1}{2}(c{+}1) & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} (x\ y\ z) - abz^2 + (abc - ab)z\,.
\end{aligned}$$

Let

$$\hat{P} = \begin{bmatrix} c & -\frac{1}{2}(c + 1) & 0 \\ -\frac{1}{2}(c + 1) & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then,

$$\hat{P}(x, y) \leqslant \lambda_{\max}(\hat{P})\|(x, y)\|^2 = \frac{1}{2}(\sqrt{2}\sqrt{c^2 + 1} + c + 1)\|(x, y)\|^2.$$

Introduce the function:

$$f(z) = -abz^2 + (abc - ab)z.$$

We have

$$f'(z) = -2abz + abc - ab.$$

When $f'(z) = 0$, this yields $z_0 = \dfrac{abc - ab}{2ab}$. Then, $f''(z_0) = -2ab < 0$. Thus,

$$\sup_{z \in \mathbb{R}} f(z) = f(z_0) = \frac{ab(c-1)}{4}.$$

It follows that,

$$\begin{aligned}
\dot{V}(X) &= -a(x\ y)^t \begin{bmatrix} c & -\frac{1}{2}(c+1) \\ -\frac{1}{2}(c+1) & 1 \end{bmatrix} (x\ y) + f(z) \\
&\leqslant -a\frac{1}{2}\left(\sqrt{2}\sqrt{c^2+1} + c + 1\right)\|X\|^2 + f(z_0) \\
&= -a\frac{1}{2}\left(\sqrt{2}\sqrt{c^2+1} + c + 1\right)\|X\|^2 + \frac{ab(c-1)}{4}.
\end{aligned}$$

Thus, we obtain the following estimation: $\dot{V}(X) \leqslant -\alpha_3(\|X\|) + \rho$, with $\alpha_3(\|X\|) = a\frac{1}{2}\left(\sqrt{2}\sqrt{c^2+1} + c + 1\right)\|X\|^2$. Hence, B_{r_2} is globally uniformly asymptotically stable, with

$$r_2 = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\rho), \qquad \rho = \frac{ab(c-1)}{4} > 0$$

and the set $\mathcal{R}_{r_2} = \mathbb{R}^3 \setminus B_{r_2}$ estimates the region of attraction. □

3.3. Illustrative example

Let consider the following special cases:

i) $\begin{cases} \dot{x} = 12(y-x) \\ \dot{y} = 0.3x - xz - y \\ \dot{z} = xy - 8z \end{cases}$ **ii)** $\begin{cases} \dot{x} = 6(y-x) \\ \dot{y} = x - xz - y \\ \dot{z} = xy - 0.5z \end{cases}$

iii) $\begin{cases} \dot{x} = 5(y-x) \\ \dot{y} = 17x - xz - y \\ \dot{z} = xy - 3z \end{cases}.$

For each case, one has the following simulation – see figures.

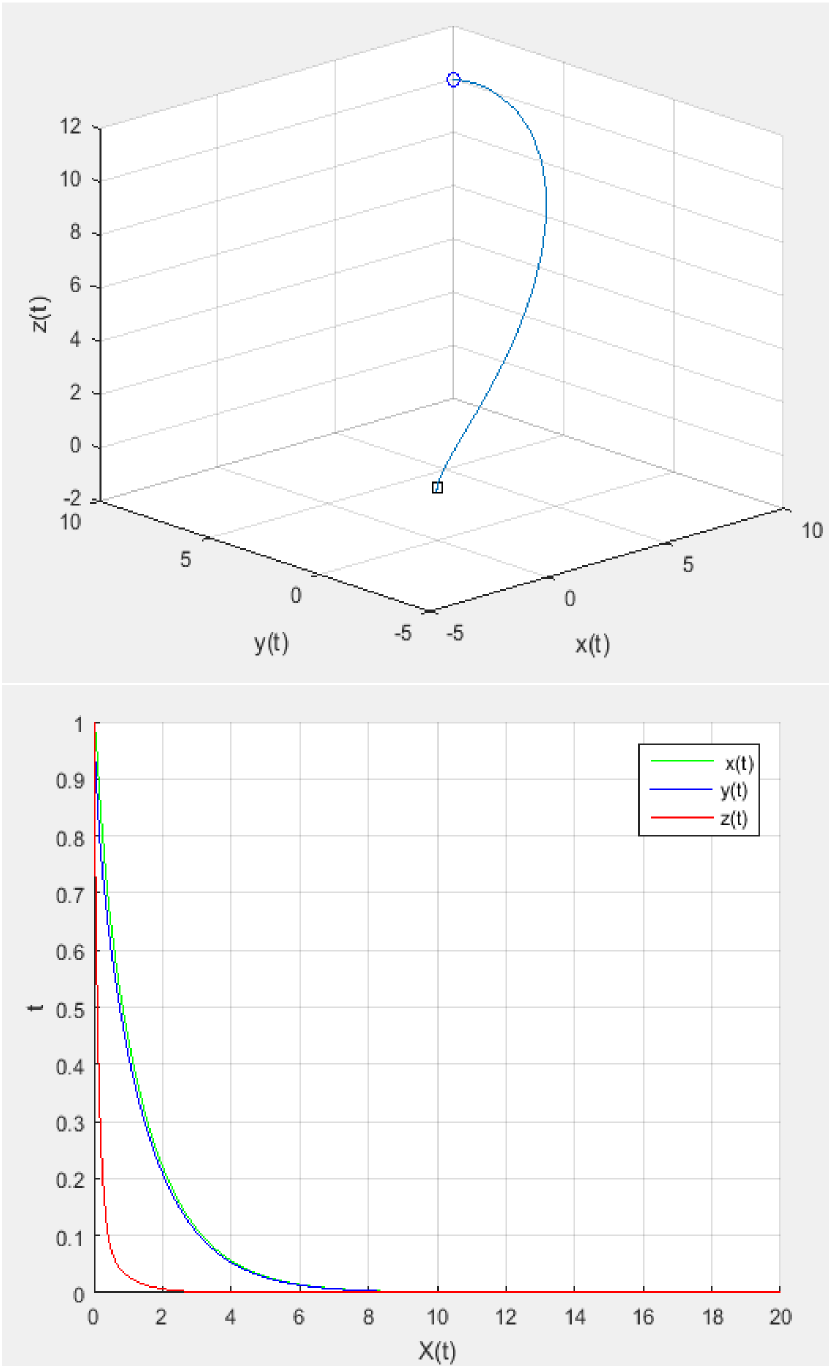


Figure 1: $a = 12, \ b = 8, \ c = 0.3$

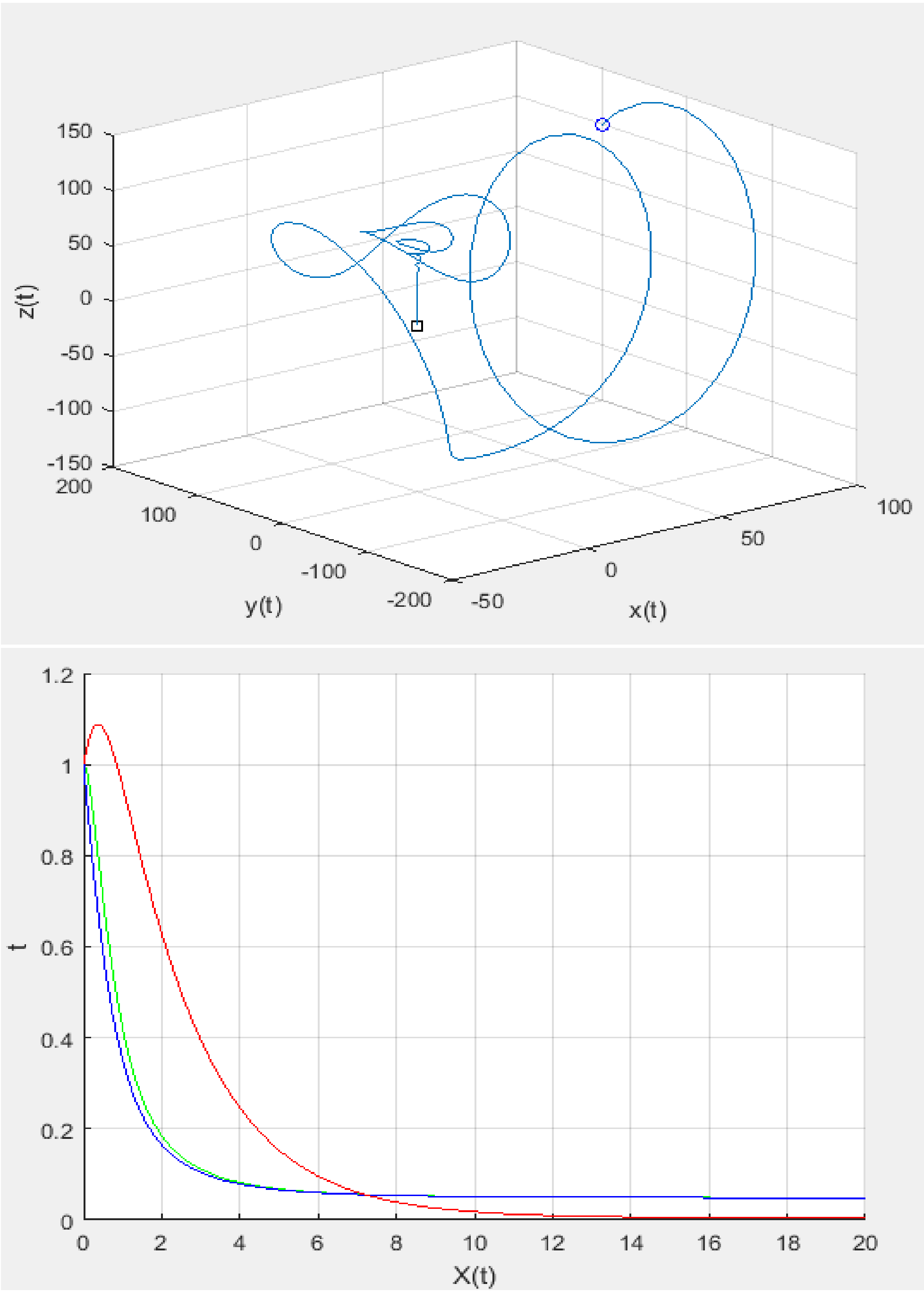
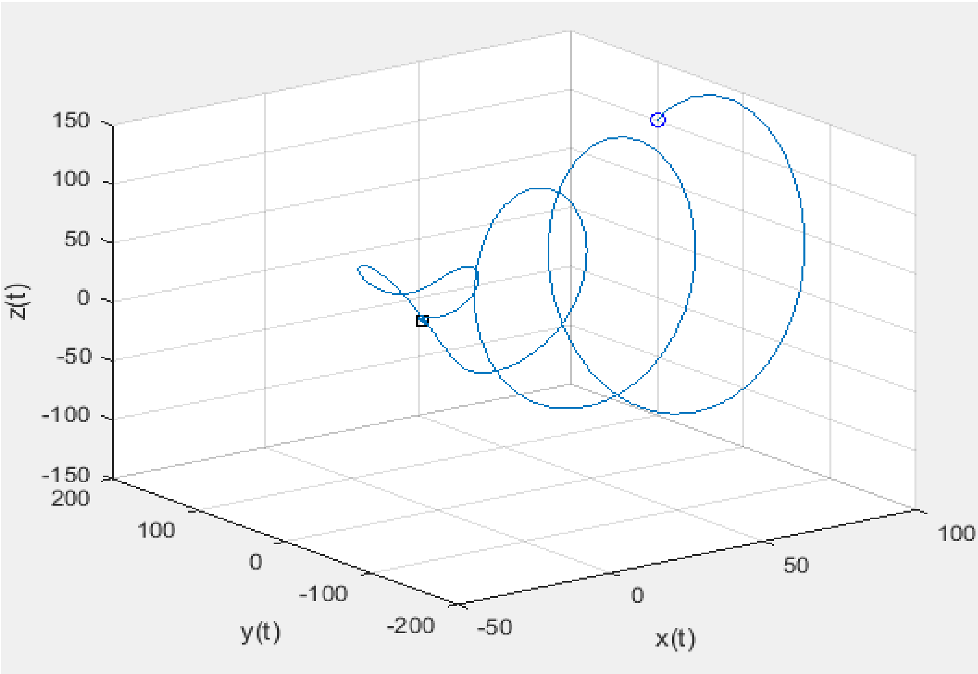


Figure 2: $a = 6,\ b = 0.5,\ c = 1$



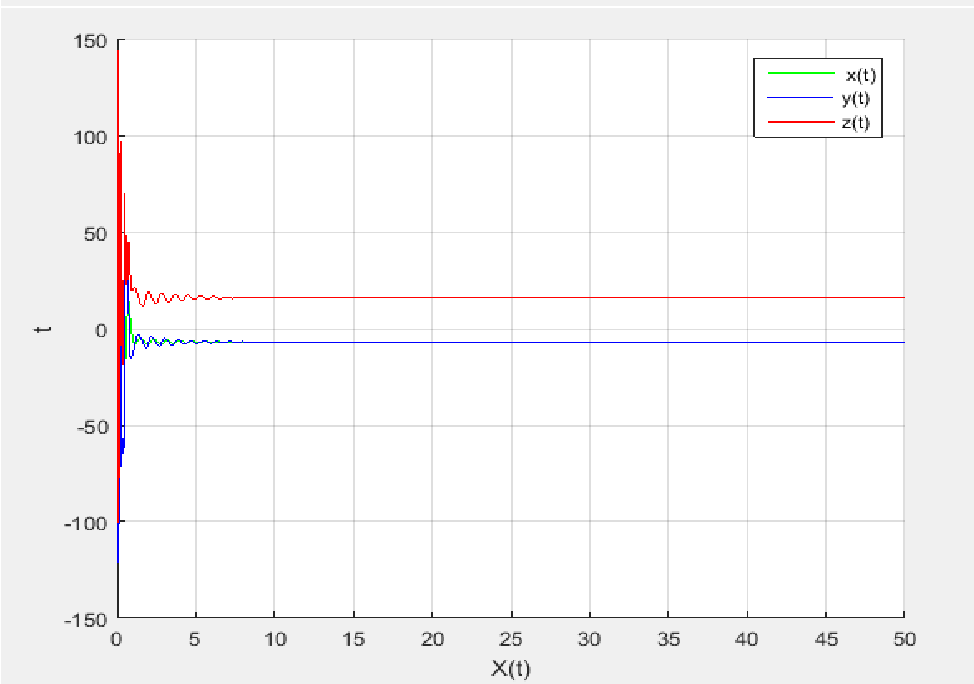


Figure 3: $a = 5,\ b = 3,\ c = 17 \quad a = 6,\ b = 0.5,\ c = 1$

3.4. A special case

We are often interested in determining how far from the origin the trajectory can be and still converge to the origin as t approaches to infinity. Note that, finding the exact region of attraction analytically might be difficult. However, Lyapunov functions can be used to estimate the region of attraction, that is, to find the optimal set contained in the region of attraction. We want to particularly study Lorenz system (1) for $a = 10$, $b = 8/3$, $c = 28$. Therefore the new Lorenz system is given by:

$$\begin{cases} \dot{x} = 10(y - x) \\ \dot{y} = 28x - xz - y \\ \dot{z} = xy - \dfrac{8}{3}z \end{cases} . \tag{4}$$

The stationary point are given by $p(0, 0, 0)$, $q_{\pm} = (\pm 6\sqrt{2}, \pm 6\sqrt{2}, 27)$.

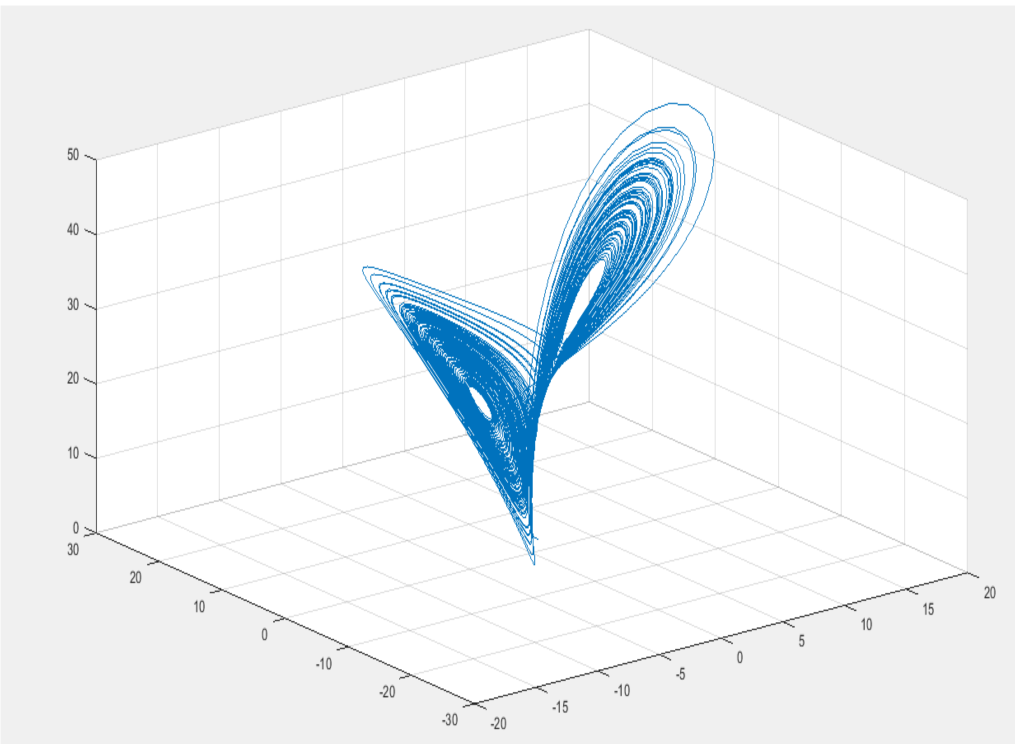


Figure 4: $a = 10$, $b = 8/3$, $c = 28$

Remark that, the line making up the curve never intersected itself and never retraced its own path. Instead, it looped around forever and ever, sometimes spending time on one wing before switching to the other side. The main global property that we shall address in the sequel is the property of being able to guarantee that solutions of the system with sufficiently close initial values remain close to each other over indefinite amounts of time in the future. It can also be viewed as a result about the long term behavior of solutions in the sense that the trajectories are bounded in a certain region of the space.

Let consider the linearization of the system around the origin.

$$A = \frac{\partial f}{\partial X}|_{X=0} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}$$

We can find a Lyapunov function for system (4) by solving the Lyapunov equation

$$PA + A^T P = -Q \quad \text{for} \quad Q = I.$$

The unique solution is the positive definite matrix:

$$P = \begin{bmatrix} 169/5940 & -32/1485 & 0 \\ -32/1485 & -307/2970 & 0 \\ 0 & 0 & 3/16 \end{bmatrix} \simeq \begin{bmatrix} 0.028 & -0.021 & 0 \\ -0.021 & 0.103 & 0 \\ 0 & 0 & 0.187 \end{bmatrix}$$

Now, we need to determine a domain D about the origin where $\dot{V}(x)$ is negative definite and a bounded set $\Omega_c \subset D$.

$$V(X) = X^T PX = 0.028x^2 + 0.103y^2 + 0.187z^2 - 0.42xy\,.$$

We calculate its derivative with respect the time along the system trajectories.

$$\begin{aligned}\dot{V}(X) &= (0.056x - 0.042y)\dot{x} + (0.206y - 0.042x)\dot{y} + 0.374z\dot{z} \\ &= -0.56x^2 - 1.176x^2 - 0.206y^2 - 0.42y^2 - 0.99z^2 + 0.042x^2z \\ &\quad + 6.328xy + 0.42xy + 0.042xy + 0.168xzy \\ &= -1.736x^2 - 0.626y^2 - 0.99z^2 + 0.042x^2z + 6.79xy + 0.168xzy\,.\end{aligned}$$

Since we can say nothing about the sign of the derivative we will search for it with polar coordinates. Taking

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

with

$$\begin{cases} \rho = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arctan\left(\dfrac{y}{x}\right) \\ \varphi = \arccos\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \end{cases}.$$

We have

$$\begin{aligned} \dot{V} &= -1.736(\rho\sin\varphi\cos\theta)^2 - 0.626(\rho\sin\varphi\sin\theta)^2 - 0.99(\rho\cos\varphi)^2 \\ &\quad + 0.042(\rho\sin\varphi\cos\theta)^2(\rho\cos\varphi) + 6.79(\rho\sin\varphi\cos\theta)(\rho\sin\varphi\sin\theta) \\ &\quad + 0.168(\rho\sin\varphi\cos\theta)(\rho\sin\varphi\sin\theta)(\rho\cos\varphi) \\ &= -1.736\rho^2\sin^2\varphi\cos^2\theta - 0.626\rho^2\sin^2\varphi\sin^2\theta - 0.99\rho^2\cos^2\varphi \\ &\quad + 0.042\rho^3\sin^2\varphi\cos^2\theta\cos\varphi + 6.79\rho^2\sin^2\varphi\cos\theta\sin\theta \\ &\quad + 0.168\rho^3\sin^2\varphi\cos\theta\sin\theta\cos\varphi \\ &\leqslant -1.736\rho^2|\sin^2\varphi\cos^2\theta| - 0.626\rho^2|\sin^2\varphi\sin^2\theta| - 0.99\rho^2|\cos^2\varphi| \\ &\quad + 0.042\rho^3|\sin^2\varphi\cos^2\theta\cos\varphi| + 6.79\rho^2|\sin^2\varphi\cos\theta\sin\theta| \\ &\quad + 0.168\rho^3|\sin^2\varphi\cos\theta\sin\theta\cos\varphi| \\ &\leqslant -1.736\rho^2 - 0.626\rho^2 - 0.99\rho^2 + 0.007\rho^3 + 2.17\rho^2 + 0.03\rho^3. \end{aligned}$$

It follows that,

$$\dot{V} \leqslant -1.182\rho^2 + 0.037\rho^3.$$

Then, $\dot{V} < 0$ for $\rho < 31.94$. So, the system is asymptotically stable. Hence, there is an open set

$$D = \{x \in \mathbb{R}^3 / \|X\| \leqslant r,\ r > 0\}$$

such that $\dot{V} < 0$ in D. We can therefore estimate the domain of attraction by choosing

$$c < \min\{V(X),\ \|X\| = r\} \simeq \lambda_{\min}(P) r^2.$$

Since $\lambda_{\min}(P) \simeq 0.0225$, c can be taken smaller than $\lambda_{\min}(P).\rho^2$. Thus, the set Ω_c, with $c = 22.95$ is an estimate of the region of attraction.

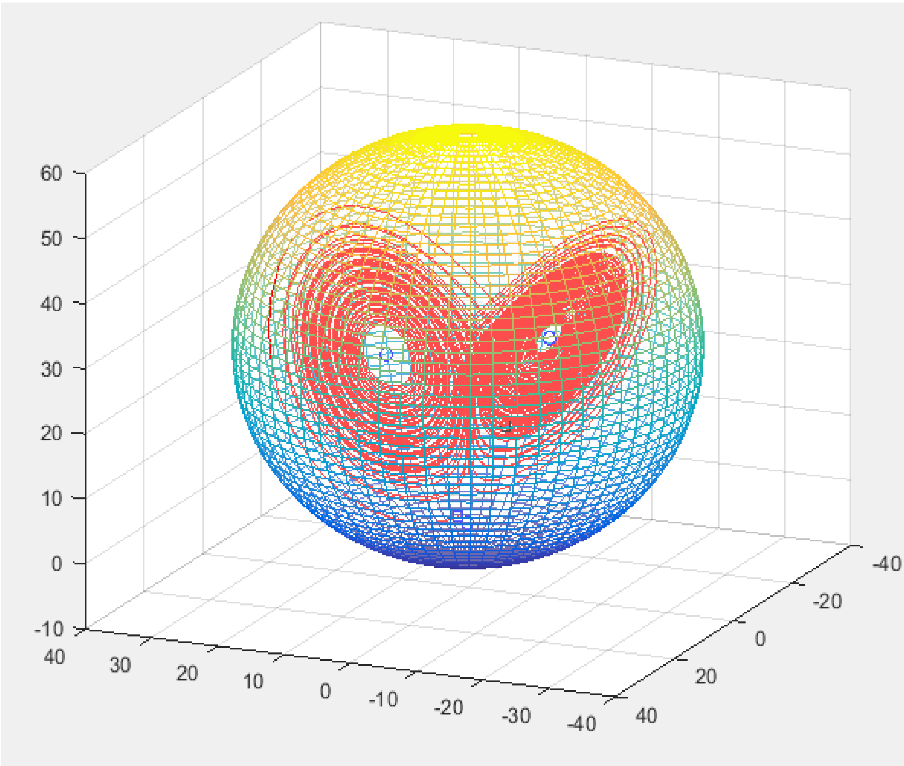


Figure 5: $a = 10, \ b = 8/3, \ c = 28$

4. Conclusion

In this paper, a new method is proposed to estimate the basin of attraction based on the construction of a suitable Lyapunov function. In particular, the relationship between the initial conditions and the regions of attractions when the origin is not necessarily an equilibrium point. Moreover, an numerical application to Lorenz system is given to verify the validity of the proposed method.

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