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Ordered Pure Multi-Pushdown Automata

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Abstract In the presented paper we discuss pure versions of pushdown automata that have no extra non-input symbols. More specifically, we study pure multi-pushdown automata, which have several pushdown lists. We restrict these automata by the total orders defined over their pushdowns or alphabets and determine the accepting power of the automata restricted in this way. Moreover, we explain the significance of the achieved results and relate them to some other results in the automata theory.

Keywords pure multi-pushdown automata; total orders; accepting power;

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1 INTRODUCTION

Formal language theory introduced pure grammars — grammars that have only terminal symbols without any extra non-terminal symbols [1, 2, 3] — several decades ago, and it has investigated them since. However, while most language-generating grammatical models have their accepting counterparts based upon automata, pure grammars lacked them. Therefore, to fill this gap, the language theory has recently introduced and discussed pure versions of pushdown automata — pushdown automata that do not possess any extra pushdown non-input symbols [4, 5] — because their classical versions, in which non-input symbols may occur, fulfil a crucially important role in automata theory and its applications, such as parsing.

More specifically, multi-pushdown automata, which may have several pushdown lists, were investigated intensively [6, 7, 8, 9, 10, 11, 12], including their pure versions [4, 5]. The presented paper continues the study of pure multi-pushdown automata by introducing and investigating three new versions.

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First, we discuss ordinary pure multi-pushdown automata. It is proved that their one-pushdown versions characterize the family of context-free languages, while their two-or-more-pushdown versions are computational complete — they are as powerful as Turing machines.

Second, we restrict the way these automata work by introducing total order \preceq over their pushdowns. In essence, during any computation, after using pushdown i , only pushdown j satisfying $i \preceq j$ can be used throughout the rest of the computation. The acceptance of an input is successfully completed by emptying all its pushdowns and entering a final state. The paper demonstrates that these pushdown automata with any number of pushdown lists characterize the family of context-free languages.

Finally, we introduce and study the total order \trianglelefteq over the alphabets of these automata. We require that during any computation, the symbols of any pushdown string are ordered according to \trianglelefteq . The study proves that the automata restricted in this way define only a proper subfamily of the family of context-free languages.

The above-mentioned concepts and results are of some interest in view of other studies on the same subject, including the publications summarized next. Multi-pushdown automata have been defined and studied in [13]; additionally, this paper have studied their stateless versions. In [5], pure multi-pushdown automata that perform complete pushdown pops are introduced and studied. Papers [14, 15, 16, 17] from the area of formal verification study multi-pushdown automata with ordered pushdowns, where the pop operation can be performed only on the first nonempty pushdown.

2 PRELIMINARIES

We assume that the reader is familiar with formal language theory (see [18, 19, 20]). For every positive integer n , let I_n denote the set $\{1, 2, \dots, n\}$. For a set P , $\text{card}(P)$ denotes the cardinality of P . A binary relation \preceq over P is a *total order* if and only if \preceq is antisymmetric, transitive, and total. For an alphabet (finite nonempty set) V , V^* represents the free monoid generated by V under the operation of concatenation. The unit of V^* is denoted by ε . Set $V^+ = V^* - \{\varepsilon\}$; algebraically, V^+ is thus the free semigroup generated by V under the operation of concatenation. If $\text{card}(V) = 1$, then V is a *unary alphabet*. For $x \in V^*$, $|x|$ denotes the length of x , $\text{reversal}(x)$ denotes the reversal (mirror image) of x , and $\text{alph}(x)$ denotes the set of symbols occurring in x . For $K \subseteq V^*$, we define $\text{alph}(K) = \bigcup_{x \in K} \text{alph}(x)$. If $\text{card}(\text{alph}(K)) = 1$, then K is a *unary language*.

A *pushdown automaton (PDA)* is a septuple

$$M = (Q, \Sigma, \Gamma, R, s, Z, F),$$

where Q is a finite set, Σ is an alphabet such that $Q \cap \Sigma = \emptyset$, Γ is an alphabet such that $\Sigma \subset \Gamma$, $R \subseteq \Gamma^* \times Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma^* \times Q$ is a finite relation, $Z \in \Gamma - \Sigma$, $s \in Q$, and $F \subseteq Q$. The components Q , Σ , Γ , R , s , Z , and F are called the set of *states*, the *input alphabet*, the *pushdown alphabet*, the set of *rules*, the *start state*, the *initial pushdown symbol*, and the set of *final states*, respectively. Instead of $(z, p, a, w, q) \in R$, we write $zpa \rightarrow wq \in R$ throughout the paper. Let $\# \notin Q \cup \Sigma \cup \Gamma$ be a *bottom marker*. The *direct move relation* over $\{\#\}\Gamma^*Q\Sigma^*$, symbolically denoted by \vdash_M , is defined as follows: $\#yzpax \vdash_M \#ywqax$ in M if and only if $\#yzpax, \#ywqax \in \{\#\}\Gamma^*Q\Sigma^*$

and $zpa \rightarrow wq \in R$. Let \vdash_M^m and \vdash_M^* denote the m th power of \vdash_M , for some $m \geq 1$, and the reflexive-transitive closure of \vdash_M , respectively. The *language accepted by M* is denoted by $L(M)$ and defined as

$$L(M) = \{w \in \Sigma^* \mid \#Zsw \vdash_M^* \#f, f \in F\}$$

A *two-pushdown automaton (2-PDA)* [11] is an octuple

$$M = (Q, \Sigma, \Gamma, R, s, Z_1, Z_2, F).$$

where Q, Σ, Γ, s , and F are defined as in a pushdown automaton, $R \subseteq \Gamma \times \Gamma \times Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma^* \times \Gamma^* \times Q$ is a finite relation, and $Z_1, Z_2 \in \Gamma - \Sigma$. The components $Q, \Sigma, \Gamma, R, s, Z_1, Z_2$, and F are called the set of *states*, the *input alphabet*, the *pushdown alphabet*, the set of *rules*, the *start state*, the *initial symbol of pushdown 1*, the *initial symbol of pushdown 2*, and the set of *final states*, respectively. Instead of $(A_1, A_2, p, a, w_1, w_2, q) \in R$, we write $A_1|A_2pa \rightarrow w_1|w_2q \in R$ throughout the paper. Let $\# \notin Q \cup \Sigma \cup \Gamma$ be a *bottom marker*. The *direct move relation* over $\{\#\}^* \Gamma^* \{\#\}^* \Gamma^* Q \Sigma^*$, symbolically denoted by \vdash_M , is defined as follows: $\#y_1A_1\#y_2A_2pax \vdash_M \#y_1w_1\#y_2w_2qx$ in M if and only if $\#y_1A_1\#y_2A_2pax, \#y_1w_1\#y_2w_2qx \in \{\#\}^* \Gamma^* \{\#\}^* \Gamma^* Q \Sigma^*$ and $A_1|A_2pa \rightarrow w_1|w_2q \in R$. Let \vdash_M^m and \vdash_M^* denote the m th power of \vdash_M , for some $m \geq 1$, and the reflexive-transitive closure of \vdash_M , respectively. The *language accepted by M* is denoted by $L(M)$ and defined as

$$L(M) = \{w \in \Sigma^* \mid \#Z_1\#Z_2sw \vdash_M^* \#\#f, f \in F\}.$$

The families of regular languages, context-free languages, and recursively enumerable languages are denoted by **RG**, **CF**, and **RE**, respectively. Recall that pushdown automata characterize **CF** and that two-pushdown automata characterize **RE**.

For every unary language L , $L \in \mathbf{RG}$ [21], there exists a finite automaton accepting L — there is no need for any pushdown storage. Thus, in what follows, we suppose every language is non-unary.

3 PURE MULTI-PUSHDOWN AUTOMATA

In this section, we define pure multi-pushdown automata and prove that with a single pushdown, they characterize the family of context-free languages. Then, we prove that pure pushdown automata with two or more pushdowns characterize the family of recursively enumerable languages — that is, they are Turing complete.

3.1 DEFINITIONS AND EXAMPLES

First, we define pure multi-pushdown automata. Next, we illustrate them by an example.

Definition 1. A *pure n -pushdown automaton (n -PPDA)*, where $n \geq 1$, is a $(5 + n)$ -tuple

$$M = (Q, \Sigma, R, s, Z_1, Z_2, \dots, Z_n, F),$$

where

- Q is a finite set of *states*;
- Σ is the *input alphabet* ($Q \cap \Sigma = \emptyset$);
- R is a finite set of *rules* of the form

$$izpa \rightarrow wq,$$

where $i \in I_n$, $p, q \in Q$, and $z, w \in \Sigma^*$;

- $s \in Q$ is the *start state*;
- $Z_i \in \Sigma$ is the *initial symbol of pushdown* i , for $i = 1, 2, \dots, n$;
- $F \subseteq Q$ is a set of *final states*. □

Definition 2. Let $M = (Q, \Sigma, R, s, Z_1, Z_2, \dots, Z_n, F)$ be an n -PPDA, for some $n \geq 1$. Let $\# \notin Q \cup \Sigma$ be a *bottom marker*. The *direct move relation* over $(\{\#\}\Gamma^*)^n Q \Sigma^*$ is denoted by \vdash_M and defined as follows:

$$\#y_1z_1\#y_2z_2 \cdots \#y_nz_n p a x \vdash_M \#y_1w_1\#y_2w_2 \cdots \#y_nw_n q x$$

if and only if

$$\#y_1z_1\#y_2z_2 \cdots \#y_nz_n p a x, \#y_1w_1\#y_2w_2 \cdots \#y_nw_n q x \in (\{\#\}\Gamma^*)^n Q \Sigma^*,$$

$iz_i p a \rightarrow w_i q \in R$ for some $i \in I_n$, and $w_j = z_j$ for $j \in I_n - \{i\}$. Let \vdash_M^m and \vdash_M^* denote the m th power of \vdash_M , for some $m \geq 1$, and the reflexive-transitive closure of \vdash_M , respectively. □

Definition 3. Let $M = (Q, \Sigma, R, s, Z_1, Z_2, \dots, Z_n, F)$ be an n -PPDA, for some $n \geq 1$. The *language accepted by* M , denoted by $L(M)$, is defined as

$$L(M) = \{w \in \Sigma^* \mid \#Z_1\#Z_2 \cdots \#Z_n s w \vdash_M^* \underbrace{\#\#\dots\#}_n f, f \in F\}. \quad \square$$

We illustrate the previous definitions by the following example.

Example 1. Consider the 2-PPDA

$$M = (\{s_0, s_1, q_0, q_1, q_2, q_3, f_0, f_1\}, \{a, b, c\}, R, s_0, c, c, \{f_1\}),$$

where R contains the following rules:

$$\begin{array}{ll} 1cs_0 \rightarrow cs_1 & 1acq_1c \rightarrow cq_2 \\ 2cs_1 \rightarrow ccq_0 & 2bcq_2 \rightarrow cq_3 \\ 1cq_0a \rightarrow acq_0 & 1acq_3c \rightarrow cq_2 \\ 2cq_0b \rightarrow bcq_1 & 1ccq_3 \rightarrow f_0 \\ 2cq_1b \rightarrow bcq_1 & 2ccf_0 \rightarrow f_1 \end{array}$$

To give an insight into the way M works, observe that M uses the first pushdown to store as . Then, it makes use of the second pushdown to store bs . Finally, it compares the number of input cs with the contents of both pushdowns. Clearly, $L(M) = \{a^n b^n c^n \mid n \geq 1\}$. For example, the string $aabbcc$ is accepted by M in the following way:

$$\begin{array}{l}
 \#c\#cs_0aabbcc \vdash_M \#cc\#cs_1aabbcc \vdash_M \#cc\#ccq_0aabbcc \vdash_M \\
 \#cac\#ccq_0abbcc \vdash_M \#caac\#ccq_0bbcc \vdash_M \#caac\#cbcq_1bcc \vdash_M \\
 \#caac\#cbbcq_1cc \vdash_M \#cac\#cbbcq_2c \vdash_M \#cac\#cbcq_3c \vdash_M \\
 \#cc\#cbcq_2 \vdash_M \#cc\#ccq_3 \vdash_M \#\#ccf_0 \vdash_M \\
 \#\#f_1
 \end{array}$$

□

For $n \geq 1$, let n **PPDA** denote the family of languages accepted by n -PPDAs. Set

$$\mathbf{PMPDA} = \bigcup_{i=1}^{\infty} i\mathbf{PPDA}.$$

3.2 ACCEPTING POWER

In this section, we prove that 1-PPDAs characterize the family of context-free languages, and that 2-PPDAs characterize the family of recursively enumerable languages.

Lemma 1. *Let K be a context-free language. Then, there is a 1-PPDA, $M = (Q, \text{alph}(K), R, s, Z, F)$, such that $L(M) = K$.*

Proof. For any context-free language K , there exists a PDA M such that $L(M) = K$. Next, we show how to simulate M by a 1-PPDA. Let

$$M = (Q, \Sigma, \Gamma, R, s, Z, F)$$

be the PDA with $\Sigma = \text{alph}(K)$. Without any loss of generality, suppose that $a, b \in \Sigma$, $a \neq b$. Let $\Sigma \cup \Gamma = \{c_0, c_1, c_2, \dots, c_n\}$, where $n = \text{card}((\Sigma \cup \Gamma) - \{Z\})$ and $c_0 = Z$. Define the homomorphism τ from $(\Sigma \cup \Gamma)^*$ to $\{a, b\}^*$ as $\tau(c_i) = ab^i$, for $i = 0, 1, 2, \dots, n$. Construct the 1-PPDA

$$M' = (Q, \Sigma, R', s, \tau(Z), F),$$

where

$$R' = \{1\tau(z)pd \rightarrow \tau(w)q \mid zpd \rightarrow wq \in R\}.$$

Notice that $\tau(Z) = a$. To prove that $L(M) = L(M')$, we establish two claims. Claim 1 demonstrates how M' simulates M . Claim 2 demonstrates the converse simulation.

Claim 1. If $\#Zsw \vdash_M^k \#uqv$, where $q \in Q$, $v, w \in \Sigma^*$, $u \in \Gamma^*$, for some $k \geq 0$, then $\#\tau(Z)sw \vdash_{M'}^* \#\tau(u)qv$.

Proof. This claim is established by induction on $k \geq 0$.

Basis. Let $k = 0$. Then, for $\#Zsw \vdash_M^0 \#Zsw$ with $w \in \Sigma^*$, there is $\#\tau(Z)sw \vdash_{M'}^0 \#\tau(Z)sw$, so the basis holds.

Induction Hypothesis. Suppose that there exists $k \geq 0$ such that the claim holds for all sequences of moves of length m , where $0 \leq m \leq k$.

Induction Step. Consider any sequence of moves

$$\#Zsw \vdash_M^{k+1} \#u'q'v',$$

where $w, v' \in \Sigma^*$, $u' \in \Gamma^*$, $q' \in Q$. Since $k + 1 \geq 1$, this sequence can be written in the form

$$\#Zsw \vdash_M^k \#uqv \vdash_M \#u'q'v',$$

where $u \in \Gamma^*$, $q \in Q$, $v \in \Sigma^*$. Then, there exists a rule $xqa \rightarrow x'q' \in R$, where $u = yx$, $u' = yx'$ and $v = av'$, which was used to perform the $(k + 1)$ th move. By the construction of M' , there exists a corresponding rule $1\tau(x)qa \rightarrow \tau(x')q' \in R'$. By the induction hypothesis,

$$\#\tau(Z)sw \vdash_{M'}^* \#\tau(u)qv.$$

Thus, by using $1\tau(x)qa \rightarrow \tau(x')q'$, M' can make the move

$$\#\tau(u)qv \vdash_{M'} \#\tau(u')q'v'.$$

Notice that $\tau(u) = \tau(y)\tau(x)$. The resulting configuration of M' corresponds to the new configuration of M and the claim holds. \square

Claim 2. If $\#\tau(Z)sw \vdash_{M'}^k \#\tau(u)qv$, where $q \in Q$, $v, w \in \Sigma^*$, $u \in \Gamma^*$, for some $k \geq 0$, then $\#Zsw \vdash_M^* \#uqv$.

Proof. This claim is established by induction on $k \geq 0$.

Basis. Let $k = 0$. Then, for $\#\tau(Z)sw \vdash_{M'}^0 \#\tau(Z)sw$, where $w \in \Sigma^*$, there is $\#Zsw \vdash_M^0 \#Zsw$, so the basis holds.

Induction Hypothesis. Suppose that there exists $k \geq 0$ such that the claim holds for all sequences of moves of length m , where $0 \leq m \leq k$.

Induction Step. Consider any sequence of moves

$$\#\tau(Z)sw \vdash_{M'}^{k+1} \#\tau(u')q'v',$$

where $w, v' \in \Sigma^*$, $u' \in \Gamma^*$, $q' \in Q$. Since $k + 1 \geq 1$, this sequence can be written in the form

$$\#\tau(Z)sw \vdash_{M'}^k \#\tau(u)qv \vdash_{M'} \#\tau(u')q'v',$$

where $u \in \Gamma^*$, $q \in Q$, $v \in \Sigma^*$. Then, there exists a rule $1\tau(x)qa \rightarrow \tau(x')q' \in R'$, where $\tau(u) = \tau(y)\tau(x)$, $\tau(u') = \tau(y)\tau(x')$, and $v = av'$, which was used to perform the $(k + 1)$ th move.

By the construction of M' , this rule was introduced from a rule $xqa \rightarrow x'q' \in R$. By the induction hypothesis,

$$\#Zsw \vdash_M^* \#uqv.$$

Thus, by using $xqa \rightarrow x'q'$, M can make the move

$$\#uqv \vdash_M \#u'q'v'.$$

The resulting configuration of M corresponds to the new configuration of M' and the claim holds. \square

Consider a special case of Claim 1 when $u = v = \varepsilon$ and $q \in F$. Then, M accepts w . Since $\tau(u) = \varepsilon$, M' also accepts w . So, $L(M) \subseteq L(M')$.

Similarly, in the special case of Claim 2 when $\tau(u) = v = \varepsilon$ and $q \in F$, M' accepts w . Since $u = \varepsilon$, M also accepts w . So, $L(M') \subseteq L(M)$.

Hence, $L(M) = L(M')$, and the lemma holds. \square

Theorem 1. ${}_1\text{PPDA} = \text{CF}$

Proof. The inclusion $\text{CF} \subseteq {}_1\text{PPDA}$ follows from Lemma 1. The opposite inclusion, ${}_1\text{PPDA} \subseteq \text{CF}$, follows directly from the definitions of 1-PPDAs and PDAs. Hence, ${}_1\text{PPDA} = \text{CF}$, so the theorem holds. \square

Next, we turn our attention to 2-PPDAs.

Lemma 2. *Let K be a recursively enumerable language. Then, there is a 2-PPDA, $M = (Q, \text{alph}(K), R, s, Z_1, Z_2, F)$, such that $L(M) = K$.*

Proof. For any recursively enumerable language K , there exists a 2-PDA M such that $L(M) = K$. We introduce a 2-PPDA M' that simulates M . Let

$$M = (Q, \Sigma, \Gamma, R, s, Z_1, Z_2, F)$$

be the 2-PDA with $\Sigma = \text{alph}(K)$. Without any loss of generality, suppose that $a, b \in \Sigma$, $a \neq b$ and i th rule in R is labeled with r_i , for $i \in I_{\text{card}(R)}$. Let $\Sigma \cup \Gamma \cup \{Z\} = \{c_0, c_1, c_2, \dots, c_n\}$, where $n = \text{card}(\Sigma \cup \Gamma)$, $Z \notin \Sigma \cup \Gamma$, and $c_0 = Z$. Define the homomorphism τ from $(\Sigma \cup \Gamma \cup \{Z\})^*$ to $\{a, b\}^*$ as $\tau(c_i) = ab^i$, for $i = 0, 1, \dots, n$. Construct the 2-PPDA

$$M' = (Q', \Sigma, R', s', \tau(Z), \tau(Z), F)$$

as follows. Initially, set $Q' = Q \cup \{s', s''\}$ ($s', s'' \notin Q$) and $R' = \emptyset$. Perform (1) and (2), given next:

(1) add $1\tau(Z)s' \rightarrow \tau(Z_1)s''$, $2\tau(Z)s'' \rightarrow \tau(Z_2)s$ to R' ;

(2) for each $r_i : a|bpd \rightarrow u|wq \in R$,

(a) add r_i to Q' ,

(b) add $1\tau(a)pd \rightarrow \tau(u)r_i, 2\tau(b)r_i \rightarrow \tau(w)q$ to R' .

Before proving that $L(M) = L(M')$, let us give an insight into the construction. Every computation of M' begins by performing two moves by using a special starting rules from (1). Since the homomorphism τ encodes the start symbols Z_1 and Z_2 with the strings containing more than one symbol, M' uses this starting rules to push $\tau(Z_1)$ and $\tau(Z_2)$ onto its pushdowns. Therefore, the resulting configuration corresponds to the starting configuration of M . Then, for every rule from R , in (2), we add two new rules to R' , which act in a similar way but use the homomorphism τ for encoding the pushdown symbols. M use both pushdowns simultaneously in every move, while M' can use only one. Therefore, every rule $r_i \in R$ is simulated with two rules.

To prove the identity $L(M) = L(M')$, we establish two claims. Claim 3 demonstrates how M' simulates M . Claim 4 shows the converse simulation.

Claim 3. If $\#Z_1\#Z_2sw \vdash_M^k \#u_1\#u_2qv$, where $q \in Q, w, v \in \Sigma^*, u_1, u_2 \in \Gamma^*$, for some $k \geq 0$, then $\#\tau(Z)\#\tau(Z)s'w \vdash_{M'}^* \#\tau(u_1)\#\tau(u_2)qv$.

Proof. This claim is established by the induction on $k \geq 0$.

Basis. Let $k = 0$. Then, for $\#Z_1\#Z_2sw \vdash_M^0 \#Z_1\#Z_2sw$, where $w \in \Sigma^*$, there is

$$\#\tau(Z)\#\tau(Z)s'w \vdash_{M'} \#\tau(Z_1)\#\tau(Z)s'' \vdash_{M'} \#\tau(Z_1)\#\tau(Z_2)sw$$

so the basis holds.

Induction Hypothesis. Suppose that there exists $k \geq 0$ such that the claim holds for all sequences of moves of length m , where $0 \leq m \leq k$.

Induction Step. Consider any sequence of moves

$$\#Z_1\#Z_2sw \vdash_M^{k+1} \#u'_1\#u'_2q'v',$$

where $w, v' \in \Sigma^*, u'_1, u'_2 \in \Gamma^*, q' \in Q$. Since $k + 1 \geq 1$, this sequence can be written in the form

$$\#Z_1\#Z_2sw \vdash_M^k \#u_1\#u_2qv \vdash_M \#u'_1\#u'_2q'v',$$

where $v \in \Sigma^*, u_1, u_2 \in \Gamma^*, q \in Q$. Then, there exists a rule

$$r_i : x_1|x_2qa \rightarrow x'_1|x'_2q' \in R,$$

where $u_1 = y_1x_1, u'_1 = y_1x'_1, u_2 = y_2x_2, u'_2 = y_2x'_2, v = av'$, which was used to perform the $(k + 1)$ th move. By the construction of M' , there exist two corresponding rules

$$1\tau(x_1)qa \rightarrow \tau(x'_1)r_i, 2\tau(x_2)r_i \rightarrow \tau(x'_2)q' \in R'.$$

By the induction hypothesis,

$$\#\tau(Z)\#\tau(Z)s'w \vdash_{M'}^* \#\tau(u_1)\#\tau(u_2)qv.$$

Consequently, by using previous two rules M' can make the moves

$$\#\tau(u_1)\#\tau(u_2)qv \vdash_{M'} \#\tau(u'_1)\#\tau(u_2)r_iv' \vdash_{M'} \#\tau(u'_1)\#\tau(u'_2)q'v',$$

and the resulting configuration of M' corresponds to the new configuration of M . Therefore, the claim holds. \square

In Claim 4, without any loss of generality, we do not consider any sequence of moves of M' of odd length. From the construction of M' , every odd move leads to an intermediate state, while simulating one move of M , which is never final. Additionally, there is always exactly one applicable rule, so the computation never ends after odd number of moves, neither fails nor accepts.

Claim 4. If $\#\tau(Z)\#\tau(Z)s'w \vdash_{M'}^{2k} \#\tau(u_1)\#\tau(u_2)qv$, where $q \in Q$, $w, v \in \Sigma^*$, $u_1, u_2 \in \Gamma^*$, for some $k \geq 1$, then $\#Z_1\#Z_2sw \vdash_M^* \#u_1\#u_2qv$.

Proof. This claim is established by the induction on $k \geq 1$.

Basis. Let $k = 1$. By the construction of M' , at the beginning of every computation, there is the single applicable rule

$$1\tau(Z)s' \rightarrow \tau(Z_1)s'' \in R',$$

and by its application M' performs the move

$$\#\tau(Z)\#\tau(Z)s'w \vdash_{M'} \#\tau(Z_1)\#\tau(Z)s''w$$

where $w \in \Sigma^*$. In state s'' , there is also the single applicable rule

$$2\tau(Z)s'' \rightarrow \tau(Z_2)s \in R'.$$

Thus, M' performs the move

$$\#\tau(Z_1)\#\tau(Z)s''w \vdash_{M'} \#\tau(Z_1)\#\tau(Z_2)sw.$$

Then, there is

$$\#Z_1\#Z_2sw \vdash_M^0 \#Z_1\#Z_2sw.$$

Therefore, the basis holds.

Induction Hypothesis. Suppose that there exists $k \geq 1$ such that the claim holds for all sequences of moves of length m , where $1 \leq m \leq k$.

Induction Step. Consider any sequence of moves

$$\#\tau(Z)\#\tau(Z)s'w \vdash_{M'}^{2(k+1)} \#\tau(u'_1)\#\tau(u'_2)q'v',$$

where $w, v' \in \Sigma^*$, $u'_1, u'_2 \in \Gamma^*$, $q' \in Q$. Since $k + 1 \geq 1$, this sequence can be written in the form

$$\#\tau(Z)\#\tau(Z)s'w \vdash_{M'}^{2k} \#\tau(u_1)\#\tau(u_2)qv \vdash_{M'},$$

$$\#\tau(u'_1)\#\tau(u_2)r_i v' \vdash_{M'} \#\tau(u'_1)\#\tau(u'_2)q'v',$$

where $v \in \Sigma^*$, $u_1, u_2 \in \Gamma^*$, $q \in Q$, $i \in I_{\text{card}(R)}$. Then, there exist a rule

$$1\tau(x_1)qa \rightarrow \tau(x'_1)r_i, 2\tau(x_2)r_i \rightarrow \tau(x'_2)q' \in R',$$

with the identities $\tau(u_1) = \tau(y_1)\tau(x_1)$, $\tau(u'_1) = \tau(y_1)\tau(x'_1)$, $\tau(u_2) = \tau(y_2)\tau(x_2)$, $\tau(u'_2) = \tau(y_2)\tau(x'_2)$, and $v = av'$, which was used to perform the $(k + 1)$ th and $(k + 2)$ th move. By the construction of M' , these rules were introduced by a rule

$$x_1|x_2qa \rightarrow x'_1|x'_2q' \in R.$$

By the induction hypothesis,

$$\#Z_1\#Z_2sw \vdash_M^k \#u_1\#u_2qv.$$

Thus, by using $x_1|x_2qa \rightarrow x'_1|x'_2q'$, M can make the move

$$\#u_1\#u_2qv \vdash_M \#u'_1\#u'_2q'v'.$$

The resulting configuration of M corresponds to the new configuration of M' and the claim holds. \square

Consider a special case of Claim 3, where $u_1 = u_2 = v = \varepsilon$ and $q \in F$. Then, M accepts w . Since $\tau(u_1) = \tau(u_2) = \varepsilon$, M' also accepts w . So, $L(M) \subseteq L(M')$.

Similarly, in a special case of Claim 4, where $\tau(u_1), \tau(u_2), v = \varepsilon$ and $q \in F$, M' accepts w . Since $u_1, u_2 = \varepsilon$, M accepts w as well. So, $L(M') \subseteq L(M)$.

Hence, $L(M) = L(M')$, and the lemma holds. \square

Theorem 2. ${}_2\text{PPDA} = \text{RE}$

Proof. The inclusion $\text{RE} \subseteq {}_2\text{PPDA}$ follows from Lemma 2. The opposite inclusion, ${}_2\text{PPDA} \subseteq \text{RE}$, can be obtained by standard simulations. Hence, ${}_2\text{PPDA} = \text{RE}$, so the theorem holds. \square

From Theorem 2, we obtain the following corollary.

Corollary 1. $\text{PMPDA} = \text{RE}$ \square

4 PURE MULTI-PUSHDOWN AUTOMATA WITH ORDERED PUSHDOWNS

In the previous section, we studied pure multi-pushdown automata. In the present section, we restrict the way they use their pushdowns. We introduce a total order \preceq over their pushdowns. After using the i th pushdown, the automaton can work only with the j th pushdown, where $i \preceq j$. We prove that these restricted versions of pure multi-pushdown automata characterize the family of context-free languages independently of the number of their pushdowns.

4.1 DEFINITIONS AND EXAMPLES

Next, we define pure multi-pushdown automata with ordered pushdowns and illustrate them by an example.

Definition 4. For $n \geq 1$, a *pure n -pushdown-ordered multi-pushdown automaton* (n -PPOPDA) is a pair

$$\mathcal{H} = (M, \preceq),$$

where M is an n -PPDA, and \preceq is a total order over I_n . \square

Definition 5. Let $\mathcal{H} = (M, \preceq)$ be an n -PPOPDA, for some $n \geq 1$, where $M = (Q, \Sigma, R, s, Z_1, Z_2, \dots, Z_n, F)$. The *direct move relation* over $I_n(\{\#\}\Gamma^*)^n Q \Sigma^*$ is denoted by $\vdash_{\mathcal{H}}$ and defined as follows:

$$k\#y_1z_1\#y_2z_2\cdots\#y_nz_npax \vdash_{\mathcal{H}} i\#y_1w_1\#y_2w_2\cdots\#y_nw_nqx,$$

if and only if

$$k\#y_1z_1\#y_2z_2\cdots\#y_nz_npax, i\#y_1w_1\#y_2w_2\cdots\#y_nw_nqx \in I_n(\{\#\}\Gamma^*)^n Q \Sigma^*,$$

$iz_i pa \rightarrow w_i q \in R$, $k \preceq i$, and $w_j = z_j$ for $j \in I_n - \{i\}$. Let $\vdash_{\mathcal{H}}^m$ and $\vdash_{\mathcal{H}}^*$ denote the m th power of $\vdash_{\mathcal{H}}$, for some $m \geq 1$, and the reflexive-transitive closure of $\vdash_{\mathcal{H}}$, respectively. \square

Definition 6. Let $\mathcal{H} = (M, \preceq)$ be an n -PPOPDA, for some $n \geq 1$, where $M = (Q, \Sigma, R, s, Z_1, Z_2, \dots, Z_n, F)$. The *language accepted by \mathcal{H}* , denoted by $L(\mathcal{H})$, is defined as

$$L(\mathcal{H}) = \{w \in \Sigma^* \mid 1\#Z_1\#Z_2\cdots\#Z_nsw \vdash_{\mathcal{H}}^* \underbrace{n\#\#\dots\#f}_{n \text{ times}}, f \in F\}. \quad \square$$

We illustrate the previous definitions by the next example.

Example 2. Let $\mathcal{H} = (M, \preceq)$, where $M = (\{q\}, \{a, b, c, d\}, R, q, c, a, \{q\})$, be a 2-PPDA with $1 \preceq 2$ and R containing the following rules:

$1cqa \rightarrow caq$	$2aqc \rightarrow acq$
$1cqb \rightarrow cbq$	$2aqd \rightarrow adq$
$1aqa \rightarrow aaq$	$2cqc \rightarrow ccq$
$1bqb \rightarrow bbq$	$2dqd \rightarrow ddq$
$1aqb \rightarrow q$	$2cqd \rightarrow q$
$1bqa \rightarrow q$	$2dq c \rightarrow q$
$1cq \rightarrow q$	$2aq \rightarrow q$

First, \mathcal{H} checks the equality of the number of the occurrences of as and bs on the first pushdown and then cs and ds on the second pushdown. The language of \mathcal{H} is

$$L(\mathcal{H}) = \left\{ w \in \{a, b, c, d\}^* \mid w = w_1w_2, w_1 \in \{a, b\}^*, \#_a(w_1) = \#_b(w_1), w_2 \in \{c, d\}^*, \#_c(w_2) = \#_d(w_2) \right\},$$

where $\#_t(v)$ denotes the number of occurrences of t in v . Notice that there is no need to change a state to ensure a separation of w_1 and w_2 . Indeed, the pushdown order given by \preceq prevents a mixture of as and bs with cs and ds . For example, the string $abbacded$ is accepted by M in the following way:

$$\begin{array}{l} 1\#c\#aqabbacded \vdash_{\mathcal{H}} 1\#ca\#aqbbacded \vdash_{\mathcal{H}} 1\#c\#aqbacded \vdash_{\mathcal{H}} \\ 1\#cb\#aqacded \vdash_{\mathcal{H}} 1\#c\#aqcded \vdash_{\mathcal{H}} 1\#\#aqcded \vdash_{\mathcal{H}} \\ 2\#\#acqdcd \vdash_{\mathcal{H}} 2\#\#aqcd \vdash_{\mathcal{H}} 2\#\#acqd \vdash_{\mathcal{H}} \\ 2\#\#aq \vdash_{\mathcal{H}} 2\#\#q \end{array}$$

□

For $n \geq 1$, let n **PPOPDA** denote the family of languages accepted by n -PPOPDAs. Set

$$\mathbf{PPOMPDA} = \bigcup_{i=1}^{\infty} i\mathbf{PPOPDA}.$$

4.2 ACCEPTING POWER

In this section, we prove that n -PPOPDAs characterize the family of context-free languages.

Lemma 3. $n\mathbf{PPOPDA} \subseteq \mathbf{CF}$, for any $n \geq 1$.

Proof. We show that we can simulate any n -PPOPDA by a PDA, for any $n \geq 1$. Let $\mathcal{H} = (M, \preceq)$ be an n -PPOPDA, for some $n \geq 1$, where

$$M = (Q, \Sigma, R, s, Z_1, Z_2, \dots, Z_n, F).$$

Without any loss of generality, we assume that $1 \preceq 2 \preceq \dots \preceq n$. We introduce the PDA

$$M' = (Q', \Sigma, \Gamma, R', s', Z, \{f\}),$$

defined in the following way. Initially, set

$$\begin{aligned} Q' &= \{q_i \mid q \in Q, i \in I_n\} \cup \{s', f\} \quad (\{s', f\} \cap Q = \emptyset) \\ \Gamma &= \Sigma \cup \{Z\} \quad (Z \notin \Sigma) \\ R' &= \emptyset \end{aligned}$$

To finish the construction, perform (1) through (4), given next:

- (1) add $Zs' \rightarrow ZZ_1s_1$ to R' ;
- (2) for each $q \in Q$ and each $i \in I_{n-1}$, add $Zq_i \rightarrow ZZ_{i+1}q_{i+1}$ to R' ;
- (3) for each $q \in F$, add $Zq_n \rightarrow f$ to R' ;
- (4) for each $izpa \rightarrow wq \in R$, add $zpa \rightarrow wq_i$ to R' .

Before proving that $L(\mathcal{H}) = L(M')$, let us give an insight into the construction. \mathcal{H} uses its pushdowns in the order given by \preceq . Every configuration of \mathcal{H} keeps the information about the current working pushdown implicitly. M' keeps this information in its states. If the current state is q_i , M' simulates the work with the i th pushdown in the given order. There are two states with a special purpose: the initial s' and final f .

We use the new symbol Z as the bottom of the pushdown of M' . It is never removed except in the last terminating rule. It is also used to determine possible moves further in the simulation to the next pushdown. When Z is on the top of the pushdown, the currently simulated pushdown is empty. \mathcal{H} can make a move to the next pushdown even if the current one is not empty yet; however, no such computation could be accepting. Indeed, the future emptying of the pushdown is

not possible. M' blocks instead of simulating the unsuccessful computation. The only permitted move in the pushdown order is when the currently simulated pushdown is empty. Then, the simulation is possibly accepting.

For every rule from R , we introduce the corresponding rules to R' . Additionally, we define rules simulating the pushdown order. Every accepting computation of \mathcal{H} is divided into n phases, and in the i th phase, for $i \in I_n$, \mathcal{H} uses only the i th pushdown, with respect to \preceq . Therefore, the simulation is also divided into n phases. The i th phase, for $i \in I_n$, is simulated as follows:

- (I) Phase 1 starts with the rule from (1), phase i for $i > 1$ with the rule from (2). The start symbol Z_i of the i th pushdown of \mathcal{H} is inserted to the pushdown of M' . Notice that except the first phase, the previously simulated $(i - 1)$ th pushdown must be empty before. We determine this by checking that Z is on the top of the pushdown.
- (II) The simulation continues with rules from (4), where for a rule of the form $z p_j a \rightarrow w q_j$, it holds that $i = j$.
- (III) If the symbol Z is on the top of the pushdown, the simulated pushdown is empty and the simulation continues back to (I) except the phase n , where the simulation can successfully finish by using a rule from (3).

To prove the identity $L(\mathcal{H}) = L(M')$, we establish two claims. Claim 5 demonstrates how to simulate \mathcal{H} by M' . Claim 6 demonstrates the converse simulation.

In Claim 5, without any loss of generality, we do not consider any computation of \mathcal{H} , which manipulates the i th pushdown before emptying the j th one, where $i > j$. Since future emptying is not possible, such computation is obviously not accepting. This restriction is without any loss of generality. In Claim 5, we only show that M' covers all accepting computations of \mathcal{H} .

Claim 5. If $1 \# Z_1 \# Z_2 \# \dots \# Z_n s w \vdash_{\mathcal{H}}^k i \#^i u \# Z_{i+1} \# \dots \# Z_n q v$, where $w, v \in \Sigma^*$, $u \in \Gamma^*$, $q \in Q$, $i \in I_n$, and all j th pushdowns, with $j > i$, contain only the symbol Z_j , for some $k \geq 0$, then $\# Z s' w \vdash_{M'}^* \# Z u q_i v$.

Proof. This claim is established by induction on $k \geq 0$.

Basis. Let $k = 0$. Then, for

$$1 \# Z_1 \# Z_2 \# \dots \# Z_n s w \vdash_{\mathcal{H}}^0 1 \# Z_1 \# Z_2 \# \dots \# Z_n s w,$$

where $w \in \Sigma^*$, there is

$$\# Z s' w \vdash_{M'} \# Z Z_1 s_1 w.$$

Therefore, the basis holds.

Induction Hypothesis. Suppose that there exists $k \geq 0$ such that the claim holds for all sequences of moves of length m , where $0 \leq m \leq k$.

Induction Step. Consider any sequence of moves

$$1 \# Z_1 \# Z_2 \# \dots \# Z_n s w \vdash_{\mathcal{H}}^{k+1} j \#^j u' \# Z_{j+1} \# \dots \# Z_n q' v',$$

where $w, v' \in \Sigma^*$, $u' \in \Gamma^*$, $q' \in Q$. Since $k + 1 \geq 1$, this sequence of moves can be written in the form

$$\begin{array}{l} 1\#Z_1\#Z_2\#\dots\#Z_nsw \quad \vdash_{\mathcal{H}}^k \quad i\#^i u\#Z_{i+1}\#\dots\#Z_nqv \\ \vdash_{\mathcal{H}} \quad j\#^j u'\#Z_{j+1}\#\dots\#Z_nq'v' \end{array}$$

where $v \in \Sigma^*$, $u \in \Gamma^*$, $q \in Q$.

First, consider $i \neq j$. Observe that i and j denote the numbers of the current working pushdown. The order \preceq ensures that $i \leq j$, so, $i < j \leq n$. If $i + 1 < j$, the $(i + 1)$ th pushdown remains nonempty and the computation is not accepting. We do not consider such a computation. Summarily, $i + 1 = j$. The $(k + 1)$ th move was then performed by using a rule $jZ_jqa \rightarrow u'q' \in R$, where $v = av'$. If the computation may be accepting, $u = \varepsilon$. Then, by the induction hypothesis,

$$\#Zs'w \vdash_{M'}^* \#Zq_i v.$$

By the construction of M' , there exist two rules

$$Zq_i \rightarrow ZZ_{i+1}q_{i+1}, Z_{i+1}q_{i+1}a \rightarrow u'q'_{i+1} \in R'.$$

Thus, by using these rules, M' can make the moves

$$\#Zq_i v \vdash_{M'} \#ZZ_{i+1}q_{i+1}v \vdash_{M'} \#Zu'q'_{i+1}v',$$

and the resulting configuration corresponds to the new \mathcal{H} 's configuration.

Second, consider $i = j$. Then, we can rewrite the \mathcal{H} 's sequence of moves to the form

$$\begin{array}{l} 1\#Z_1\#Z_2\#\dots\#Z_nsw \quad \vdash_{\mathcal{H}}^k \quad i\#^i u\#Z_{i+1}\#\dots\#Z_nqv \\ \vdash_{\mathcal{H}} \quad i\#^i u'\#Z_{i+1}\#\dots\#Z_nq'v' \end{array}$$

The $(k + 1)$ th move was performed by a rule $ixqa \rightarrow x'q' \in R$, where $u = yx$, $u' = yx'$. By the construction of M' , there exists a rule from (4), $xq_ia \rightarrow x'q_i \in R'$. By the induction hypothesis,

$$\#Zs'w \vdash_{M'}^* \#Zuq_i v.$$

By using $xq_ia \rightarrow x'q_i$, M' can perform the move

$$\#Zuq_i v \vdash_{M'} \#Zu'q'_i v'.$$

The resulting configuration corresponds to the new \mathcal{H} 's configuration.

By combining the previous two cases, Claim 5 holds. \square

Claim 6. If $\#Zs'w \vdash_{M'}^k \#Zuq_i v$, where $w, v \in \Sigma^*$, $u \in \Gamma^*$, $q_i \in Q'$, $i \in I_n$, for some $k \geq 1$. Then, $1\#Z_1\#Z_2\#\dots\#Z_nsw \vdash_{\mathcal{H}}^* j\#^j u\#Z_{i+1}\#\dots\#Z_nqv$, $j \leq i$.

Proof. This claim is established by induction on $k \geq 1$.

Basis. Let $k = 1$. By the construction of M' , at the beginning of every computation, there is the single applicable rule

$$Zs' \rightarrow ZZ_1s_1 \in R',$$

and by its application M' performs the move

$$\#Zs'w \vdash_{M'} \#ZZ_1sw,$$

where $w \in \Sigma^*$. Then, there is

$$1\#Z_1\#Z_2\#\dots\#Z_nsw \vdash_{\mathcal{H}}^0 1\#Z_1\#Z_2\#\dots\#Z_nsw,$$

and the basis holds.

Induction Hypothesis. Suppose that there exists $k \geq 1$ such that the claim holds for all sequences of moves of length m , where $1 \leq m \leq k$.

Induction Step. Consider any sequence of moves

$$\#Zs'w \vdash_{M'}^{k+1} \#Zu'q'_jv',$$

where $w, v' \in \Sigma^*$, $u' \in \Gamma^*$, $q'_j \in Q'$. Since $k+1 \geq 1$, this sequence can be written in the form

$$\#Zs'w \vdash_{M'}^k \#Zuq_iv \vdash_{M'} \#Zu'q'_jv',$$

where $v \in \Sigma^*$, $u \in \Gamma^*$, $q_i \in Q'$. Next, we cover these two cases: $i = j$ and $i+1 = j$. By the construction of M' , there is no other possibility.

First, suppose that $i+1 = j$. Then, the $(k+1)$ th move of M' was performed by a rule from (2), which is of the form

$$Zq_i \rightarrow ZZ_{i+1}q_{i+1},$$

so, $u = \varepsilon$, $u' = Z_{i+1}$, $q' = q$, $v' = v$. Thus, the sequence of moves of M' can be written as

$$\#Zs'w \vdash_{M'}^k \#Zq_iv \vdash_{M'} \#ZZ_{i+1}q_{i+1}v.$$

By the induction hypothesis we have

$$1\#Z_1\#Z_2\#\dots\#Z_nsw \vdash_{\mathcal{H}}^* i\#^i u\#Z_{i+1}\#\dots\#Z_nqv.$$

Since $u = \varepsilon$, we can rewrite this sequence of moves to the form

$$1\#Z_1\#Z_2\#\dots\#Z_nsw \vdash_{\mathcal{H}}^* i\#^{i+1} Z_{i+1}\#\dots\#Z_nqv$$

which corresponds to the new configuration of M' .

Second, suppose that $i = j$. Then, the sequence of moves of M' is

$$\#Zs'w \vdash_{M'}^k \#Zuq_iv \vdash_{M'} \#Zu'q'_i v'.$$

The $(k+1)$ th move of M' was performed by a rule from (4), $xq_ia \rightarrow x'q'_i \in R'$, where $u = yx$, $u' = yx'$, $v = av'$. By the construction of M' , this rule was added to R' from a rule $ixqa \rightarrow x'q' \in R$. By the induction hypothesis,

$$1\#Z_1\#Z_2\#\dots\#Z_nsw \vdash_{\mathcal{H}}^* i\#^i u\#Z_{i+1}\#\dots\#Z_nqv.$$

Therefore, by using $ixqa \rightarrow x'q'$, \mathcal{H} can perform an additional move

$$i\#^i u\#Z_{i+1}\#\dots\#Z_nqv \vdash_{\mathcal{H}} i\#^i u'\#Z_{i+1}\#\dots\#Z_nq'v',$$

and enter the configuration corresponding to the new configuration of M' .

Hence, we covered both possible cases and the claim holds. \square

Consider a special case of Claim 5 when

$$1\#Z_1\#Z_2\#\dots\#Z_nsw \vdash_{\mathcal{H}}^k n\#^n uqv,$$

where $u = v = \varepsilon$, $q \in F$, for some $k \geq 0$. Then, \mathcal{H} accepts w . By this claim

$$\#Zs'w \vdash_{M'}^* \#Zq_n.$$

Since $q \in F$ and by the construction of M' , there exists a rule from (3), $Zq_n \rightarrow f \in R'$, and by using this rule, M' can also accept w . So, $L(\mathcal{H}) \subseteq L(M')$.

As a special case of Claim 6, M' can make the sequence of moves

$$\#Zs'w \vdash_{M'}^k \#Zq_n,$$

for some $k \geq 1$. Then, if $q \in F$, by the construction of M' , there exists a rule from (3), $Zq_n \rightarrow f \in R'$, and M' can accept w by its application. However, by this claim

$$\#Z_1\#Z_2\#\dots\#Z_nsw \vdash_{\mathcal{H}}^* \#^n q.$$

So, \mathcal{H} accepts w as well. Hence, $L(M') \subseteq L(\mathcal{H})$.

Since $L(\mathcal{H}) \subseteq L(M')$ and $L(M') \subseteq L(\mathcal{H})$, we have that $L(\mathcal{H}) = L(M')$, and the lemma holds. \square

Lemma 4. *Let K be a context-free language. Then, there is a 1-PPOPDA, $\mathcal{H} = (M, \preceq)$, where $M = (Q, \text{alph}(K), R, s, Z, F)$, such that $L(M) = K$.*

Proof. This Lemma follows directly from Lemma 1. Indeed, observe that every 1-PPDA is a 1-PPOPDA. \square

Theorem 3. $n\text{PPOPDA} = \text{CF}$, for any $n \geq 1$.

Proof. Let $n \geq 1$. By Lemma 3, $n\text{PPOPDA} \subseteq \text{CF}$. By Lemma 4, $\text{CF} \subseteq n\text{PPOPDA}$. Hence, $n\text{PPOPDA} = \text{CF}$, so the theorem holds. \square

From Theorem 3, we obtain the following corollary.

Corollary 2. $\text{PPOMPDA} = \text{CF}$ \square

5 ORDERED PURE PUSHDOWN AUTOMATA

In the previous section, we studied pure multi-pushdown automata with a total order over their pushdowns. In the present section, we turn our attention to an order over the input symbols. We introduce an order \preceq and restrict the way the automaton pushes symbols onto its pushdown. If a is the topmost pushdown symbol, only b such that $a \preceq b$ can be pushed onto the pushdown. We prove that such restricted pure pushdown automata characterize only a proper subfamily of the family of context-free languages.

5.1 DEFINITIONS AND EXAMPLES

Next, we define ordered pure pushdown automata and illustrate them by an example.

Definition 7. An *ordered pure pushdown automaton (OPPPDA)* is a pair

$$\mathcal{H} = (M, \trianglelefteq),$$

where $M = (Q, \Sigma, R, s, Z, F)$ is a 1-PPDA, and \trianglelefteq is a total order over Σ . Since there is only a single pushdown, instead of $izpa \rightarrow wq \in R$, we write just $zpa \rightarrow wq \in R$. \square

Definition 8. Let $\mathcal{H} = (M, \trianglelefteq)$ be an OPPDA, where $M = (Q, \Sigma, R, s, Z, F)$. The *direct move relation* over $\{\#\}\Gamma^*Q\Sigma^*$ is denoted by $\vdash_{\mathcal{H}}$ and defined as

$$\#yzpax \vdash_{\mathcal{H}} \#ywqx,$$

if and only if $\#yzpax, \#ywqx \in \{\#\}\Gamma^*Q\Sigma^*$, $zpa \rightarrow wq \in R$, and

$$c_1 \trianglelefteq c_2 \trianglelefteq \cdots \trianglelefteq c_{|yw|},$$

where $yw = c_1c_2 \cdots c_{|yw|}$. Let $\vdash_{\mathcal{H}}^m$ and $\vdash_{\mathcal{H}}^*$ denote the m th power of $\vdash_{\mathcal{H}}$, for some $m \geq 1$, and the reflexive-transitive closure of $\vdash_{\mathcal{H}}$, respectively. \square

Definition 9. Let $\mathcal{H} = (M, \trianglelefteq)$ be an OPPDA, where $M = (Q, \Sigma, R, s, Z, F)$. The *language accepted by \mathcal{H}* , denoted by $L(\mathcal{H})$, is defined as

$$L(\mathcal{H}) = \{w \in \Sigma^* \mid \#Zsw \vdash_{\mathcal{H}}^* \#f, f \in F\}. \quad \square$$

We illustrate the previous definitions by an example.

Example 3. Let $\mathcal{H} = (M, \trianglelefteq)$, where $M = (\{s, q, f\}, \{a, b, c\}, R, s, a, \{f\})$, be an OPPDA with $a \trianglelefteq b \trianglelefteq c$ and R containing the following rules:

$$\begin{array}{ll}
 sa \rightarrow as & aqa \rightarrow q \\
 sb \rightarrow bs & bqb \rightarrow q \\
 sc \rightarrow cs & cq c \rightarrow q \\
 s \rightarrow q & aq \rightarrow f
 \end{array}$$

First, \mathcal{H} pushes some symbols from the input onto the pushdown. Then, it nondeterministically changes its state and checks the equality of the reversal of the pushdown string with the rest of the input string. The language of \mathcal{H} is

$$L(\mathcal{H}) = \{w_1w_2 \mid w_1 \in \{a\}^*\{b\}^*\{c\}^*, w_2 = \text{reversal}(w_1)\}.$$

The given order \trianglelefteq ensures a separation of sequences of as , bs , and cs . For example, the string $aabccbaa$ is accepted by \mathcal{H} in the following way:

$$\begin{array}{l}
 \#asaabccbaa \vdash_{\mathcal{H}} \#aasabccbaa \vdash_{\mathcal{H}} \#aaasbccbaa \vdash_{\mathcal{H}} \\
 \#aaabsccbaa \vdash_{\mathcal{H}} \#aaabscbaa \vdash_{\mathcal{H}} \#aaabcqcbaa \vdash_{\mathcal{H}} \\
 \#aaabqbaa \vdash_{\mathcal{H}} \#aaaqaa \vdash_{\mathcal{H}} \#aaqa \vdash_{\mathcal{H}} \\
 \#aq \vdash_{\mathcal{H}} \#f
 \end{array}$$

□

Let **OPPDA** denote the family of languages accepted by OPPDAs.

5.2 ACCEPTING POWER

In this section, we prove that OPPDAs characterize only a proper subfamily of the family of context-free languages.

Lemma 5. $\text{OPPDA} \subseteq \text{CF}$

Proof. Consider an OPPDA $\mathcal{H} = (M, \trianglelefteq)$, where

$$M = (Q, \Sigma, R, s, Z, F).$$

We show how to simulate \mathcal{H} by a PDA. First, we give a construction of such a PDA. Then, we describe the idea underlying this construction. However, even before, without any loss of generality, we remove from R all rules of the form $vpa \rightarrow wq$, where w does not satisfy \trianglelefteq . Observe that such rules are never applicable. Let

$$M' = (Q', \Sigma, \Gamma, R', s', Z', F),$$

be the PDA constructed as follows. Initially, set $Q' = Q \cup \{s'\}$ ($s' \notin Q$), $\Gamma = \Sigma \cup \{Z'\}$ ($Z' \notin \Sigma$), and $R' = \emptyset$. Perform (1) through (4), given next:

- (1) add $Z's' \rightarrow Z'Zs$ to R' ;
- (2) for each $f \in F$, add $Z'f \rightarrow f$ to R' ;
- (3) for each $vpa \rightarrow wq \in R$, add $Z'vpa \rightarrow Z'wq$ to R' ;
- (4) for each $vpa \rightarrow wq \in R$ and for each $c \in \Sigma$, if $cw = cd_1d_2 \dots d_{|w|}$, where $d_i \in \Sigma$, and $c \trianglelefteq d_1 \trianglelefteq d_2 \trianglelefteq \dots \trianglelefteq d_{|w|}$, add $cvpa \rightarrow cwq$ to R' .

Before we prove the identity $L(\mathcal{H}) = L(M')$, let us give an insight into the construction. M' has to simulate the order \trianglelefteq . We introduce a new pushdown symbol Z' to denote the bottom of the simulated pushdown. It is the start pushdown symbol and it is never removed until the computation succeeds.

The set of states Q is extended with a new start state s' to ensure the consistence of the starting configuration of \mathcal{H} and M' . For this purpose, we introduce the rule $Z's' \rightarrow Z'Zs$ in (1).

Then, the simulation continues rule by rule in accordance with \mathcal{H} , but unlike \mathcal{H} , M' does not ensure the order of the pushdown symbols naturally. However, the rules are designed to satisfy

the simulated order \trianglelefteq . Roughly speaking, M' has to look one symbol deeper into the pushdown whether the following move does not violate \trianglelefteq . More precisely, every rule $vpa \rightarrow wq \in R$ is replaced with rules from (3) and (4). The resulting rules extend all the rules in R . The pushdown strings on both sides of each new rule contain an additional pushdown symbol. It is either Z' or $c \in \Sigma$ that preserves \trianglelefteq . Therefore, the rule to be applied is chosen according to the top pushdown string with one additional symbol, which remains in the pushdown. Then, the resulting pushdown configuration corresponds to the simulated one.

Finally, whenever Z' occurs on the top of the pushdown and the current state is $f \in F$, by the construction of M' , there exists the rule $Z'f \rightarrow f \in R'$ from (2) and M' can accept.

To prove that $L(\mathcal{H}) = L(M')$, we establish two claims. Claim 7 shows how M' simulates \mathcal{H} . Claim 8 shows the converse simulation.

Claim 7. If $\#Zsw \vdash_{\mathcal{H}}^k \#uqv$, where $v, w \in \Sigma^*$, $u \in \Gamma^*$, $q \in Q$, for some $k \geq 0$, then $\#Z's'w \vdash_{M'}^* \#Z'uqv$.

Proof. This claim is established by induction on $k \geq 0$.

Basis. Let $k = 0$. Then, for $\#Zsw \vdash_{\mathcal{H}}^0 \#Zsw$, where $w \in \Sigma^*$, there is

$$\#Z's'w \vdash_{M'} \#Z'Zsw,$$

by the rule from (1), so the basis holds.

Induction Hypothesis. Suppose that there exists $k \geq 0$ such that the claim holds for all sequences of moves of length m , where $0 \leq m \leq k$.

Induction Step. Consider any sequence of moves $\#Zsw \vdash_{\mathcal{H}}^{k+1} \#u'q'v'$, where $w, v' \in \Sigma^*$, $u' \in \Gamma^*$, $q' \in Q$. Since $k+1 \geq 1$, this sequence can be written in the form

$$\#Zsw \vdash_{\mathcal{H}}^k \#uqv \vdash_{\mathcal{H}} \#u'q'v'$$

where $v \in \Sigma^*$, $u \in \Gamma^*$, $q \in Q$. Then, there exists a rule $xqa \rightarrow x'q' \in R$, where $u = yx$, $u' = yx'$, $v = av'$, which was used to perform the $(k+1)$ th move. Since both yx and yx' satisfy \trianglelefteq and $y = y'c$, where $c \in \Sigma \cup \{\varepsilon\}$ and $y' \in \Gamma^*$, by the construction of M' , there exists a rule $c'xqa \rightarrow c'x'q' \in R'$, where, if $c = \varepsilon$, $c' = Z'$, otherwise $c' = c$. By the induction hypothesis,

$$\#Z's'w \vdash_{M'}^* \#Z'uqv.$$

Consequently, by using $c'xqa \rightarrow c'x'q'$, M' can make the move

$$\#Z'uqv \vdash_{M'} \#Z'u'q'v'.$$

The resulting configuration of M' corresponds to the new configuration of \mathcal{H} , and the claim holds. \square

Claim 8. If $\#Z's'w \vdash_{M'}^k \#Z'uqv$, where $v, w \in \Sigma^*$, $u \in \Gamma^*$, $q \in Q$, $u = c_1c_2 \dots c_{|u|}$ and $c_1 \trianglelefteq c_2 \trianglelefteq \dots \trianglelefteq c_{|u|}$, for some $k \geq 1$, then $\#Zsw \vdash_{\mathcal{H}}^* \#uqv$.

Proof. This claim is established by induction on $k \geq 1$.

Basis. Let $k = 1$. From the construction of M' , there exists the only applicable rule from (1), $Z's' \rightarrow Z'Zs$. Since Z satisfies \trianglelefteq , then for

$$\#Z's'w \vdash_{M'} \#Z'Zsw,$$

where $w \in \Sigma^*$, there is $\#Zsw \vdash_{\mathcal{H}}^0 \#Zsw$ and the basis holds.

Induction Hypothesis. Suppose that there exists $k \geq 1$ such that the claim holds for all sequences of moves of length m , where $1 \leq m \leq k$.

Induction Step. Consider any sequence of moves

$$\#Z's'w \vdash_{M'}^{k+1} \#Z'u'q'v',$$

where $w, v' \in \Sigma^*$, $u' \in \Gamma^*$, $q' \in Q$. Since $k + 1 \geq 1$, this sequence can be written in the form

$$\#Z's'w \vdash_{M'}^k \#Z'uqv \vdash_{M'} \#Z'u'q'v',$$

where $v \in \Sigma^*$, $u \in \Gamma^*$, $q \in Q$. Then, there exists a rule $cxqa \rightarrow cx'q' \in R'$, where $u = ycx$, $u' = ycx'$, $c \in \Gamma \cup \{Z'\}$, $v = av'$, which was used to perform the $(k + 1)$ th move. Since yc and cx' satisfy \trianglelefteq , the order \trianglelefteq is preserved. Then, by the construction of M' , used rule was introduced to R' according to some rule $xqa \rightarrow x'q' \in R$. By the induction hypothesis,

$$\#Zsw \vdash_{\mathcal{H}}^k \#uqv.$$

Consequently, by using $xqa \rightarrow x'q'$, \mathcal{H} can make the move

$$\#uqv \vdash_{\mathcal{H}} \#u'q'v'.$$

The resulting configuration of \mathcal{H} corresponds to the new configuration of M' and the claim holds. \square

Consider a special case of Claim 7 $\#Zsw \vdash_{\mathcal{H}}^k \#uqv$ when $u, v = \varepsilon$, $q \in F$. \mathcal{H} accepts w . Then,

$$\#Z's'w \vdash_{M'}^* \#Z'q,$$

and, by the construction of M' , there exists a rule $Z'q \rightarrow q$ from (2). M' can also accept w . Hence, $L(\mathcal{H}) \subseteq L(M')$.

In a special case of Claim 8, M' performs the sequence of moves

$$\#Z's'w \vdash_{M'}^k \#Z'uqv,$$

where $u, v = \varepsilon$, $q \in F$. Then, by the construction of M' , there exists a rule $Z'q \rightarrow q$ from (2). By using this rule, M' can accept w . However, on the basis of $\#Zsw \vdash_{\mathcal{H}}^* \#uqv$ and, since $u = v = \varepsilon$ and $q \in F$, \mathcal{H} accepts w as well. Hence, $L(M') \subseteq L(\mathcal{H})$.

Consequently, $L(\mathcal{H}) = L(M')$, and the lemma holds. \square

Lemma 6. $\mathbf{CF} - \mathbf{OPPDA} \neq \emptyset$

Proof. Consider the language $L_R = \{w \text{ reversal}(w) \mid w \in \{a, b\}^*\}$. Obviously, $L_R \in \mathbf{CF}$. Suppose that there exists an OPPDA $\mathcal{H} = (M, \trianglelefteq)$, where

$$M = (Q, \Sigma, R, s, Z, F),$$

and $L(\mathcal{H}) = L_R$. Additionally, let $uv \in L(\mathcal{H})$, where $|u| = |v|$. To check the equality of u with its reversal v , \mathcal{H} has to remember u first. So, with every symbol of u read from the input, \mathcal{H} has to enter a unique configuration different from all the previous configurations.

Now, suppose that $u = (a^i b^j)^k$, where $i, j > \text{card}(Q)$ and $k > \text{card}(\Sigma)$. To save the information about the lengths of the sequences of as and bs , \mathcal{H} has to insert nonempty sequences of the symbols into the pushdown. Indeed, states are not enough to encode this. Each sequence on the pushdown must differ from the previous one—contain different symbols. With the m th sequence, where $m > \text{card}(\Sigma)$, \mathcal{H} has to use the previously used symbol to differentiate the sequences. However, it is in the conflict with \trianglelefteq , which is a contradiction. Therefore, the lemma holds. \square

Lemma 7. $\mathbf{RG} \subseteq \mathbf{OPPDA}$

Proof. For any regular language K , there is a finite automaton $M = (Q, \Sigma, R, s, F)$ satisfying $L(M) = K$. Then, we can define the OPPDA $\mathcal{H} = (M', \trianglelefteq)$, where $M' = (Q, \Sigma, R', s, a, F)$ with a being an arbitrary symbol from Σ and $R' = R \cup \{as \rightarrow s\}$. Obviously, $L(\mathcal{H}) = K$, so the lemma holds. \square

Lemma 8. $\mathbf{OPPDA} - \mathbf{RG} \neq \emptyset$

Proof. This lemma follows from Example 3 (recall that $L(\mathcal{H}) \notin \mathbf{RG}$). \square

Theorem 4. $\mathbf{RG} \subset \mathbf{OPPDA} \subset \mathbf{CF}$

Proof. By Lemma 5, $\mathbf{OPPDA} \subseteq \mathbf{CF}$. By Lemma 6, $\mathbf{CF} - \mathbf{OPPDA} \neq \emptyset$. By Lemma 7, $\mathbf{RG} \subseteq \mathbf{OPPDA}$. By Lemma 8, $\mathbf{OPPDA} - \mathbf{RG} \neq \emptyset$. Hence, $\mathbf{RG} \subset \mathbf{OPPDA} \subset \mathbf{CF}$, so the theorem holds. \square

6 CONCLUSIONS

In this final section, we briefly conclude all the achieved results. First, we studied pure versions of PDAs and proved that the absence of special pushdown symbols does not affect their accepting power; one-pushdown PPDA's still characterize \mathbf{CF} and PPDA's with two or more pushdown lists characterize \mathbf{RE} — that is they are Turing-complete. Next, we introduced an order above the pushdown lists of multi-pushdown PPDA's and proved that they characterize \mathbf{CF} regardless of a number of their pushdown lists. Finally, we introduce PPDA's with ordered pushdown symbols. We proved that they characterize a superfamily of \mathbf{RG} , however, only a subfamily of \mathbf{CF} .

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