10.24425/acs.2020.134676

Archives of Control Sciences
Volume 30(LXVI), 2020
No. 3, *pages 523–552*

Exact determinantions of maximal output admissible set for a class of semilinear discrete systems

AMINE EL BHIH, YOUSSEF BENFATAH and MOSTAFA RACHIK

Consider the semilinear system defined by

$$\begin{cases} x(i+1) & = Ax(i) + f(x(i)), \qquad i \geqslant 0 \\ x(0) & = x_0 \in \mathbb{R}^n \end{cases}$$

and the corresponding output signal $y(i) = Cx(i)$, $i \geqslant 0$, where A is a $n \times n$ matrix, C is a $p \times n$ matrix and f is a nonlinear function. An initial state $x(0)$ is output admissible with respect to A, f, C and a constraint set $\Omega \subset \mathbb{R}^p$, if the output signal $(y(i))_i$ associated to our system satisfies the condition $y(i) \in \Omega$, for every integer $i \geqslant 0$. The set of all possible such initial conditions is the maximal output admissible set $\Gamma(\Omega)$. In this paper we will define a new set that characterizes the maximal output set in various systems (controlled and uncontrolled systems). Therefore, we propose an algorithmic approach that permits to verify if such set is finitely determined or not. The case of discrete delayed systems is taken into consideration as well. To illustrate our work, we give various numerical simulations.

Key words: discrete-time, output admissible set, semilinear system, asymptotic stability, uncontrolled system, controlled system, delayed system

1. Introduction

The characterization of admissible set has several significant usages in the analysis and design of closed-loop systems with state and control constraints [8, 10, 11, 13–17]. The initial aim of this paper is to contribute to the study of the maximal output admissible set for a class of semilinear system.

In linear case we have the following references: Gilbert et al. [3] have given some properties and characterizations of maximal output admissible set (MOAS)

All authors are with Laboratory of Analysis Modeling and simulation, Department of Mathematics and Computer Science, Faculty of Sciences Ben M'Sik, Hassan II University Casablanca, BP 7955, Sidi Othman, Casablanca, Morocco. E-mails: elbhihamine@gmail.com, youssef.benfatah@gmail.com, m_rachik@yahoo.fr

Received 20.09.2019.

and they have shown that this set can be represented by a finite number of functional, they have also given practical algorithms for generating these functions. A. Feuer et al. (1976) [1] have defined the maximally admissible set in regulator design, and they presented an efficient computational method for determining these sets. M. Rachik et al. [11] have consider the linear discrete time-delayed system described by $x_{i+1} = \sum_{j=0}^{m} A_j x_{i-j}$, $x_k = \alpha_k$ for $-m \leqslant k < 0$ where $\alpha = (x_0, \alpha_{-1}, \ldots, \alpha_{-m})$ is given in $\mathbb{R}^{n(m-1)}$ and A_i is an $n \times n$ real matrix. They have concluded some results connected to the set MOAS, and they have shown also that MOAS is stable by a small perturbation of the constraint set as well. In (2014) Moritz Schulze Darup, Martin Mönnigmann [12] have presented a new method for the approximation of the largest constraint admissible set for linear continuous-time systems with state and input constraints. Faultlessly, the maximal output admissible set has been completely determined for linear systems with state and control constraints, and methods to exactly find maximal output admissible set have been established [9]. The issue of linear perturbation has been studied by Rachik et al. [10]. For further details, the reader can reach to the following references [4, 8, 10].

Numerous algorithms have presented in the literature for determining the maximal state constraint sets. In (1986) M. Cwikel and P.-O. Gutman [2] have studied the Conversence of an algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states. While in (1987) P.O. Gutman et al. [14] have given an algorithm to find polyhedral approximation to the maximal state constraint set. In (2018) a stochastic approach to maximal output admissible set and reference governor has given by Joycer Osorio and Hamid R. Ossareh [6]. Later Bouyaghroumni et al. have improved some results by introducing a bilinear term in the mathematical model considered in [5].

However, in most of the available studies, the problem for semilinear systems is not considered, hence their applicability is severely limited. Therefore, we propose an explicit procedure to exactly determine the maximal output admissible set for this class.

In this paper, we study the concept of maximal output set for a semilinear systems described by

$$\begin{cases} x(i+1) &= Ax(i) + f(x(i)) \qquad i \in \mathbb{N} \\ x(0) &= x_0 \in \mathbb{R}\,. \end{cases}$$

$A \in \mathcal{L}(\mathbb{R}^n)$, $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function and the corresponding output signal is given by

$$y(i) = Cx(i), \qquad i \geqslant 0, \quad C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)\,.$$

An initial state $x(0)$ is output admissible with respect to A, f, C and a constraint set $\Omega \subset \mathbb{R}^p$, if the output signal $(y(i))_i$ associated to our system satisfies the condition $y(i) \in \Omega$, for every integer $i \geqslant 0$. The set of all possible that verifies such initial conditions is the maximal output admissible set $\Gamma(\Omega)$. In this paper, we limit our research to the study of the set $\Gamma^\rho(\Omega) = \Gamma(\Omega) \cap B(0, \rho)$ where $B(0, \rho) = \{x \in \mathbb{R}^n : \|x\| \leqslant \rho\}$; ρ is a real positive. We limited ourselves to the determination of the initial data x_0, which verify

$$\|x_0\| \leqslant \rho, \qquad \rho > 0 \tag{1}$$

such that

$$y_i \in \Omega, \quad \forall i \in \mathbb{N}. \tag{2}$$

An initial state x_0 satisfying the conditions (1) and (2) is said to be ρ-admissible. The set of all such initial states is said to be the maximal ρ-admissible set. The adapted method has been generalized to the discrete controlled semilinear system given by

$$\begin{cases} x(i+1) &= Ax(i) + f(x(i)) + Bu_i + g(v(i)) \qquad i \in \mathbb{N} \\ x(0) &= x_0\,, \end{cases} \tag{3}$$

where u_i, $v_i \in \mathbb{R}^m$ are the feedback controls given by $u_i = kx_i$ and $v_i = h(x_i)$, $\forall i \in \mathbb{N}$.

A is a $n \times n$ real matrix, B is a $n \times m$ real matrix, f, g, h are supposed to be continuous nonlinear appropriate functions.

This work is organised as following. In section 2, we give some preliminary results related to the stability asymptotic of our system. In section 3, we define and characterize the maximal output ρ-admissible set in the case of uncontrolled system. Moreover, in section 4, we propose an algorithm to determine if the maximal state constraint sets are finitely determined or not. While in section 5, we give some sufficient conditions for finite determination of $\Gamma^\rho(\Omega)$ in addition to some examples to illustrate our results. In section 6, the maximal output ρ-admissible set for semilinear discrete delayed system is considered. Lastly, in section 7, we give some characterization results of ρ-admissible set in the case of a controlled system illustrating our results by using some examples.

2. Preliminary results

Consider the uncontrolled semilinear discrete system described by

$$\begin{cases} x(i+1) &= Ax(i) + f(x(i)), \qquad i \geqslant 0 \\ x(0) &= x_0 \in \mathbb{R}^n\,, \end{cases} \tag{4}$$

where $A \in \mathcal{L}(\mathbb{R}^n)$, $f : \mathbb{R}^n \to \mathbb{R}^n$ nonlinear function, and the observation variable $y(i) \in \mathbb{R}^p$, satisfing the output constraint

$$y_i = Cx_i \in \Omega, \qquad i \geqslant 0, \tag{5}$$

where C is a $p \times n$ real matrix.

An initial condition $x_0 \in \mathbb{R}^n$ is output ρ-admissible if $x_0 \in B(0,\rho)$ and if the resulting output function $(y_i)_i$ satisfies the constraint (5). The set of all such initial state is the maximal output ρ-admissible set $\Gamma^\rho(\Omega)$.

We show that under hypothesis on the matrix A and the function f, the maximal output ρ-admissible set $\Gamma^\rho(\Omega)$ is determined by a finite number of functional inequalities and leads to algorithmic procedures for its computation.

Let Ω be the constraint set. Let Ψ be the function defined by

$$\begin{aligned} \Psi : \mathbb{R}^n &\to \quad \mathbb{R}^n \\ x \quad &\to Ax + f(x) \end{aligned}$$

Then

$$\Gamma(\Omega) = \left\{x_0 \in \mathbb{R}^n \middle/ C\,\Psi^i(x_0) \in \Omega\,, \quad \forall i \in \mathbb{N}\right\}$$

and the set of all output ρ-admissible initial states is formally given by

$$\Gamma^\rho(\Omega) = \left\{x_0 \in B(0,\rho) \cap \mathbb{R}^n / C\,\Psi^i(x_0) \in \Omega, \quad \forall i \in \mathbb{N}\right\}. \tag{6}$$

2.1. Notation and hypotheses

Now, we give some conditions which are sufficient to ensure, for convenient initial states x_0, the asymptotic stability of system (4). The following proposition will be useful in the sequel.

Proposition 1 *If we suppose the following to hold*

(i) $\|f(x) - f(y)\| \leqslant L\,\|x - y\|^\alpha$, *for all* $x, y \in \mathbb{R}^n$ *and for some* $L, \alpha > 0$.

(ii) $f(0) = 0$.

(iii) There exists $\beta \in]0,1[$ *and* $\gamma \geqslant 1$ *such that* $\|(A)^n\| \leqslant \gamma\,\beta^n$, $\forall n \in \mathbb{N}$.

(iii) $L\rho^{\alpha-1}\gamma^\alpha < 1 - \beta$.

Then system (4) is asymptotically stable in the region $\{x \in \mathbb{R}^n / \|x\| \leqslant \rho\}$, *i.e*

$$\lim_{n\to\infty} \|\Psi^n(x_0)\| = 0, \qquad \textit{for every} \;\; x_0 \in B(0,\rho).$$

Proof. Let $x_0 \in B(0,\rho)$. Then the solution of (4) at time n can be written as the following

$$x_n = A^n x_0 + \sum_{k=0}^{n-1} A^{n-k-1} f(x_k) \quad \forall n \geqslant 1.$$

Thus

$$\|x_n\| \leqslant \|A^n\|\,\|x_0\| + \sum_{k=0}^{n-1} \left\|A^{n-k-1}\right\| \|f(x_k)\|, \quad \forall n \geqslant 1.$$

Using (i), (ii) and (iii) we get

$$\|x_n\| \leqslant \gamma\beta^n \|x_0\| + \sum_{k=0}^{n-1} \gamma\beta^{n-k-1} L \|x_k\|^{\alpha}.$$

Take $z_n = \dfrac{x_n}{\rho}$. Then we can prove that

$$\|z_0\| \leqslant \gamma \quad \text{and} \quad \|z_n\| \leqslant \gamma \quad \text{for all } n \in \mathbb{N}^*.$$

Indeed $\|z_0\| = \dfrac{\|x_0\|}{\rho} \leqslant \dfrac{\rho}{\rho} = 1 \leqslant \gamma$ and

$$\|z_n\| \leqslant \gamma\beta^n \|z_0\| + \gamma\beta^{n-1} L \sum_{k=0}^{n-1} \beta^{-k} \|z_k\| \rho^{\alpha-1} \|z_k\|^{\alpha-1}, \qquad \forall n \geqslant 1.$$

Now, assume that $\|z_n\| \leqslant \gamma$, $\forall n \in \{0, 1, \ldots, N\}$ where $N \in \mathbb{N}$. Then

$$\|z_n\| \leqslant \gamma\beta^n \|z_0\| + \gamma^{\alpha}\beta^{n-1} L\rho^{\alpha-1} \sum_{k=0}^{n-1} \beta^{-k} \|z_k\|, \qquad \forall n \in \{1, 2, \ldots N+1\}. \tag{7}$$

Take $a_n = \dfrac{\|z_n\|}{\beta^n}$, $\forall n \geqslant 0$ and $Y_n = r + p\sum_{k=0}^{n-1} a_k$, $n \geqslant 1$, where $r = \gamma\|z_0\|$ and $p = \gamma^{\alpha}\beta^{-1}L\rho^{\alpha-1}$.

We conclude from (7) that $a_n \leqslant Y_n$ for all $n \in \{1, 2, \ldots N+1\}$, which implies

$$a_n = \frac{Y_{n+1} - Y_n}{p} \leqslant Y_n \quad \text{for} \quad n \in \{1, 2, \ldots N+1\}.$$

This implies the inequality

$$a_n \leqslant Y_n \leqslant (1+p)^{n-1} Y_1, \quad \forall \ n \in \{1, 2, \ldots N+1\}.$$

In this time we replace a_n and Y_1 by their values, we get

$$\|z_n\| \leqslant \beta^n \left(\gamma^\alpha \beta^{-1} L\rho^{\alpha-1} + 1\right)^{n-1} \left(\gamma \|z_0\| + \gamma^\alpha \beta^{-1} L\rho^{\alpha-1} \|z_0\|\right), \quad \forall n \in \{1, 2, \ldots N+1\} \tag{8}$$

$$= \left(L\gamma^\alpha \rho^{\alpha-1} + \beta\right)^n \frac{\gamma + \gamma^\alpha \beta^{-1} L\rho^{\alpha-1}}{\gamma^\alpha \beta^{-1} L\rho^{\alpha-1} + 1} \|z_0\| . \tag{9}$$

Since, $L\gamma^\alpha \rho^{\alpha-1} + \beta < 1$ and $\|z_0\| \leqslant 1$, the inequality (8) implies that

$$\|z_n\| \leqslant \gamma \qquad \forall n \in \{1, 2, \ldots N+1\}.$$

Therefore

$$\|z_n\| \leqslant \gamma \qquad \forall n \in \mathbb{N}.$$

Using inequality (8) for all $n \geqslant 0$, we conclude that

$$\lim_{n\to\infty} \|z_n\| = 0 \quad \text{since} \quad L\gamma^\alpha \rho^{\alpha-1} < 1 - \beta,$$

and consequently

$$\lim_{n\to\infty} \|\Psi^n(x_0)\| = \lim_{n\to\infty} \|x_n\| = 0.$$

□

We assume hereafter that $0 \in int\,\Omega$, this assumption is satisfied in any reasonable application and has nice consequences. Imposing special conditions on A, f and Ω which imposes corresponding conditions on $\Gamma^\rho(\Omega)$. Some implications of this type are summarized in the following proposition.

Proposition 2

(i) If Ω is closed then the set $\Gamma^\rho(\Omega)$ is also closed.

(ii) If we suppose the following to hold

1. $\|f(x) - f(y)\| \leqslant L\|x - y\|^\alpha$, *for all* $x, y \in \mathbb{R}^n$ *and for some* $L, \alpha > 0$.
2. $f(0) = 0$.
3. *There exists* $\beta \in (0, 1)$ *and* $\gamma \geqslant 1$ *such that* $\|A^n\| \leqslant \gamma\beta^n$ *for all* $n \in \mathbb{N}$.
4. $L\rho^{\alpha-1}\gamma^\alpha < 1 - \beta$.
5. $0 \in int\,\Omega$.

Then, $0 \in int\,\Gamma^\rho(\Omega)$.

Proof.

(i) $\Gamma^\rho(\Omega)$ can be written as

$$\Gamma^\rho(\Omega) = \bigcap_{n\in\mathbb{N}} \{x_0 \in B(0,\rho) \cap \mathbb{R}^n \,/ C\,\Psi^n(x_0) \in \Omega\},$$
$$= \bigcap_{n\in\mathbb{N}} \Phi_n^{-1}(\Omega),$$

where
$$\begin{array}{rcl} \Phi_n : B(0,\rho)\cap\mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ x & \longmapsto & \Phi_n(x) = C\,\Psi^n(x). \end{array}$$

Since Ω is closed and Φ_n continuous for all $n \in \mathbb{N}$ (f continuous), we conclued that $\Phi_n^{-1}(\Omega)$ is closed and then $\Gamma^\rho(\Omega)$ is closed.

(ii) From hypotheses 1, 2, 3 and 4 and from Proposition 1 we have

$$\lim_{n\to\infty} \|\Psi^n(x_0)\| = 0 \quad \forall x_0 \in B(0,\rho).$$

Let $x_0 \in B(0,\rho)$ and $\varepsilon > 0$. Then

$$\exists n_0 \in \mathbb{N}: \quad \Psi^n(x_0) \in B(0,\varepsilon), \quad \forall n \geqslant n_0$$
$$0 \in int\,\Omega \Longrightarrow \exists \varepsilon_1 > 0: \quad B(0,\varepsilon_1) \subset \Omega.$$

Take $\varepsilon = \dfrac{\varepsilon_1}{\|C\|}$, we get then

$$\exists n_0 \in \mathbb{N}: \quad C\,\Psi^n(x_0) \in B(0,\varepsilon_1) \subset \Omega \quad \forall n \geqslant n_0.$$

In what follows we show that $\forall\, x_0 \in B(0,\delta)$, $C\,\Psi^n(x_0) \in \Omega\, \forall\, n \in \{0,\ldots,n_0\}$ for some $\delta > 0$.

Since Ψ, $\Psi^2,\ldots,$ Ψ^{n_0} are continuous and $\varepsilon(0) = 0$ we have for $\dfrac{\varepsilon_1}{\|C\|}$

$$\forall n \in \{1,\ldots,n_0\} \quad \left[\exists\, \delta_n > 0\colon \forall\, x_0 \in B(0,\delta_n) \text{ we have } \Psi^n(x_0) \in B\left(0, \frac{\varepsilon_1}{\|C\|}\right)\right]$$

$i.e.$ $\forall n \in \{1,\ldots,n_0\}$ $[\exists\, \delta_n > 0\colon\ \forall\, x_0 \in B(0,\delta_n)$ we have $C\Psi^n(x_0) \in B(0,\varepsilon_1)]$

for $n = 0$ we have $\forall\, x_0 \in B\left(0, \dfrac{\varepsilon_1}{\|C\|}\right)$, $\|Cx_0\| \leqslant \|C\|\|x_0\| \leqslant \varepsilon_1$ i.e., $Cx_0 \in B(0,\varepsilon_1)$.

If we choose $\delta = \inf\left\{\{\delta_n, n \in \{1,\ldots,n_0\}\}, \dfrac{\varepsilon_1}{\|C\|}\right\}$ we obtain

$$\forall\ x_0 \in B(0,\delta),\ C\Psi^n\,(x_0) \in B(0,\varepsilon_1) \subset \Omega \quad \text{for all} \quad n \in \{0,\ldots,n_0\}.$$

We conclude that

$$\forall x_0 \in B(0,\tau),\quad C\Psi^n(x_0) \in \Omega \ \text{ for all } \ n \in \mathbb{N}$$

where $\tau = \inf\{\delta,\ \rho\}$. Thus $B(0,\tau) \subset \Gamma^\rho(\Omega)$, and consequently $0 \in int\Gamma^\rho(\Omega)$. □

3. Characterization of the maximal output ρ-admissible set

In order to characterize the maximal output ρ-admissible set given formally by (6), we define for each integer k the set

$$\Gamma_k^\rho(\Omega) = \{x_0 \in B(0,\rho) \cap \mathbb{R}^n \,/ C\Psi^n(x_0) \in \Omega, \quad \forall n \in \{0,\ldots,k\}\}\,.$$

Definition 1 *The set $\Gamma^\rho(\Omega)$ is finitely determined if there exists an integer k such that $\Gamma^\rho(\Omega)$ is nonempty and $\Gamma^\rho(\Omega) = \Gamma_k^\rho(\Omega)$. Let k^* be the smallest integer such that $\Gamma^\rho(\Omega) = \Gamma_{k^*}^\rho(\Omega)$, we call k^* the output ρ-admissibility index.*

Remark 1

(i) We remark that

$$\forall\ k_1,\ k_2 \in \mathbb{N} \ \text{ such that } \ k_1 \leqslant k_2 \text{ we have } \Gamma^\rho(\Omega) \subset \Gamma_{k_2}^\rho(\Omega) \subset \Gamma_{k_1}^\rho(\Omega).$$

(ii) Assume that $\Gamma^\rho(\Omega)$ is finitely determined and let k_0 be the smallest k such that $\Gamma_k^\rho(\Omega) = \Gamma_{k+1}^\rho(\Omega)$, then

$$\Gamma^\rho(\Omega) = \Gamma_{k0}^\rho(\Omega) = \Gamma_k^\rho(\Omega) \quad \forall\ k \geqslant k_0$$

Proposition 3

(i) If $\Gamma^\rho(\Omega)$ is finitely determined then there exists an integer k such that

$$\Gamma^\rho(\Omega) \text{ is nonempty and } \ \Gamma_k^\rho(\Omega) = \Gamma_{k+1}^\rho(\Omega).$$

(ii) If $\Psi\,(B(0,\rho)) \subset B(0,\rho)$ and $\Gamma_k^\rho(\Omega) = \Gamma_{k+1}^\rho(\Omega)$ for some integer k then $\Gamma^\rho(\Omega)$ is finitely determined.

Proof.

(i) $\Gamma^\rho(\Omega)$ finitely determined implies $\exists k \in \mathbb{N}$: $\Gamma_k^\rho(\Omega) = \Gamma^\rho(\Omega)$ and $\Gamma^\rho(\Omega)$ is nonempty.

Clearly $\Gamma_{k+1}^\rho(\Omega) \subset \Gamma_k^\rho(\Omega)$. But $\Gamma_k^\rho(\Omega) = \Gamma^\rho(\Omega) \subset \Gamma_{k+1}^\rho(\Omega)$.

Therefore $\Gamma_k^\rho(\Omega) = \Gamma_{k+1}^\rho(\Omega)$ for some $k \in \mathbb{N}$.

(ii) Let $x_0 \in \Gamma_k^\rho(\Omega) = \Gamma_{k+1}^\rho(\Omega)$. Then $x_0 \in B(0,\rho)$ and $C\Psi^i(x_0) \in \Omega$, $\forall i \in \{0,\ldots,k+1\}$

i.e. $x_0 \in B(0,\rho)$ and $Cx_0 \in \Omega$, $C\Psi^{i+1}(x_0) \in \Omega$, $\forall i \in \{0,\ldots,k\}$.

Thus $\Psi(x_0) \in \Gamma_k^\rho(\Omega)$ since $\Psi(x_0) \in \Psi\left(B(0,\rho)\right) \subset B(0,\rho)$.

By iteration, $x_0 \in \Gamma_k^\rho(\Omega) \implies \Psi^j(x_0) \in \Gamma_k^\rho(\Omega)$ $\forall j \in \mathbb{N}$.

i.e., $\left[x_0 \in B(0,\rho) \text{ and } C\Psi^i\left(\Psi^j(x_0)\right) \in \Omega,\ \forall i \in \{0,\ldots,k\}\right] \quad \forall j \in \mathbb{N}$.

i.e. $\left[x_0 \in B(0,\rho) \text{ and } C\Psi^{i+j}(x_0) \in \Omega,\ \forall i \in \{0,\ldots,k\}\right] \ \forall j \in \mathbb{N}$.

i.e. $x_0 \in B(0,\rho)$ and $C\Psi^i(x_0) \in \Omega$, $\forall i \in \mathbb{N}$.

i.e. $x_0 \in \Gamma^\rho(\Omega)$.

Therefore $\Gamma_k^\rho(\Omega) \subset \Gamma^\rho(\Omega)$ and $\Gamma_k^\rho(\Omega) = \Gamma^\rho(\Omega)$ for some $k \in \mathbb{N} \implies \Gamma^\rho(\Omega)$ is finitely determined. □

4. Algorithmic determination

As a natural consequence of the previous proposition, we shall give the following conceptual algorithm for determining the output ρ-admissibility index k^* such that $\Gamma^\rho(\Omega) = \Gamma_{k^*}^\rho(\Omega)$ and consequently the characterization of the set $\Gamma^\rho(\Omega)$.

Algorithm I

Step 1. Set $k = 0$;
Step 2. If $\Gamma_k^\rho(\Omega) = \Gamma_{k+1}^\rho(\Omega)$ then set $k^* = k$ and stop, else continue;
Step 3. Replace k by $k+1$ and return to step 2.

Clearly, Algorithm I will produce k^* and $\Gamma^\rho(\Omega)$ if and only if $\Gamma^\rho(\Omega)$ is finitely determined. There appears to not to be finite algorithmic procedure for showing that $\Gamma^\rho(\Omega)$ is not finitely determined.

Algorithm I is not practical because it does not describe how the test $\Gamma_k^\rho(\Omega) = \Gamma_{k+1}^\rho(\Omega)$ is implemented. In order to overcome this difficulty, let $\mathbb{R}^n$ be endowed with the following norm

$$\|x\| = \max_{1\leqslant i\leqslant n} |x_i|, \qquad \forall x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

If Ω is defined by

$$\Omega = \{y \in \mathbb{R}^p;\ h_i(y) \leqslant 0, \quad i = 1, \dots, s\}$$

where $h_i \colon \mathbb{R}^p \to \mathbb{R}$ are a given functions. Such sets have much more importance in a practical view. In this case, for every integer k, $\Gamma_k^\rho(\Omega)$ is given by

$$\Gamma_k^\rho(\Omega) = \left\{x_0 \in B(0,\rho); \quad h_j\left(C\Psi^i(x_0)\right) \leqslant 0, \ \ j = 0, \dots, s; \ \ i = 0, \dots, k\right\},$$

on the other hand

$$\begin{aligned}\Gamma_{k+1}^\rho(\Omega) &= \left\{x_0 \in \Gamma_k^\rho(\Omega); \quad C\Psi^{k+1}(x_0) \in \Omega\right\}\\ &= \left\{x_0 \in \Gamma_k^\rho(\Omega); \quad h_j\left(C\Psi^{k+1}(x_0)\right) \leqslant 0, \ \text{ for } \ j = 1, \dots, s\right\}.\end{aligned}$$

Now, since $\Gamma_{k+1}^\rho(\Omega) \subset \Gamma_k^\rho(\Omega)$ for every integer k, then

$$\begin{aligned}\Gamma_{k+1}^\rho(\Omega) = \Gamma_k^\rho(\Omega) &\Longleftrightarrow \Gamma_k^\rho(\Omega) \subset \Gamma_{k+1}^\rho(\Omega)\\ &\Longleftrightarrow \forall x_0 \in \Gamma_k^\rho(\Omega), \ \ x_0 \in \Gamma_{k+1}^\rho(\Omega)\\ &\Longleftrightarrow \forall x_0 \in \Gamma_k^\rho(\Omega), \ \ \Big\{x_0 \in \Gamma_k^\rho(\Omega) \ \text{ and } \ h_j\left(C\Psi^{k+1}(x_0)\right) \leqslant 0\\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall j \in \{1, \dots, s\}\Big\}\\ &\Longleftrightarrow \forall x_0 \in \Gamma_k^\rho(\Omega), \ \ h_j\left(C\Psi^{k+1}(x_0)\right) \leqslant 0, \ \ \forall j \in \{1, \dots, s\}\\ &\Longleftrightarrow \sup_{\substack{x_0 \in B(0,\rho), \ h_i(C\Psi^l(x_0)) \leqslant 0\\ \forall i \in \{1,\dots,s\}, \forall l \in \{0,\dots,k\}}} h_j\left(C\Psi^{k+1}(x_0)\right) \leqslant 0, \ \forall j \in \{1, \dots, s\}\end{aligned}$$

with $(h_j)_{j\in\{1,\dots,2p\}} \colon \mathbb{R}^p \to \mathbb{R}$ is described for all $x = \left(x_1, \dots, x_p\right) \in \mathbb{R}^p$ by

$$\begin{cases} h_{2m-1}(x) = x_m - t, & \text{for } \ m \in \{1, 2, \dots, p\}, \ \ t \in \mathbb{R}\\ h_{2m}(x) = -x_m - t, & \text{for } \ m \in \{1, 2, \dots, p\}, \ \ t \in \mathbb{R}.\end{cases}$$

Consequently, the test $\Gamma_k^\rho(\Omega) = \Gamma_{k+1}^\rho(\Omega)$ leads to a set of mathematical programming problems, and Algorithm I can be implemented as follows:

Algorithm II

Step 1. Let $k = 0$;

Step 2. For $i = 1, \dots, 2p$, do:

Maximize $J_i(x) = h_i\left(C\Psi^{k+1}(x_0)\right)$

$$\begin{cases} h_j\left(C\Psi^l(x_0)\right) \leqslant 0, & x_0 \in B(0,\rho)\\ \forall j \in \{1, \dots, 2p\}, & \forall l \in \{l = 0, \dots, k\}.\end{cases}$$

Let J_i^* be the maximum value of $J_i(x)$.
If $J_i^* \leqslant 0$, for $i = 1, 2, \ldots, 2p$ then set $k^* := k$ and stop.
Else continue.

Step 3. Replace k by $k + 1$ and return to Step 2.

5. Sufficient conditions for finite determination of $\Gamma^\rho(\Omega)$

It is desirable to have simple conditions which ensure the finite determination of $\Gamma^\rho(\Omega)$. Our main results in this direction are the following two theorems.

Theorem 1 *Suppose the following hypothesis to hold*

1. $\|f(x) - f(y)\| \leqslant L\|x - y\|^\alpha$, *for all* $x, y \in \mathbb{R}^n$ *and for some* $L, \alpha > 0$.
2. $f(0) = 0$.
3. *There exist* $\beta \in]0, 1[$ *and* $\gamma \geqslant 1$ *such that* $\|A^n\| \leqslant \gamma\,\beta^n$, $\forall\, n \in \mathbb{N}$.
4. $L\rho^{\alpha-1}\gamma^\alpha < 1 - \beta$.
5. $0 \in int\,\Omega$.
6. $\Psi\,(B(0, \rho)) \subset B(0, \rho)$.

Then, $\Gamma^\rho(\Omega)$ *is finitely determined.*

Proof. By hypothesis (1), (2), (3), (4) and (5), we can show from Proposition 2 that

$$\forall x_0 \in B(0, \rho), \quad \exists\, n_0 \geqslant 0: \quad C\,\Psi^n(x_0) \in B(0, \varepsilon_1) \subset \Omega \quad \forall n \geqslant n_0.$$

For $n = n_0$, we get

$$\forall x_0 \in B(0, \rho), \quad \text{we have} \quad C\Psi^{n_0}(x_0) \in B(0, \varepsilon_1) \subset \Omega.$$

Clearly $\Gamma^\rho_{n_0}(\Omega) \subset \Gamma^\rho_{n_0-1}(\Omega)$.
Let $x_0 \in \Gamma^\rho_{n_0-1}(\Omega)$, then $x_0 \in B(0, \rho)$ and $C\Psi^n(x_0) \in \Omega\,\forall n \in \{0, \ldots, n_0{-}1\}$.
Since $x_0 \in B(0, \rho)$ and $C\,\Psi^{n_0}(x_0) \in \Omega$, we conclude that $x_0 \in \Gamma^\rho_{n_0}(\Omega)$ and

$$\Gamma^\rho_{n_0}(\Omega) = \Gamma^\rho_{n_0-1}(\Omega).$$

Therefore $\Gamma^\rho(\Omega)$ is finitely determined (using hypothesis 6 and Proposition 3 in this time). $\square$

Theorem 2 *If we suppose that*

(i) $\|\Psi(x)\| \leqslant M\|x\|$, *for all* $x \in \mathbb{R}^n$ *and* $M \in]0,1[$.

(ii) $0 \in int\,\Omega$.

Then $\Gamma^{\rho}(\Omega)$ *is finitely determined.*

Proof. Let $x_0 \in \mathbb{R}^n$. Then,

$$\|\Psi(x_0)\| \leqslant M\,\|x_0\| \quad \text{and} \quad \|\Psi^n(x_0)\| \leqslant M^n\,\|x_0\|\,, \quad \forall n \in \mathbb{N}$$

$$\|C\,\Psi^n(x_0)\| \leqslant \|C\|\,\|\Psi^n(x_0)\| \leqslant \|C\|M^n\,\|x_0\|\,, \quad \forall n \in \mathbb{N}$$

Let $x_0 \in B(0,\rho)$. Then

$$\|C\Psi^n(x_0)\| \leqslant \|C\|\rho M^n.$$

or

$$\lim_{n\to\infty} \|C\|\rho M^n = 0 \ \text{ implies } \ \exists n_0 \in \mathbb{N}: \ \forall n \geqslant n_0 \ \ \|C\|\rho M^n \leqslant \varepsilon_1\,.$$

Take $n = n_0$ we get, $\|C\|\rho M^{n_0} \leqslant \varepsilon_1$. Then,

$$\forall x_0 \in B(0,\rho), \ \ \exists\, n_0 \in \mathbb{N}: \ C\,\Psi^{n_0}(x_0) \in B(0,\varepsilon_1) \subset \Omega. \tag{10}$$

Let $x_0 \in \Gamma^{\rho}_{n_0-1}(\Omega)$. Then

$$x_0 \in B(0,\rho), \quad \text{and} \quad C\,\Psi^n(x_0) \in \Omega \quad \forall n = \{0,\ldots,n_0-1\}.$$

Using (10) we conclude that $x_0 \in \Gamma^{\rho}_{n_0}(\Omega)$ and $\Gamma^{\rho}_{n_0}(\Omega) = \Gamma^{\rho}_{n_0-1}(\Omega)$.
Let $z \in \Psi\left(B(0,\rho)\right)$. Then $z = \Psi(x_0)$ with $x_0 \in B(0,\rho)$

$$\|z\| = \|\Psi(x_0)\| \leqslant M\,\|x_0\| \leqslant M\rho \leqslant \rho\,.$$

Thus

$$z \in B(0,\rho) \quad \text{and} \quad \Psi\left(B(0,\rho)\right) \subset B(0,\rho).$$

From Proposition 3, we conclude the result. □

Example 1. Let A, f, C and ρ is given by

$$A = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 \end{pmatrix},$$

$$f(x,y) = \begin{pmatrix} 0 \\ \frac{y}{4} \end{pmatrix}, \qquad \text{for every } \ x,y \in \mathbb{R}, \ \ \rho = 1 \ \text{ and } \ t = \frac{1}{2}.$$

Then we use Algorithm II to prove that $k^* = 1$ and we have

$$\Omega = \left\{x \in \mathbb{R}/\ |x| \leqslant \frac{1}{2}\right\} = \left[-\frac{1}{2}, \frac{1}{2}\right],$$

$$\Gamma^{\rho}(\Omega) = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \Big/ |x| \leqslant 1;\ \ |y| \leqslant 1;\ \ |x + 2y| \leqslant \frac{1}{2};\ \ \left|\frac{5}{4}x + y\right| \leqslant \frac{1}{2}\right\}.$$

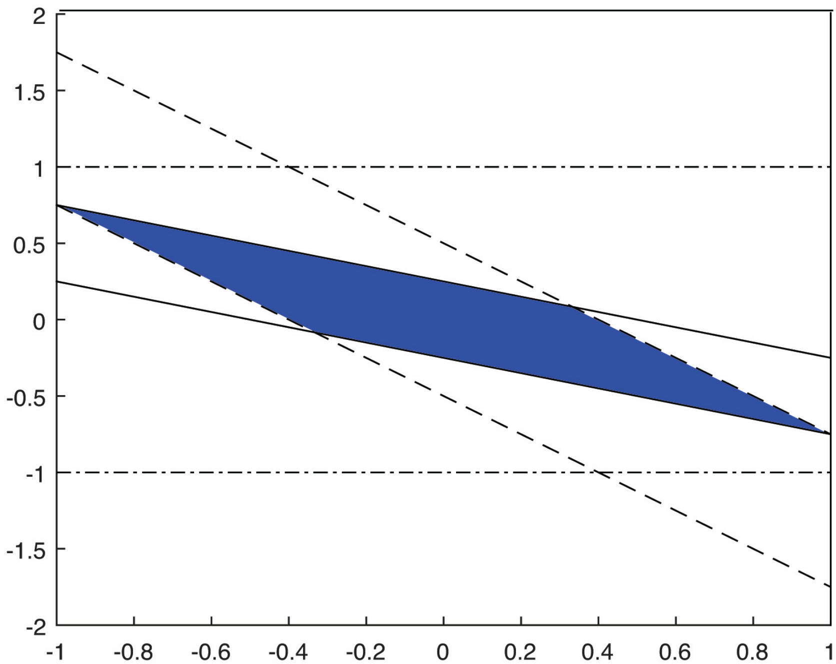

Figure 1: The dotted region is the set $\Gamma^{\rho}(\Omega)$ corresponding to Example 1

Proof. We have,

$$\Psi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{y}{4} \end{pmatrix} \quad \text{and} \quad \Psi\left(B(0,1)\right) \subset B(0,1).$$

Our set Ω is given by

$$\begin{aligned} \Omega &= \left\{y \in \mathbb{R}\ /\ h_j(y) \leqslant 0\ \ \forall j \in \{1,2\}\right\} \\ &= \left\{y \in \mathbb{R}/\ x - \frac{1}{2} \leqslant 0,\ \ -x - \frac{1}{2} \leqslant 0\right\} \\ &= \left\{y \in \mathbb{R}/\ |x| \leqslant \frac{1}{2}\right\} = \left[-\frac{1}{2}; \frac{1}{2}\right]. \end{aligned}$$

Using Algorithm II we obtain $k^* = 1$ and then $\Gamma^\rho(\Omega) = \Gamma_1^\rho(\Omega)$.
Or

$$\Gamma_1^\rho(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in B\left(0, \frac{1}{2}\right) : h_j\left(C\Psi^i \begin{pmatrix} x \\ y \end{pmatrix}\right) \leqslant 0 \ \forall\, j \in \{1,2\}\ \forall i \in \{1,2\} \right\}$$

and

$$C\begin{pmatrix} x \\ y \end{pmatrix} = x + 2y$$

$$C\Psi\begin{pmatrix} x \\ y \end{pmatrix} = \frac{5}{4}x + y$$

$$C\Psi^2\begin{pmatrix} x \\ y \end{pmatrix} = \frac{13}{16}x + \frac{y}{2}$$

$$C\Psi^3\begin{pmatrix} x \\ y \end{pmatrix} = \frac{29}{64}x + \frac{y}{4}\,.$$

Therefore

$$\begin{aligned}\Gamma_1^\rho(\Omega) &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 :\ |x| \leqslant 1,\ |y| \leqslant 1,\ x + 2y \leqslant \frac{1}{2},\ \frac{5}{4}x + y \leqslant \frac{1}{2}, \right.\\ &\qquad\qquad \left. -x - 2y \leqslant \frac{1}{2},\ -\frac{5}{4}x - y \leqslant \frac{1}{2} \right\}\\ &= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 :\ |x| \leqslant 1,\ |y| \leqslant 1,\ |x + 2y| \leqslant \frac{1}{2},\ \left|\frac{5}{4}x + y\right| \leqslant \frac{1}{2} \right\}.\end{aligned}$$

Algorithm

$k = 0$
$i = 1$

Maximize $\qquad J_1(x) = \frac{5}{4}x + y - \frac{1}{2}$

sc $\qquad x + 2y - \frac{1}{2} \leqslant 0; \quad -x - 2y - \frac{1}{2} \leqslant 0$

$\qquad x \leqslant 1; \quad y \leqslant 1; \quad -x \leqslant 1; -y \leqslant 1$

$i = 2$

Maximize $\qquad J_1(x) = -\frac{5}{4}x - y - \frac{1}{2}$

sc $\qquad x + 2y - \frac{1}{2} \leqslant 0; \quad -x - 2y - \frac{1}{2} \leqslant 0$

$\qquad x \leqslant 1; \quad y \leqslant 1; \quad -x \leqslant 1; -y \leqslant 1$

we have $J_1^*, J_2^* \geqslant 0$, we then go to the next step

$k = 1$
$j = 1$

$$\textit{Maximize} \qquad J_1(x) = \frac{13}{16}x + \frac{y}{2} - \frac{1}{2}$$

$$sc \quad \begin{array}{ll} x + 2y - \frac{1}{2} \leqslant 0; & -x - 2y - \frac{1}{2} \leqslant 0 \\ \frac{5}{4}x + y - \frac{1}{2} \leqslant 0; & -\frac{5}{4}x - y - \frac{1}{2} \leqslant 0 \\ x \leqslant 1; \quad y \leqslant 1; & -x \leqslant 1; -y \leqslant 1 \end{array}$$

$j = 2$

$$\textit{Maximize} \qquad J_2(x) = -\frac{13}{16}x - \frac{y}{2} - \frac{1}{2}$$

$$sc \quad \begin{array}{ll} x + 2y - \frac{1}{2} \leqslant 0; & -x - 2y - \frac{1}{2} \leqslant 0 \\ \frac{5}{4}x + y - \frac{1}{2} \leqslant 0; & -\frac{5}{4}x - y - \frac{1}{2} \leqslant 0 \\ x \leqslant 1; \quad y \leqslant 1; & -x \leqslant 1; -y \leqslant 1 \end{array}$$

since $J_1^*, J_2^* \leqslant 0$, we stop. □

Example 2. Let

$$A = \begin{pmatrix} \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}, \qquad C = \begin{bmatrix} 1 & 2 \end{bmatrix},$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{2} \\ 0 \end{pmatrix}$$

and $\rho = \frac{1}{2}$, $t1 = \frac{2}{5}$, $t2 = \frac{1}{2}$ then

$$\Psi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{5}{6}x \\ \frac{1}{2}x + \frac{1}{3}y \end{pmatrix} \quad \text{and} \quad \Psi\left(B(0,\rho)\right) \subset B(0,\rho).$$

Using Algorithm II we get $k^* = 3$ and then

$$\Omega = [-0.5,\ 0.4]$$

$$\Gamma^{\rho}(\Omega) = \left\{ \begin{array}{l} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 / |x| \leqslant 0.5; \quad |y| \leqslant 0.5; \quad x + 2y \leqslant 0.4; \quad -x - 2y \leqslant 0.5; \\ \frac{11}{6}x + \frac{2}{3}y \leqslant 0.4; \quad -\frac{11}{6}x - \frac{2}{3}y \leqslant 0.5; \quad \frac{67}{36}x + \frac{2}{9}y \leqslant 0.4; \\ -\frac{67}{36}x - \frac{2}{9}y \leqslant 0.5; \quad \frac{359}{216}x + \frac{2}{27}y \leqslant 0.4; \; -\frac{359}{216}x - \frac{2}{27}y \leqslant 0.5. \end{array} \right\}$$

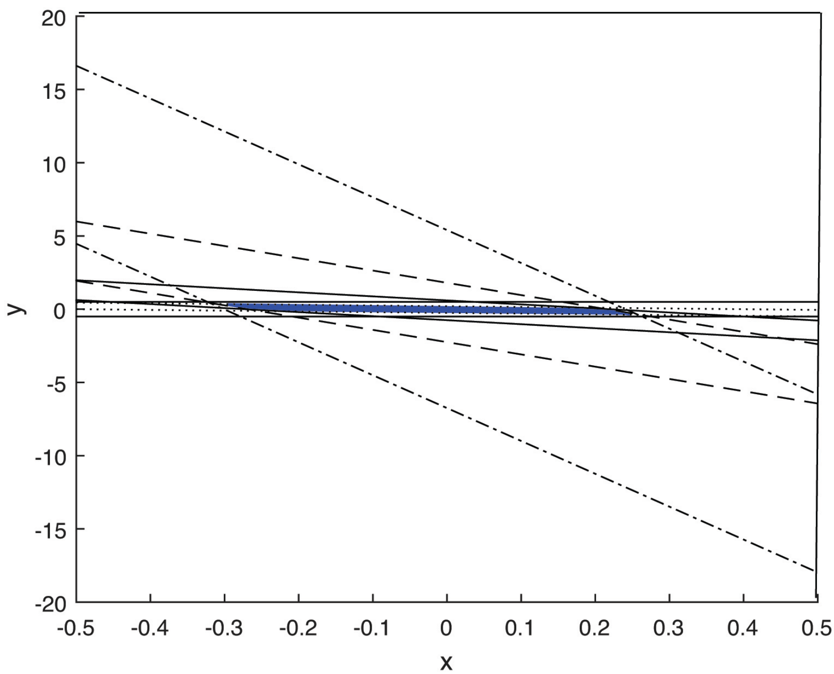

Figure 2: The dotted region is the set $\Gamma^{\rho}(\Omega)$ corresponding to Example 2

Example 3. For

$$A = \begin{pmatrix} \frac{1}{6} & 0 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \end{pmatrix},$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{x\exp\left(-x^2\right)}{4} \\ \dfrac{y}{4} \end{pmatrix}$$

and $\rho = \frac{3}{4}$, $t = \frac{1}{12}$, then

$$\Psi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{6}x + \frac{1}{4}xe^{-x^2} \\ \frac{1}{4}x + \frac{1}{2}y \end{pmatrix} \qquad \text{and} \qquad \Psi\left(B(0,\rho)\right) \subset B(0,\rho).$$

Using Algorithm II we get $k^* = 1$ and then

$$\Omega = \left\{ x \in \mathbb{R} / \ |x| \leqslant \frac{1}{12} \right\} = \left[-\frac{1}{12}, \frac{1}{12} \right]$$

$$\Gamma^{\rho}(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \Big/ |x| \leqslant \frac{3}{4}; \ \ |y| \leqslant \frac{3}{4}; \ \ |x+y| \leqslant \frac{1}{12};. \right.$$
$$\left. \left| \frac{5}{12}x + \frac{1}{2}y + \frac{1}{4}xe^{-x^2} \right| \leqslant \frac{1}{12}; \right\}.$$

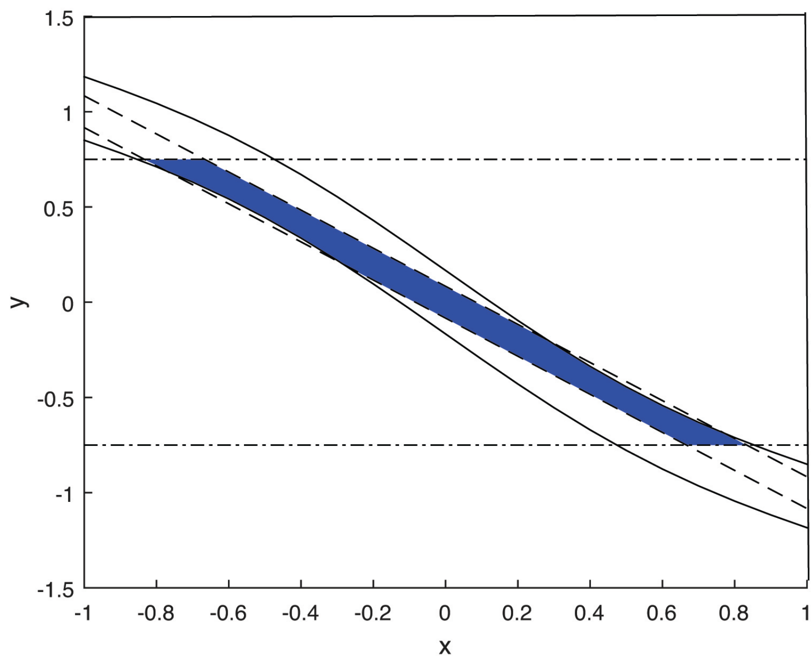

Figure 3: The dotted region is the set $\Gamma^{\rho}(\Omega)$ corresponding to Example 3

Example 4.

$$A = \begin{pmatrix} \frac{1}{4} & \frac{1}{6} \\ 0 & \frac{1}{3} \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 \end{pmatrix},$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x}{4} \\ \frac{y\exp\left(-y^2\right)}{4} \end{pmatrix}\begin{pmatrix} \frac{1}{4} & \frac{1}{3} \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{x}{4} \\ \frac{y\exp\left(-y^2\right)}{2} \end{pmatrix}$$

and $\rho = 1,\ t1 = \frac{1}{10},\ t2 = \frac{1}{2}$ then

$$\Psi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x + \frac{1}{3}y \\ \frac{1}{2}y\left(e^{-y^2} + 1\right) \end{pmatrix} \quad \text{and} \quad \Psi\left(B(0,\rho)\right) \subset B(0,\rho)$$

and

$$\begin{aligned}
C\Psi\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{2}x + \frac{4}{3}y + ye^{-y^2}, \\
C\Psi^2\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{4}x + \frac{7}{12}y + \frac{1}{4}y\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right) + \frac{5}{12}ye^{-y^2} \\
&\quad + \frac{1}{4}y\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right)e^{-y^2}, \\
C\Psi^3\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{8}x + \frac{19}{72}y + \frac{5}{48}y\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right) \\
&\quad + \frac{1}{12}y\exp\left(-\frac{1}{576}y^2\left(3\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right)+4\right)^2\left(e^{-y^2}+1\right)^2\right) \\
&\quad + \frac{13}{72}ye^{-y^2} + \frac{1}{16}y\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right) \\
&\quad \cdot\exp\left(-\frac{1}{576}y^2\left(3\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right)+4\right)^2\left(e^{-y^2}+1\right)^2\right) \\
&\quad + \frac{5}{48}y\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right)e^{-y^2} \\
&\quad + \frac{1}{12}y\exp\left(-\frac{1}{576}y^2\left(3\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right)+4\right)^2\left(e^{-y^2}+1\right)^2\right)e^{-y^2} \\
&\quad + \frac{1}{16}y\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right) \\
&\quad \cdot\exp\left(-\frac{1}{576}y^2\left(3\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right)+4\right)^2\left(e^{-y^2}+1\right)^2\right)e^{-y^2}.
\end{aligned}$$

Using Algorithm II we get $k^* = 2$ and then

$$\Omega = [-0.1,\ 0.5],$$

$$\Gamma^{\rho}(\Omega) = \left\{ \begin{array}{l} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \Big/ |x| \leqslant 1;\ \ |y| \leqslant 1;\ \ x + 2y \leqslant 0.1;\ \ -x - 2y \leqslant 0.5; \\ \frac{1}{2}x + \frac{4}{3}y + ye^{-y^2} \leqslant 0.1;\ \ -\frac{1}{2}x - \frac{4}{3}y - ye^{-y^2} \leqslant 0.5; \\ \frac{1}{4}x + \frac{7}{12}y + \frac{1}{4}y\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right) + \frac{5}{12}ye^{-y^2} \\ \qquad + \frac{1}{4}y\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right)e^{-y^2} \leqslant 0.1 \\ -\left(\frac{1}{4}x + \frac{7}{12}y + \frac{1}{4}y\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right) + \frac{5}{12}ye^{-y^2} \right. \\ \qquad \left. + \frac{1}{4}y\exp\left(-\frac{1}{4}y^2\left(e^{-y^2}+1\right)^2\right)e^{-y^2}\right) \leqslant 0.5 \end{array} \right\}.$$

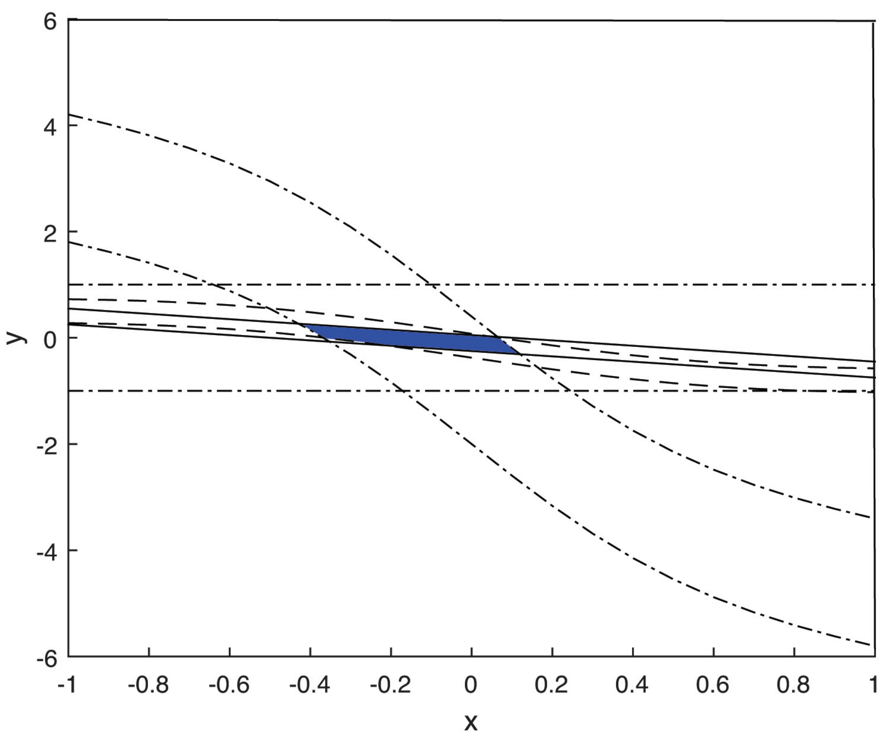

Figure 4: The dotted region is the set $\Gamma^{\rho}(\Omega)$ corresponding to Example 4

Example 5. For

$$A = \begin{pmatrix} \frac{1}{6} & \frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{x\exp\left(-x^2\right)}{4} \\ \dfrac{y}{2} \end{pmatrix}$$

and $\rho = \frac{3}{4}$, $t = \frac{1}{3}$, then

$$\Psi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{6}x + \frac{1}{4}y + \frac{1}{4}xe^{-x^2} \\ y \end{pmatrix} \quad \text{and} \quad \Psi\left(B(0,\rho)\right) \subset B(0,\rho).$$

and

$$C\Psi\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6}x + \frac{5}{4}y + \frac{1}{4}xe^{-x^2},$$

$$C\Psi^2\begin{pmatrix} x \\ y \end{pmatrix} = \frac{5}{72}x + \frac{65}{48}y + \frac{1}{24}xe^{-x^2} + \frac{1}{16}x\exp\left(-\frac{1}{144}\left(2x+3y+3xe^{-x^2}\right)^2\right)e^{-x^2},$$

$$\begin{aligned} C\Psi^3\begin{pmatrix} x \\ y \end{pmatrix} = {} & \frac{25}{864}x + \frac{805}{576}y + \frac{5}{288}xe^{-x^2} + \frac{1}{96}x\exp\left(-\frac{1}{144}\left(2x+3y+3xe^{-x^2}\right)^2\right)e^{-x^2} \\ & + \frac{1}{64}x\exp\left(-\frac{1}{144}\left(2x+3y+3xe^{-x^2}\right)^2\right)e^{-x^2} \\ & \cdot\exp\left(-\frac{1}{20\,736}\left(10x+51y+6xe^{-x^2}\right.\right. \\ & \left.\left. +9x\exp\left(-\frac{1}{144}\left(2x+3y+3xe^{-x^2}\right)^2\right)e^{-x^2}\right)^2\right). \end{aligned}$$

Using Algorithm II we get $k^* = 2$ and then

$$\Omega = \left\{x \in \mathbb{R} / \ |x| \leqslant \frac{1}{3}\right\} = \left[-\frac{1}{3}, \frac{1}{3}\right],$$

$$\Gamma^\rho(\Omega) = \left\{ \begin{array}{l} \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \Big/ |x| \leqslant \frac{3}{4};\ |y| \leqslant \frac{3}{4};\ |x+y| \leqslant \frac{1}{3};\ \left|\frac{1}{6}x + \frac{5}{4}y + \frac{1}{4}xe^{-x^2}\right| \leqslant \frac{1}{3}; \\ \left|\frac{5}{72}x+\frac{65}{48}y+\frac{1}{24}xe^{-x^2}+\frac{1}{16}x\exp\left(-\frac{1}{144}\left(2x+3y+3xe^{-x^2}\right)^2\right)e^{-x^2}\right| \leqslant \frac{1}{3} \end{array} \right\}.$$

6. ρ-admissible set for a semilinear controlled discrete system

Now we consider the semilinear controlled discrete system described by

$$\begin{cases} x(i+1) = Ax(i) + f(x(i)) + Bu_i + g(v(i)) & i \in \mathbb{N} \\ x(0) = x_0 , \end{cases} \tag{11}$$

the corresponding output function is

$$y(i) = Cx(i) + Du_i , \quad i \in \mathbb{N},$$

where u_i, $v_i \in \mathbb{R}^m$ are the feedback controls given by $u_i = kx_i$ and $v_i = h(x_i)$, $\forall i \in \mathbb{N}$.

A is a $n \times n$ real matrix, B is a $n \times m$ real matrix, f, g, h are supposed to be continuous nonlinear appropriate functions. $y(i) \in \mathbb{R}^q$, C is $q \times n$ real matrix, D is $q \times m$ real matrix and $x(i) \in \mathbb{R}^n$ is the state variable.

The purpose of this section is to characterize, under certain hypotheses, the set

$$F^\rho(\Omega) = \left\{ x(0) \in B(0,\rho) \cap \mathbb{R}^n | (C + DK)\Phi^i(x_0) \in \Omega, \;\; \forall i \in \mathbb{N} \right\},$$

where Φ is the nonlinear function defined by

$$\Phi(z) = (A + BK)z + (f + g \circ h)(z), \quad \forall z \in \mathbb{R}^n.$$

The set $F^\rho(\Omega)$ is derived from an infinite number of inequalities and it is difficult to characterize. However, we propose some sufficient conditions which ensure $F^\rho(\Omega)$ to be finitely determined, i.e., there exists an integer k such that $F^\rho(\Omega) = F_k^\rho(\Omega)$ where

$$F_k^\rho(\Omega) = \left\{ x_0 \in B(0,\rho) \cap \mathbb{R}^n \; | (C + DK)\Phi^i(x_0) \in \Omega, \;\; \forall i \in \{0, 1, \ldots, k\} \right\}.$$

Similarly as Proposition 1, the following result gives sufficient conditions to ensure the asymptotic stability of system (11) for all $x_0 \in B(0,\rho)$.

Proposition 4 *If we suppose the following assumptions to hold*

1. $\|f(x) - f(y)\| \leqslant L\|x - y\|^\alpha$, *for all* x, $y \in \mathbb{R}^n$ *and for some* $L, \alpha > 0$.
2. $f(0) = 0$.
3. $\|g(x)\| \leqslant L'\|x\|^{\alpha'}$, $\|h(x)\| \leqslant L''\|x\|^{\alpha''}$, *for all* $x \in \mathbb{R}^n$ *and for some* L', L'', α', $\alpha'' > 0$.
4. *There exist* $\beta \in]0,1[$ *and* $\gamma \geqslant 1$ *such that* $\left\|(A + BK)^i\right\| \leqslant \gamma\, \beta^i$ *for all* $i \in \mathbb{N}$.
5. $L\rho^{\alpha-1}\gamma^\alpha + L'(L'')^{\alpha'} \rho^{\alpha'\alpha''-1}\gamma^{\alpha'\alpha''} < 1 - \beta$.

Then, the system (11) is asymptotically stable in the region $\{x \in \mathbb{R}^n / \|x\| \leqslant \rho\}$.

We can use the techniques developed in the previous section to give sufficient conditions make $F^{\rho}(\Omega)$ finitely determined and we deduce the following theorems.

Theorem 3 *Suppose the following hypothesis to hold*

1. $\|f(x) - f(y)\| \leqslant L\|x - y\|^{\alpha}$, *for all* $x,\ y \in \mathbb{R}^n$ *and for some* $L, \alpha > 0$.
2. $f(0) = 0$.
3. *There exist* $\beta \in]0, 1[$ *and* $\gamma \geqslant 1$ *such that* $\|(A + BK)^n\| \leqslant \gamma\, \beta^n \ \ \forall\, n \in \mathbb{N}$.
4. $\|g(x)\| \leqslant L'\|x\|^{\alpha'}$, $\|h(x)\| \leqslant L''\|x\|^{\alpha''}$, *for all* $x \in \mathbb{R}^n$ *and for some* L', L'', α', $\alpha'' > 0$.
5. $L\rho^{\alpha-1}\gamma^{\alpha} + L'\,(L'')^{\alpha'}\,\rho^{\alpha'\alpha''-1}\gamma^{\alpha'\alpha''} < 1 - \beta$.
6. $0 \in int\,\Omega$.
7. $\Phi\left(B(0, \rho)\right) \subset B(0, \rho)$.

Then, there exists an integer k *such that* $F^{\rho}(\Omega) = F_k^{\rho}(\Omega)$.

Theorem 4 *If we suppose that*

(i) $\|\Phi(x)\| \leqslant M\|x\|$, *for all* $x \in \mathbb{R}^n$ *and* $M \in]0, 1[$

(ii) $0 \in int\,\Omega$.

Then $F^{\rho}(\Omega)$ *is finitely determined.*

The determination of the integer k cited in Theorem (3) is obtained by applying algorithm II with change $\Psi \to \Phi$.

Example 6. For

$$A = \begin{pmatrix} -1 & 0 \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}, \qquad B = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \end{pmatrix}, \qquad K = \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{3}{4} & 1 \end{pmatrix}, \qquad D = \frac{1}{2}, \qquad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{xy}{4} \\ \frac{1}{18} \end{pmatrix},$$

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{xy}{4} \\ \frac{1}{18} \end{pmatrix}$$

and $\rho = t = \frac{1}{3}$, then

$$\Phi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{-5}{6}x + \frac{xy}{2} \\ \frac{5}{12}x + \frac{y}{6} + \frac{1}{9} \end{pmatrix} \quad \text{and} \quad \Phi\left(B(0,\rho)\right) \subset B(0,\rho).$$

Using Algorithm II we get $k^* = 0$ and then

$$\Omega = \left\{ x \in \mathbb{R} / \ |x| \leqslant \frac{1}{3} \right\} = \left[-\frac{1}{3}, \frac{1}{3} \right],$$

$$\Gamma^\rho(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \Big/ |x| \leqslant \frac{1}{3}; \ \ |y| \leqslant \frac{1}{3}; \ \ |x+y| \leqslant \frac{1}{3} \right\}.$$

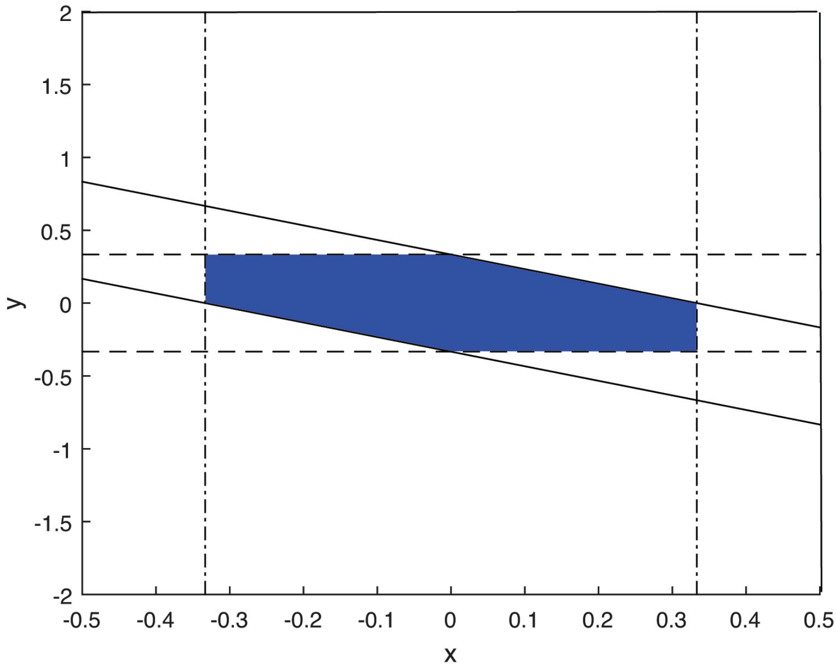

Figure 5: The dotted region is the set $\Gamma^\rho(\Omega)$ corresponding to Example 6

Example 7. For

$$A = \begin{bmatrix} -\frac{1}{10} & 0 \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix},$$

$$K = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{x\exp\left(-x^2\right)}{8} \\ \dfrac{y}{12} \end{pmatrix}, \qquad g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad h\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{x\exp\left(-x^2\right)}{8} \\ \dfrac{y}{12} \end{pmatrix},$$

$$\widetilde{A} = A + BK = \begin{bmatrix} -\dfrac{1}{10} & 0 \\ -\dfrac{1}{5} & \dfrac{2}{5} \end{bmatrix}, \qquad \widetilde{C} = C + DK = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

and $\rho = \dfrac{4}{5}$, $t1 = \dfrac{1}{50}$, $t2 = \dfrac{1}{10}$ then

$$\Phi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \dfrac{1}{4}xe^{-x^2} - \dfrac{1}{10}x \\ \dfrac{17}{30}y - \dfrac{1}{5}x \end{pmatrix} \quad \text{and} \quad \Phi\left(B(0,\rho)\right) \subset B(0,\rho).$$

$$\widetilde{C}\Phi\begin{pmatrix} x \\ y \end{pmatrix} = \frac{17}{30}y - \frac{1}{10}x - \frac{1}{4}xe^{-x^2},$$

$$\widetilde{C}\Phi^2\begin{pmatrix} x \\ y \end{pmatrix} = \frac{289}{900}y - \frac{31}{300}x + \frac{1}{40}x\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right)$$
$$- \frac{1}{40}xe^{-x^2} - \frac{1}{16}xe^{-x^2}\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right),$$

$$\widetilde{C}\Phi^3\begin{pmatrix} x \\ y \end{pmatrix} = \frac{4913}{27\,000}y - \frac{97}{1800}x + \frac{1}{400}x\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right)$$
$$- \frac{1}{400}x\exp\Bigg(-\frac{1}{160\,000}x^2\left(5e^{-x^2}-2\right)^2$$
$$\times\left(5\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right)-2\right)^2\Bigg) - \frac{31}{1200}xe^{-x^2}$$
$$+ \frac{1}{160}x\exp\Bigg(-\frac{1}{160\,000}x^2\left(5e^{-x^2}-2\right)^2$$
$$\times\left(5\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right)-2\right)^2\Bigg)e^{-x^2}$$

$$
\begin{aligned}
&+\frac{1}{160}x\exp\left(-\frac{1}{160\,000}x^2\left(5e^{-x^2}-2\right)^2\right.\\
&\times\left(5\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right)-2\right)^2\Bigg)\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right)\\
&-\frac{1}{160}xe^{-x^2}\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right)\\
&-\frac{1}{64}x\exp\left(-\frac{1}{160\,000}x^2\left(5e^{-x^2}-2\right)^2\right.\\
&\times\left(5\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right)-2\right)^2\Bigg)e^{-x^2}\\
&\times\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right).
\end{aligned}
$$

Using Algorithm II we get $k^* = 2$ and then

$$\Omega = [-0.02,\ 0.1],$$

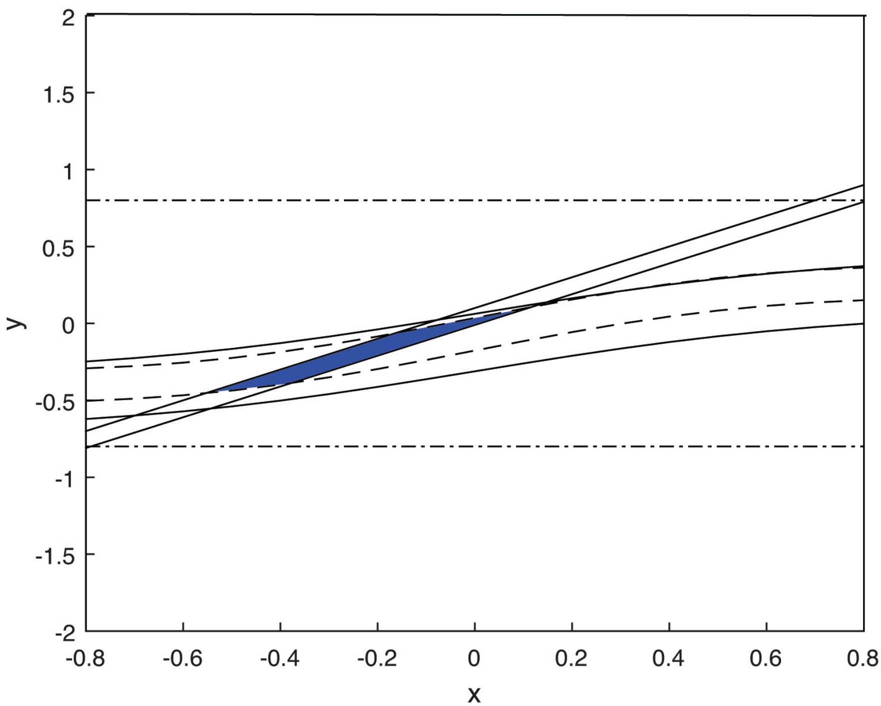

Figure 6: The dotted region is the set $\Gamma^{\rho}(\Omega)$ corresponding to Example 7

$$\Gamma^{\rho}(\Omega)=\left\{\begin{array}{l}\begin{pmatrix}x\\y\end{pmatrix}\in\mathbb{R}^2\Big/|x|\leqslant 0.8;\ \ |y|\leqslant 0.8;\ \ x-y\leqslant 0.01;\ \ -x+y\leqslant 0.1;\\ \frac{17}{30}y-\frac{1}{10}x-\frac{1}{4}xe^{-x^2}\leqslant 0.02;\ \ -\frac{17}{30}y+\frac{1}{10}x+\frac{1}{4}xe^{-x^2}\leqslant 0.1;\\ \frac{289}{900}y-\frac{31}{300}x+\frac{1}{40}x\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right)-\frac{1}{40}xe^{-x^2}\\ -\frac{1}{16}xe^{-x^2}\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right)\leqslant 0.02;\\ -\left(\begin{array}{l}\frac{289}{900}y-\frac{31}{300}x+\frac{1}{40}x\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right)\\ -\frac{1}{40}xe^{-x^2}-\frac{1}{16}xe^{-x^2}\exp\left(-\frac{1}{400}x^2\left(5e^{-x^2}-2\right)^2\right)\end{array}\right)\leqslant 0.1\end{array}\right\}.$$

7. Maximal output ρ-admissible sets for semilinear discrete delayed system

Consider the uncontrolled semilinear discrete delayed system described by

$$\begin{cases} x(i+1) = \sum_{j=0}^{p} A_j x(i-j) + f(x(i),\ldots,x(i-p)) & i\geqslant 0\\ x(0) = x_0 & \\ x(s) = \beta_s, & -p\leqslant s\leqslant -1.\end{cases}$$

The corresponding output is

$$y(i)=\sum_{j=0}^{m} C_j x(i-j),\qquad i\in\mathbb{N},\tag{12}$$

where the state variable $x(i)$ is in $\mathbb{R}^n$, $\left(A_j\right)_{0\leqslant j\leqslant p}$ are $n\times n$ real matrices and f is continuous nonlinear functions on $\mathbb{R}^{n\times(p+1)}$ and C_j is a $q\times n$ real matrix. p and m are integer such that $m\leqslant p$.

The observation variable $y(i)\in\mathbb{R}^q$, satisfies the output constraint

$$y(i)\in\Omega,\qquad \forall\, i\in\mathbb{N}\tag{13}$$

an initial condition $\beta=\left(x_0,\ \ \beta_{-1},\ \ \ldots,\ \ \beta_{-p}\right)\in\mathbb{R}^{n\times(p+1)}$ is output ρ-admissible if $\beta\in B(0,\rho)$ and the corresponding output (12) satisfies (13). The set of all such initial conditions is the maximal output ρ-admissible set $\widetilde{\Gamma}^{\rho}(\Omega)$.

We prove that under certain hypotheses on A_j, f the maximal output ρ-admissible set is finitely determined by a finite number of functional inequalities and leads to algorithmic procedures for its computation.

Define the state variables $\Lambda(i)$ by

$$\Lambda(i) = (x(i), x(i-1), \ldots, x(i-p))^{tr}$$

where *tr* denote the transpose of a matrix. We show that $(\Lambda(i))_{i\geqslant 0}$ is the solution of the following system

$$\begin{cases} \Lambda(i+1) = \widetilde{A}\Lambda(i) + \widetilde{f}(\Lambda(i)), & i \geqslant 0 \\ \Lambda(0) = \beta, \end{cases}$$

where $\widetilde{A}$ is the matrix define by

$$\widetilde{A} = \begin{pmatrix} A_0 & A_1 & \cdots & \cdots & A_p \\ I_n & 0_{n\times n} & \cdots & \cdots & 0_{n\times n} \\ 0_{n\times n} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{n\times n} & \cdots & 0_{n\times n} & I_n & 0_{n\times n} \end{pmatrix}$$

and $\widetilde{f}$ is the nonlinear function defined for all $z = (z_0, \ \cdots, \ z_p) \in \mathbb{R}^{n\times(p+1)}$ by

$$\widetilde{f}(z) = \begin{pmatrix} f(z(0), \ldots, z(p)) \\ 0_n \\ \vdots \\ 0_n \end{pmatrix},$$

where I_n and $0_{n\times n}$ are respectively the $n \times n$-identity matrix and the $n \times n$-zero matrix.

Indeed, $\Lambda(0) = (x(0), x(-1), \ldots, x(-p)) = \beta$ and

$$\begin{aligned}
\Lambda(i+1) &= (x(i+1), x(i), \ldots; x(i-p+1))^{tr} \\
&= \left(\sum_{j=0}^{p} A_j x(i-j) + f(x(i), \ldots, x(i-p)), \quad x(i), \quad \ldots, \quad x(i-p+1) \right)^{tr} \\
&= \begin{pmatrix} A_0 & A_1 & \cdots & \cdots & A_p \\ I_n & 0_{n\times n} & \cdots & \cdots & 0_{n\times n} \\ 0_{n\times n} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{n\times n} & \cdots & 0_{n\times n} & I_n & 0_{n\times n} \end{pmatrix} \begin{pmatrix} x(i) \\ x(i-1) \\ \vdots \\ \vdots \\ x(i-p) \end{pmatrix} + \begin{pmatrix} f(x(i), \ldots, x(i-p)) \\ 0_n \\ \vdots \\ \vdots \\ 0_n \end{pmatrix} \\
&= \widetilde{A}\Lambda(i) + \widetilde{f}(\Lambda(i)).
\end{aligned}$$

If we define the matrix $\widetilde{C}$ and the nonlinear function $\widetilde{\Psi}$ by

$$\widetilde{C} = \left(C_0| \ \cdots \ C_m| \ 0_{q\times n} \ \cdots \ 0_{q\times n}\right) \in \mathcal{L}\left(\mathbb{R}^{n(p+1)}, \mathbb{R}^q\right)$$

and

$$\widetilde{\Psi}(z) = \widetilde{A}z + \widetilde{f}(z), \quad \text{for every } \ z \in \mathbb{R}^{n(p+1)},$$

then the output function $y(i)$ is described in terms of the new state variable $\Lambda(i)$, i.e

$$y(i) = \widetilde{C}\Lambda(i) = \widetilde{C}\widetilde{\Psi}^i(\beta).$$

Thus, the set of all output ρ-admissible initial states is formally given by

$$\widetilde{\Gamma}^\rho(\Omega) = \left\{\beta \in B(0,\rho) \cap \mathbb{R}^{n(p+1)} \Big/ \widetilde{C}\,\widetilde{\Psi}^i(\beta) \in \Omega, \quad \forall i \in \mathbb{N}\right\}. \tag{14}$$

In order to characterize the maximal output sets given formally by (14), we define for each integer k the set

$$\widetilde{\Gamma}^\rho_k(\Omega) = \left\{\beta \in B(0,\rho) \cap \mathbb{R}^{n(p+1)} \Big/ \widetilde{C}\,\widetilde{\Psi}^i(\beta) \in \Omega, \quad \forall i \in \{0,1,\ldots,k\}\right\}$$

and then we can use results of Theorems 1 and 2 of Section 5 to characterize the set $\widetilde{\Gamma}^\rho(\Omega)$ described by Eq. (14).

8. Conclusion

In this work we have characterirezed the maximal output ρ-admissible set $\Gamma^\rho(\Omega)$ for a class of discrete semilinear system in the case of uncontrolled and controlled system. Sufficients conditions to ensure the finitely determined of such set have given. In addition to that we have proposed an algorithm approach to verify if $\Gamma^\rho(\Omega)$ is finitely determined. The characterization of maximal output ρ-admissible set for semilinear discrete delayed system has taken into consideration. To illustrate our results various examples have given. In the future one can characterize the set $\Gamma^\rho(\Omega)$ for a linear or nonlinear system in case of infinite dimensional spaces.

References

[1] A. Feuer and M. Heymann: Admissible sets in linear feedback systems with bounded controls, *International Journal of Control*, **23**(3) (1976), 381–392, DOI: 10.1080/00207177608922165.

[2] M. Cwikel and P.-O. Gutman: Convergence of an algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states, in Pror. 24th IEEE Conf. Derision Conrr. Fort Lauderdale. FL, Dec. 1985; also in *IEEE Trans. Automat. Contr.*, **31**(5) (1986), 457–459.

[3] E.G. Gilbert and K. Tin Tan: Linear Systems with State and Control con straints: The Theory and Application of Maximal Output Admissible Sets, *IEEE Trans. Automat. Contr.*, **36**(9) (1991), 1008–1020.

[4] F. Tahir and I.M. Jaimoukha: Robust Positively Invariant Sets for Linear Systems subject to model-uncertainty and disturbances, 4th IFAC Nonlinear Model Predictive Control Conference International Federation of Automatic Control Noordwijkerhout, NL, August 23-27, 2012.

[5] J. Bouyaghroumni, A. El Jai, and M. Rachik: Admissible disturbances set for discrete perturbed systems, *Int. J. Appl. Math. Comput. Sci.*, **11**(2) (2001), 349–367.

[6] J. Osorio and H.R. Ossareh: A Stochastic Approach to Maximal Output Admissible Sets andReference Governors, *Control Technology and Applications (CCTA) 2018IEEE Conference*, pp. 704–709, 2018.

[7] K. Hirata and Y. Ohta: Exact determinations of the maximal output admissible set for a class of nonlinear systems, *Automatica*, **44**(2) (2008), 526–533.

[8] I. Kolmanovsky and E.G. Gilbert: Maximal output admissible sets for discrete-time systems with disturbance inputs, *Proceedings of 1995 American Control Conference-ACC'95*, Vol. 3. IEEE, 1995.

[9] I. Kolmanovsky and E.G. Gilbert: Theory and computation of disturbance invariant sets for discrete-time linear systems, *Mathematical Problems in Engineering*, **4** (1998), 317–367.

[10] M. Rachik, E. Labriji, A. Abkari, and J. Bouyaghroumni: Infected discrete linear systems: On the admissible sources, *Optimization*, **48** (2000) 271–289.

[11] M. Rachik, A. Abdelhak, and J. Karrakchou: Discrete systems with delays in state, control and observation: The maximal output sets with state and control constraints. *Optimization*, **42** (1997), 169–183.

[12] M.S. Darup and M. Mönnigmann: Computation of the Largest Constraint Admissible Set for Linear Continuous-Time Systems with State and Input Constraints, *Proceedings of the 19th World Congress The International Federation of Automatic ControlCape Town*, South Africa. August 24–29, 2014.

[13] P. Kapasouri, M. Athans, and G. Stein: Design of feedback control systems for stable plants with saturating actuator, In: *Proc. 27th Conf. Decision Contr.*, Austin, TX (1988), pp. 469–479.

[14] P.O. Gutman and M. Cwikel: An algorithm to find maximal state con straint sets for discrete-time linear dynamical systems with bounded con trols and states, *IEEE Trans. Automat. Contr.*, **32**(3) (1987), 251–254.

[15] P.O. Gutman and P. Hagander: A new design of constrained controllers for linear systems. *IEEE Trans. Automat. Contr.*, **30**(1) (1985), 22–23.

[16] S.S. Keerthi: Optimal feedback control of discrete-time systems with state-control constrains and general cost functions, Ph.D. dissertation, Computer Informat. Contr. Eng., Univ. Michigan, Ann Arbor, MI, 1986.

[17] S.S. Keerthi and E.G. Gilbet: Optimal infinite horizon feedback laws for a general class of constrained discrete-time systems, Stability and moving horizon approximations, *J. Optimiz. Theory Appl.*, **57**(2) (1988), 265–293.