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Fractional order, discrete model of heat transfer process using time and spatial Grünwald-Letnikov operator

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Abstract. In the paper a new, state space, fully discrete, fractional model of a heat transfer process in one dimensional body is addressed. The proposed model derives directly from fractional heat transfer equation. It employes the discrete Grünwald-Letnikov operator to express the fractional order differences along both coordinates: time and space. The practical stability and numerical complexity of the model are analysed. Theoretical results are verified using experimental data.

Key words: fractional order systems; heat transfer equation; fractional order state equation; Fractional Order Backward Difference; Grünwald-Letnikov operator; practical stability.

1. Introduction

It is known that the non integer order calculus can be applied in modeling of processes and phenomena hard to analyse with the use of other tools. Non integer order (NIO) or fractional order (FO) models of different physical phenomena have been presented by many Authors. A amount of FO models of various processes is collected in the book [1]. The book [2] presents fractional order models of chaotic systems and Ionic Polymer Metal Composites (IPMC). Fractional models of ultracapacitors are "classics" of FO modeling. They are given for example by [3]. Distributed parameter systems can be also described using FO approach. As an example diffusion processes discussed in [4–6] can be given. A collection of recent results employing new Atangana-Baleanu operator can be found in [7]. In this book i.e., the FO blood alcohol model, the Christov diffusion equation and fractional advection-dispersion equation for groundwater transport process are presented.

Heat transfer processes can also be described using non integer order approach. For example a temperature–heat flux relationship for heat flow in semi-infinite conductor is presented in [1], the beam heating problem is given in [3], the FO transfer function temperature models in a room are presented by [8], temperature models in three dimensional solid body are given in [9]. The use of fractional order approach to the modeling and control of heat systems is also presented in [10].

This paper is devoted to presenting a new, discrete, FO model of heat transfer process in one dimensional body. Such a process is described by a partial differential equation (PDE) of parabolic type. In the considered case the both partial derivatives along time and length are fractional and expressed by the Grünwald- Letnikov operator. This operator best describes the discrete fractional differentiation and it is discrete "from its nature".

The paper is organized as follows. Preliminaries recall some elementary ideas and definitions from fractional calculus. Next, the discrete state space model using Grünwald- Letnikov operator along time and space is proposed and its practical stability is discussed. Finally, the experimental validation of theoretical results is given.

2. Preliminaries

At the beginning the non integer order, integro-differential operator is presented (see for example [1, 11–13]).

Definition 1. (The elementary non integer order operator) The non integer order integro-differential operator is defined as follows:

$$ {}_aD_t^{\alpha} g(t) = \begin{cases} \dfrac{\mathrm{d}^{\alpha} g(t)}{\mathrm{d}t^{\alpha}} & \alpha > 0, \\ g(t) & \alpha = 0, \\ \displaystyle\int_a^t g(\tau)(\mathrm{d}\tau)^{\alpha} & \alpha < 0, \end{cases} . \tag{1} $$

where a and t denote time limits for operator calculation, $\alpha \in \mathbb{R}$ denotes the non integer order of the operation.

Next an idea of complete Gamma Euler function is recalled (see for example [12]):

Definition 2. (The complete Gamma function)

$$ \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \,\mathrm{d}t. \tag{2} $$

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The fractional order, integro-differential operator (1) is described by different definitions, given by Grünwald and Letnikov (GL definition), Riemann and Liouville (RL definition) and Caputo (C definition). Relations between Caputo and Riemann-Liouville, between Riemann-Liouville and Grünwald-Letnikov operators are given for example in [1, 14]. Discrete versions of these operators are analysed with details in [15]. In the further consideration the GL definition along time and space is used and it needs to be recalled.

The GL derivative along the time from function $g(t)$ is defined as follows ([2, 16]):

Definition 3. (The time Grünwald-Letnikov definition)

$$ {}_0^{GL}D_t^{\alpha} g(t) = \lim_{h\to 0} h^{-\alpha} \sum_{l=0}^{\left[\frac{t}{h}\right]} (-1)^l \binom{\alpha}{l} g(t-lh). \tag{3} $$

In (3) $0.0 < \alpha \leq 1.0$ is the fractional order along the time, h is the sample time, $[..]$ is the nearest integer value, $\binom{\alpha}{l}$ is the binomial coefficient:

$$ \binom{\alpha}{l} = \begin{cases} 1, & l = 0, \\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-l+1)}{l!}, & l > 0. \end{cases} \tag{4} $$

Analogically the space GL derivative of the function $g(x)$ is defined as follows (see [17]):

Definition 4. (The space Grünwald-Letnikov definition)

$$ {}_0^{GL}D_1^{\beta} g(x) = \lim_{\Delta x\to 0} \Delta x^{-\beta} \sum_{n=0}^{N} (-1)^n \binom{\beta}{n} g(t-n\Delta x). \tag{5} $$

In (5) $1.0 < \beta \leq 2.0$ is the fractional order along the space, N is the number of space divisions, Δx is the elementary step of length. If the length of an object is equal 1, then $\Delta x = \dfrac{1}{N}$. The binomial coefficient $\binom{\beta}{n}$ is defined by (4).

The GL definition is limit case for $h \to 0$, $\Delta x \to 0$ of the Fractional Order Backward Difference (FOBD), commonly employed in discrete FO calculations (see for example [18], p. 68). It can be defined for time and length separately.

Definition 5. (The Fractional Order Backward Difference along the time – FOBDT)

$$ \Delta^{\alpha} g(t) = \frac{1}{h^{\alpha}} \sum_{l=0}^{L} (-1)^l \binom{\alpha}{l} g(t-lh). \tag{6} $$

Denote coefficients $(-1)^l \binom{\alpha}{l}$ by d_l:

$$ d_l = (-1)^l \binom{\alpha}{l}. \tag{7} $$

The coefficients (7) can be also calculated with the use of the following, equivalent recursive formula (see for example [2], p. 12), useful in numerical calculations:

$$ \begin{aligned} d_0 &= 1, \\ d_l &= \left(1 - \frac{1+\alpha}{l}\right) d_{l-1}, \quad l = 1,\ldots,L. \end{aligned} \tag{8} $$

It is proven in [19] that:

$$ \sum_{l=1}^{\infty} d_l = 1 - \alpha. \tag{9} $$

In (6) L denotes a memory length necessary to correct approximation of a non integer order operator. Unfortunately good accuracy of approximation requires to use a long memory L what can make difficulties during implementation.

The FOBD along the space (FOBDS) can be defined analogically. It is given for example in [20]:

Definition 6. (The Fractional Order Backward Difference along the space – FOBDS)

$$ \Delta^{\beta} g(x) = \frac{1}{\Delta x^{\beta}} \sum_{n=0}^{N} (-1)^n \gamma_n g(x-n\Delta x). \tag{10} $$

The coefficients γ_n are as follows:

$$ \gamma_n = \frac{\Gamma(n-\beta)}{\Gamma(-\beta)\Gamma(n+1)}. \tag{11} $$

The discrete, fractional order state equation using definition (6) is written as follows (see for example [15, 21]):

$$ \begin{cases} \Delta_L^{\alpha} q(t+h) = A^{+} q(t) + B^{+} u(t), \\ y(t) = C^{+} q(t), \end{cases} \tag{12} $$

where $q(t) \in \mathbb{R}^N$ is the state vector, $u(t) \in \mathbb{R}^P$ is the control, $y(t) \in \mathbb{R}^M$ is the output. A^+, B^+ and C^+ are state, control and output matrices respectively. If we shortly denote k-th time instant: hk by k, then Eq. (12) turns to:

$$ \begin{cases} \Delta_L^{\alpha} q(k+1) = A^{+} q(k) + B^{+} u(k), \\ y(k) = C^{+} q(k), \end{cases} \tag{13} $$

where:

$$ A^{+} = h^{\alpha} A, \tag{14} $$

$$ B^{+} = h^{\alpha} B, \tag{15} $$

$$ C^{+} = C. \tag{16} $$

The practical implementation of FO system (13) takes the form of state equation with L delays in the state:

$$ q(k+1) = P^{+} q(k) - \sum_{l=2}^{L} A_l^{+} q(k-l) + B^{+} u(k), \tag{17} $$

where:

$$P^{+} = A^{+} + \alpha I = h^{\alpha} A + \alpha I, \tag{18}$$

$$A_l^{+} = d_l I_{N\times N}. \tag{19}$$

Next an idea of practical stability is recalled (see for example [19])

Definition 7. (The practical stability)
The fractional order system (13) is called practically stable if its practical implementation (17) is asyptotically stable.

3. The experimental system and its PDE model

Figure 1 shows the simplified scheme of the considered heat system. This is a thin copper rod heated by an electric heater x_u long, localized at one end of rod. The output temperature is measured using miniature RTD sensors x_s long. Assume to simplify that the length of the rod equals 1. Then the centers of sensors are in: $x_1 = 0.29$, $x_2 = 0.50$ and $x_3 = 0.73$ of the rod length. The length of each sensor is equal: $x_s = 0.06$. More details about the construction of this laboratory system are given in the section "Experimental Results".

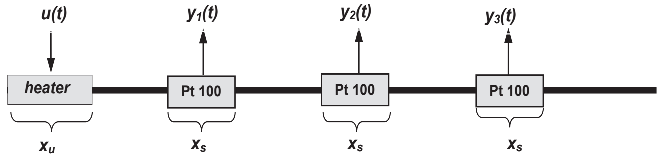

Fig. 1. The simplified scheme of the experimental system

The fundamental mathematical model describing the heat transfer in the rod is the PDE of the parabolic type. Both frontal surfaces of the rod are much smaller than its side surface. This allows to assume the homogeneous Neumann boundary conditions at the both ends as well as the heat exchange along the length needs to be considered. The control and observation are distributed because the size of heater and sensors should be also considered. Such a model with integer orders of both differentiations has been considered in papers [22–24]. Its fractional version is as follows:

$$\begin{cases} {}_0^C D_t^{\alpha} Q(x,t) = a_w \dfrac{\partial^{\beta} Q(x,t)}{\partial x^{\beta}} - R_a Q(x,t) + b(x)u(t), \\ \dfrac{\partial Q(0,t)}{dx} = 0, \quad t \geq 0, \\ \dfrac{\partial Q(1,t)}{dx} = 0, \quad t \geq 0, \\ Q(x,0) = Q_0, \quad 0 \leq x \leq 1, \\ y(t) = k_0 \displaystyle\int_0^1 Q(x,t)c(x)\,\mathrm{d}x. \end{cases} \tag{20}$$

In (20) α and β are non integer orders of the system, $a_w > 0$, $R_a \geq 0$ denote coefficients of heat conduction and heat exchange, k_0 is a steady-state gain of the model, $b(x)$ and $c(x)$ are heater and sensor functions. They take the following form:

$$b(x) = \begin{cases} 1, & x \in [0, x_u], \\ 0, & x \notin [0, x_u]; \end{cases} \tag{21}$$

$$c(x) = \begin{cases} 1, & x \in [x_j - 0.5x_s, x_j + 0.5x_s], \\ 0, & x \notin [x_j - 0.5x_s, x_j + 0.5x_s], \end{cases} \tag{22}$$

where $j = 1, 2, 3$ denotes the output sensor.

4. The proposed discrete model

The proposed discrete FO model follows directly from parabolic equation (20) after use (6) and (10).

Firstly estimate the spatial derivative using (10). In order to do it, divide the length of the rod into N short sectors, each $\Delta x = \dfrac{1}{N}$ long. Next denote temperature in the n-th point by: $Q(n\Delta x, t) = Q_n(t)$, $n = 1, \ldots, N$. The homogenous Neumann boundary condition in (20) can be written as follows:

$$\begin{aligned} \Delta^{\beta} Q_1(t) &= Q_0(t), \\ \Delta^{\beta} Q_N(t) &= Q_{N-1}(t). \end{aligned} \tag{23}$$

Using (10) and with respect to (20) the spatial derivative from temperature in the n-th node ($n = 2, \ldots, N-1$) is as follows:

$$\Delta^{\beta} Q_n(t) \approx \sum_{m=2}^{n+1} e_m Q_{n-k+1}(t) + e_1 Q_n(t) + e_0 Q_{n+1}(t), \qquad n = 2, \ldots, N-1, \tag{24}$$

where:

$$e_n = \frac{\gamma_n a_w}{\Delta x^{\beta}} = \gamma_n a_w N^{\beta}. \tag{25}$$

In (25) γ_n are expressed by (11).

Using (23) and (24) the single FOPDE (20) can be transformed to the following set of ordinary FO differential equations:

$$\begin{cases} D^{\alpha} Q_1(t) = (e_1 + e_2 - R_a) Q_1(t) + b_1 u(t), \\ \ldots \\ D^{\alpha} Q_n(t) = \displaystyle\sum_{m=2}^{n+1} a_n Q_{n-m+1}(t) + \\ \qquad + (e_1 - R_a) Q_n(t) + e_0 Q_{n+1}(t) + b_n u(t), \\ \ldots \\ D^{\alpha} Q_{N-1}(t) = \displaystyle\sum_{m=2}^{N} a_n Q_{n-m+1}(t) + \\ \qquad + (e_0 + e_1 - R_a) Q_{N-1}(t) + e_0 Q_N(t) + b_{N-1} u(t). \end{cases} \tag{26}$$

Equation (26) can be expressed as the finite, time-continuous FO state equation:

$$\begin{cases} D^{\alpha}Q(t) = AQ(t) + Bu(t), \\ y(t) = CQ(t), \end{cases} \tag{27}$$

where $Q(t) = [Q_1(t), \ldots, Q_{N-1}(t)]^T$ is the state vector, the state matrix A is as follows:

$$A = \begin{bmatrix} e_1+e_2-R_a & e_0 & 0 & \ldots & \ldots & 0 \\ e_2+e_3 & e_1-R_a & e_0 & \ldots & \ldots & 0 \\ e_3+e_4 & e_2 & e_1-R_a & e_0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ e_n+e_{n+1} & e_{n-1} & \ldots & e_1-R_a & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & e_0 \\ e_{N-1}+e_N & e_{N-1} & e_{N-2} & \ldots & e_2 & e_0+e_1-R_a \end{bmatrix}. \tag{28}$$

The control matrix B takes the following form:

$$B = \left[b_1, \ldots, b_N\right]^T. \tag{29}$$

Each element b_n is defined as follows:

$$b_n = \begin{cases} 1, & n\Delta x \in [0, x_u] \\ 0, & n\Delta x \notin [0, x_u] \end{cases} \quad n = 0, 1, \ldots, N. \tag{30}$$

The output matrix C takes the following form:

$$C = \left[C_1, C_2, C_3\right]^T. \tag{31}$$

Each row of matrix (31) is associated to one sensor:

$$C_j = \left[c_{j,0}, \ldots, c_{j,N}\right], \quad j = 1, 2, 3. \tag{32}$$

The elements of each row C_1-C_3 are defined as follows:

$$c_{j,n} = \begin{cases} 1, & x \in [x_j - 0.5x_s, x_j + 0.5x_s] \\ 0, & n\Delta x \notin [x_j - 0.5x_s, x_j + 0.5s] \end{cases} \quad \text{for} \begin{cases} n = 0, 1, \ldots, N, \\ j = 1, 2, 3. \end{cases} \tag{33}$$

Next replace the time continuous FO derivative in Eq. (27) by FOBDT (6). Then we obtain the fractional order, difference equation along both coordinates. It takes the form (13) with matrices A^+ and B^+ defined by (14) and (15). The solution of this equation is as (17) with matrix P^+ defined as follows:

$$P^+ = \begin{bmatrix} R_1^+ & e_0^+ & 0 & \ldots & \ldots & 0 \\ e_2^+ + e_3^+ & R^+ & e_0^+ & \ldots & \ldots & 0 \\ e_3^+ + e_4^+ & e_2 & R^+ & e_0^+ & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ e_n^+ + e_{n+1}^+ & e_{n-1}^+ & \ldots & R^+ & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ e_{N-1}^+ + e_N^+ & e_{N-1}^+ & e_{N-2}^+ & \ldots & e_2^+ & R_N^+ \end{bmatrix}, \tag{34}$$

where:

$$e_n^+ = h^{\alpha} e_n, \quad n = 0, 1, \ldots, N, \tag{35}$$

$$\begin{aligned} R_1^+ &= h^{\alpha}\big((e_1 + e_2) - R_a\big) + \alpha, \\ R^+ &= h^{\alpha}(e_1 - R_a) + \alpha, \\ R_N^+ &= h^{\alpha}\big((e_0 + e_1) - R_a\big) + \alpha. \end{aligned} \tag{36}$$

Matrices B^+ and C^+ are as follows:

$$B^+ = h^{\alpha} B = [b_1^+, \ldots, b_N^+]^T, \tag{37}$$

$$C^+ = C. \tag{38}$$

With respect to (26), (34), (35) and (36) the solution of the state equation (17) in the n-th point and $k+1$ time instant takes the following form:

$$\begin{cases} Q_1(k+1) = R_1^+ Q_1(k) + e_0^+ Q_2^+(k) \\ \quad - \sum_{l=2}^{L} d_l Q_1(k-l) + b_1^+ u(k), \\ Q_2(k+1) = \\ = (e_2^+ + e_3^+) Q_1(k) + R^+ Q_2(k) + e_0^+ Q_3(k) \\ \quad - \sum_{l=2}^{L} d_l Q_2(k-l) + b_2^+ u(k), \\ \ldots \\ Q_n(k+1) = \\ = (e_n^+ + e_{n+1}^+) Q_1(k) + \ldots + R Q_n(k) + e_0^+ Q_{n+1}(k) \\ \quad - \sum_{l=2}^{L} d_l Q_n(k-l) + b_n^+ u(k), \\ \ldots \\ Q_{N-1}(k+1) = \\ = (e_{N-1}^+ + e_N^+) Q_1(k) + \ldots + R Q_{N-1}(k) + e_0^+ Q_N(k) \\ \quad - \sum_{l=2}^{L} d_l Q_{N-1}(k-l) + b_N^+ u(k). \end{cases} \tag{39}$$

The output of the discrete model is as follows:

$$y^+(k) = C^+ Q(k). \tag{40}$$

Equations (39) and (40) can be written as the practical implementation of FO model analogically as (17):

$$\begin{cases} Q(k+1) = P^+ Q(k) - \sum_{l=2}^{L} A_l Q(k-l) + B^+ U(k), \\ y+(k) = C^+ Q(k), \end{cases} \tag{41}$$

where P^+ is expressed by (34), matrices A_l are as follows:

$$A_l = d_l I. \tag{42}$$

Equation (41) can be applied to compute time responses of the considered plant. Particularly it can be employed to calculate the step response.

Next write the state equation with L delays in state (41) as the equivalent equation without delay, analogically as in [25]. To do it denote the vector delayed by l steps $Q(k-l)$ by $Q_l(k)$, $l=1,\ldots,L$ and introduce the new state vector $\mathbb{Q}$, NL long:

$$\mathbb{Q}=\begin{bmatrix} Q(k) \\ Q(k-1) \\ \ldots \\ Q(k-i) \\ \ldots \\ Q(k-L) \end{bmatrix}=\begin{bmatrix} \mathbb{Q}_1(k) \\ \mathbb{Q}_2(k) \\ \ldots \\ \mathbb{Q}_l(k) \\ \ldots \\ \mathbb{Q}_L(k) \end{bmatrix}. \tag{43}$$

Consequently the state equation (41) with delay can be rewritten in the equivalent form without delay:

$$\begin{cases} \mathbb{Q}(k+1)=\mathbb{A}\mathbb{Q}(k)+\mathbb{B}u(k), \\ y(k)=\mathbb{C}\mathbb{Q}(k), \end{cases} \tag{44}$$

where:

$$\mathbb{A}=\begin{bmatrix} P^{+} & -A_2 & -A_3 & \ldots & -A_L \\ I & 0 & 0 & \ldots & 0 \\ 0 & I & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I & 0 \end{bmatrix}_{NL\times NL}, \tag{45}$$

$$\mathbb{B}=\begin{bmatrix} B^{+} & 0 & 0 & \ldots & 0 \end{bmatrix}^T, \tag{46}$$

$$\mathbb{C}=\begin{bmatrix} C^{+} & 0 & 0 & \ldots & 0 \end{bmatrix}. \tag{47}$$

Equation (44) is the discrete, integer order equation equivalent to the continuous FO equation (20). It can be solved using standard tools, for example *step* function in MATLAB. Its main disadvantage is high dimension, equal NL, where the memory length L needs to be $L>100$ to assure good accuracy of the model. However its form is convenient to practical stability analysis due to (44) is the typical, integer order, discrete state equation. The analysis of the practical stability using such an approach is given in the next section.

5. Practical stability of the proposed model

The practical stability of the model we deal with, depends on the number of length divisions N, sample time h and memory length L. Analysis of other parameters: α, β a_w and R_a does not make sense because they are determined by the modeled process. At the beginning consider the practical stability as a function of N. It is described by the following proposition:

Proposition 1. (The maximum number of N assuring the practical stability)

Consider the discrete fractional order system described by (41) or equivalently by (44)–(47).

This system is practically stable if (but not iff) the number of length divisions N meets the following inequality:

$$N<\left(\frac{1+\alpha-h^{\alpha}R_a-d_s}{(1+\beta+\gamma_{N12}+\gamma_{3N})a_w h^{\alpha}}\right)^{\frac{1}{\beta}}, \tag{48}$$

where:

$$\begin{aligned} \gamma_{N12}&=|\gamma_{N+2}+\gamma_{N+1}|, \\ \gamma_{3N}&=\sum_{n=3}^{N}|\gamma_n|. \end{aligned} \tag{49}$$

Proof. The condition (48) is proven using model without delay (43)-(47) and Gershgorin theorem. Let $D_n(s_n,r_n)$ denote Gershgorin disc centered at s_n and with radius equal r_n. The form of the state matrix (45) allows to note that the stronger limitation of its eigenvalues will be given by discs associated to rows. For rows $N+1$ to NL the only one, unit Gershgorin disc is defined: $D_I(0,1)$. This disc is not critical for stability because it describes the stability condition of each discrete model indepenendently on its parameters. The stability as a function of model parameters is determined by other discs, associated to rows $1-N$ in matrix (45). Denote these discs by $D_1(s_1,r_1)$–$D_N(s_N,r_N)$, where n is the number of row. Location of these discs in the complex plane is determined by the parameters of the model and it estimates its stability. With respect to (25), (34), (35) and (36) center of the s_n-th disc is as follows:

$$s_n=\begin{cases} \alpha-h^{\alpha}(\beta N^{\beta}a_w+R_a-N^{\beta}a_w\gamma_2), & n=1, \\ \alpha-h^{\alpha}(\beta N^{\beta}a_w+R_a), & n=2,\ldots,N-1, \\ \alpha-h^{\alpha}(\beta N^{\beta}a_w+R_a-N^{\beta}a_w), & n=N. \end{cases} \tag{50}$$

Notice that all the Gershgorin discs have only 3 centres independently on N, because the main diagonal of $\mathbb{A}$ contains only three different elements: in 1st row, N-th row and the same elements in rows 2 to $N-1$. The radii of discs are as follows:

$$r_n=\begin{cases} h^{\alpha}N^{\beta}a_w+d_s, \quad n=1, \\ \ldots \\ h^{\alpha}N^{\beta}a_w\left(1+|\gamma_{N+2}+\gamma_{N+1}|+\sum_{n=3}^{N}|\gamma_n|\right)+d_s, \\ \qquad n=N-1, \\ h^{\alpha}N^{\beta}a_w\sum_{n=3}^{N}|\gamma_n|+d_s, \quad n=N, \end{cases} \tag{51}$$

where:

$$d_s=\sum_{l=2}^{L}|d_l|. \tag{52}$$

The discrete system will be asymptotically stable if and only if its whole spectrum is located inside the unit disc D_I. This is expressed as follows:

$$\forall n=1,\ldots,N:\ D_n(s_n,r_n)\subset D_I(0,1). \tag{53}$$

The form of all discs, described by (50) and (51) suggests that not all need to be analyzed because the stability is determined by a biggest disc with center furthest from zero. This is disc $D_{N-1}(s_{N-1}, r_{N-1})$ and it will be used in further analysis.

For this disc the condition (53) can be written as follows:

$$\begin{cases} s_{N-1} + r_{N-1} < 1.0, \\ s_{N-1} - r_{N-1} > -1.0. \end{cases} \tag{54}$$

Using (50) and (51) in (54) yields:

$$\begin{cases} N < \left(\dfrac{1 - \alpha + h^{\alpha} R_a - d_s}{(1 - \beta + \gamma_{N12} + \gamma_{3N}) a_w h^{\alpha}} \right)^{\frac{1}{\beta}}, \\ N < \left(\dfrac{1 + \alpha - h^{\alpha} R_a - d_s}{(1 + \beta + \gamma_{N12} + \gamma_{3N}) a_w h^{\alpha}} \right)^{\frac{1}{\beta}}. \end{cases} \tag{55}$$

Selection the stronger limitation in (55) gives directly condition (48) and the proof is completed. □

The conclusion from the above proposition is a little bit surprising, because the shortening a length discretization step $\Delta x = \dfrac{1}{N}$ causes the loss of practical stability.

Next the practical stability as a function of sample time h can be considered. It can be estimated as follows:

Proposition 2. (The maximum size of sample time h assuring the practical stability) Consider the discrete fractional order system described by (41) or equivalently by (44)–(47).

This system is practically stable if (but not iff) the sample time h meets the following inequality:

$$h < \left(\frac{1 + \alpha - d_s}{N^{\beta} a_w (1 + \beta + \gamma_{N12} + \gamma_{3N}) + R_a} \right)^{\frac{1}{\alpha}}, \tag{56}$$

where γ_{N12} and γ_{3N} are expressed by (49).

Proof. The proof is analogical as previous one. We start from condition (54). Using (50) and (51) we obtain:

$$\begin{cases} h^{\alpha} < \left(\dfrac{1 - \alpha - d_s}{N^{\beta} a_w (1 - \beta + \gamma_{N12} + \gamma_{3N}) - R_a} \right), \\ h^{\alpha} < \left(\dfrac{1 + \alpha - d_s}{N^{\beta} a_w (1 + \beta + \gamma_{N12} + \gamma_{3N}) + R_a} \right). \end{cases} \tag{57}$$

The first inequality in (57) does not have solution due to its right side is negative. The second one gives directly (56) and the proof is completed. □

The condition (48) allows to conclude that shortening the sample time h allows to increase the number of length divisions N. The same is seen in condition (56), where increasing N causes descreasing the permissible range of sample time h. For $N \to \infty$ the sample time $h \to 0$.

An interesting issue is to estimate the sensitivity of stability estimations (48) and (56) to uncertainty of plant parameters a_w and R_a. It can be generally defined as the ratio of the disturbed value to the nominal one:

$$s_{Mp} = \frac{M_{dist}}{M_{nom}}, \tag{58}$$

where $M = N, h$ denotes the considered model parameter: number of length divisions N or sample time h, $p = a_w, R_a$ denotes the considered parameter, indices "$_{nom}$" and "$_{dist}$" denote values of N or h for nominal or disturbed parameters a_w or R_a respectively. Denote the deviations of parameter a_w or R_a from nominal value by Δa and ΔR respectively. Then the sensitivity functions are equal:

$$\begin{aligned} s_{Na_w} &= \left(\frac{1}{1 + \Delta a} \right)^{\frac{1}{\beta}}, \\ s_{NR_a} &= \left(\frac{1 + \alpha - d_s - h^{\alpha} R_a}{1 + \alpha - d_s - h^{\alpha} R_a (1 + \Delta R)} \right)^{\frac{1}{\beta}}, \\ s_{ha_w} &= \left(\frac{a_w N^{\beta} (1 + \beta + \gamma_{N12} + \gamma_{3N}) + R_a}{a_w (1 + \Delta a) N^{\beta} (1 + \beta + \gamma_{N12} + \gamma_{3N}) + R_a} \right)^{\frac{1}{\alpha}}, \\ s_{hR_a} &= \left(\frac{a_w N^{\beta} (1 + \beta + \gamma_{N12} + \gamma_{3N}) + R_a}{a_w N^{\beta} (1 + \beta + \gamma_{N12} + \gamma_{3N}) + R_a (1 + \Delta R)} \right)^{\frac{1}{\alpha}}. \end{aligned} \tag{59}$$

All the above sensitivity functions can be also expressed in percents:

$$S_{Mp} = \left| 1 - s_{Mp} \right| * 100\%. \tag{60}$$

Finally deal with the impact of memory length L on practical stability. This factor is the hardest to consider due to fact that the sum $d_s(L)$ is a strongly implicit function of memory length L (see (8), (52)). On the other hand, its value for $L \to \infty$ is bounded by (9). For range of α correctly describing the considered heat process the value of d_s is small and it can be ignored during stability analysis.

6. Experimental validation of results

Experiments were done using laboratory system shown in Fig. 2. It is supervised by PLC SIEMENS S7 1500. The input signal is given as standard current 0- 20 [mA] by analog output from PLC. It is amplified to range 0–1.5 [A] and sent to the heater. The temperature of the rod is measured using the

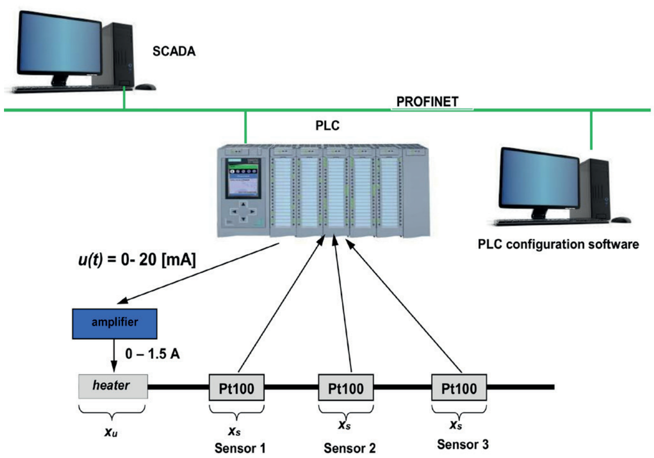

Fig. 2. The experimental system

miniature RTD-s of Pt-100 type. Signals from RTD-s are read directly by universal analog inputs of PLC. Trends of temperature and control signal are collected by SCADA application built with the use of INTOUCH v12. The time-spatial temperature distribution in the plant is shown in Fig. 3.

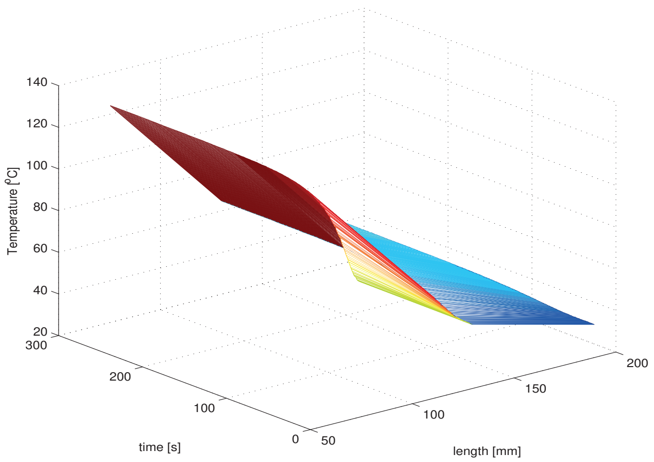

Fig. 3. The time-spatial temperature distribution in the plant

The parameters of the model expressed by (34)–(42) can be identified by minimization the Mean Square Error (MSE) cost function describing the difference between step response of plant and model at the same time mesh:

$$MSE = \frac{1}{3K}\sum_{k=1}^{K}(y(k) - y_e(k))^2. \tag{61}$$

In (61) K is the number of all collected samples, $y(k)$ is the step response of the model, computed using (13) and (41), $y_e(k)$ is the experimental response measured at the same time moments k. Parameters of the model minimizing the cost function (61) are given in Table 1. Orders N and L were assigned to assure the practical stability of the model with respect to Propositions 11 and 2. The step responses of the proposed model compared to experimental results are presented in Fig. 4. Next the

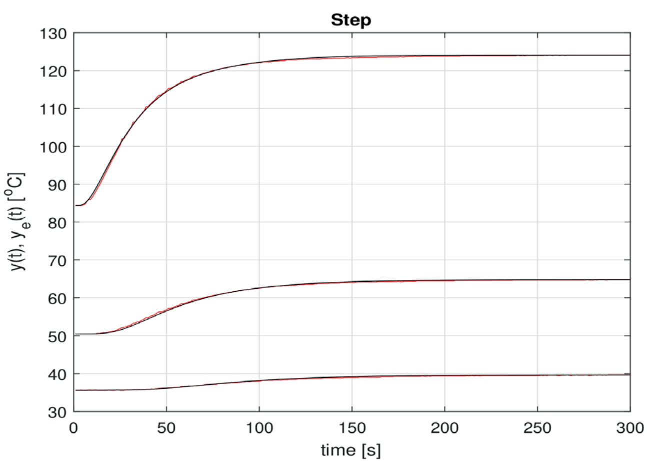

Fig. 4. The comparison of step responses for model and plant. The model is marked by black, experimental data are marked by red

Table 1
Parameters of the model and the *MSE* cost function

α	β	a_w	R_a	N	L	MSE
0.9070	1.9987	0.0003	0.0222	22	100	0.0445

practical stability of the proposed model was tested using conditions (48) and (56). Using parameters from Table 1 we obtain the following limit values: $N = 39$ for $h = 1$ and $h = 3.3878[s]$ for $N = 22$. To verify this result in Figs. 5–7 are shown Gershgorin discs for different combinations of N and h and other parameters given in Table 1. The distribution of spectrum of the state matrix $\mathbb{A}$ is shown in Figs. 8–10.

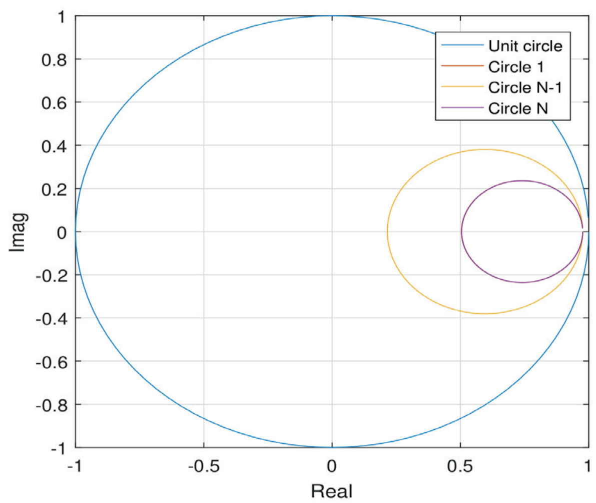

Fig. 5. Gershgorin circles for $N = 22$, $h = 1$, the both criteria met

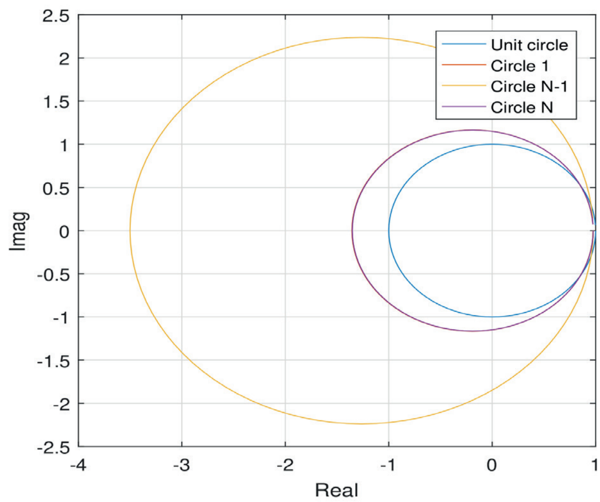

Fig. 6. Gershgorin circles for $N = 60$, $h = 1$, the 1st criterion not met

Next the the robustness analysis using sensitivity functions (58)–(60) will be given. To do it consider the plant parameters given in the Table 1 and assume that the deviations from nominal values for the both parameters are equal 10%. All the sensitivity functions are collected in the Table 2.

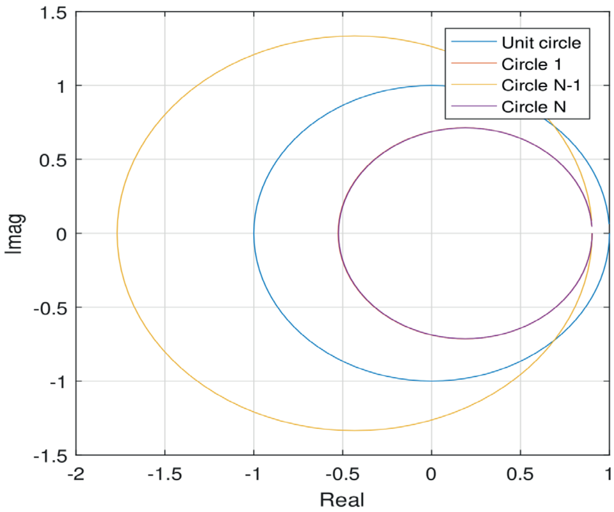

Fig. 7. Gershgorin circles for $N = 22$, $h = 5$, the 2nd criterion not met

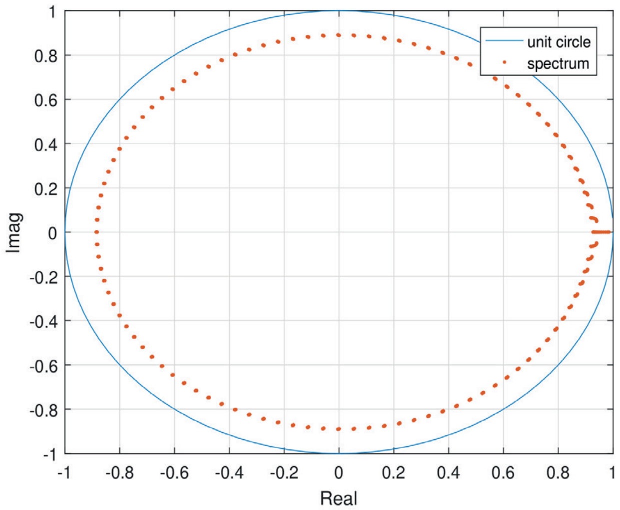

Fig. 8. Spectrum of the system (44) for $N = 22$, $h = 1$, the system is stable

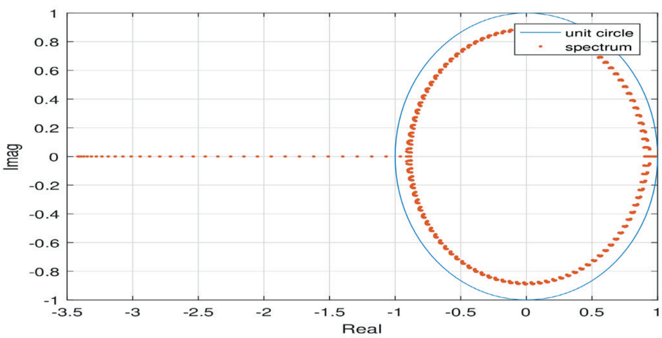

Fig. 9. Spectrum of the system (44) for for $N = 60$, $h = 1$, the system is unstable

Results given in Table 2 allow to conclude that the proposed stability criterions are practically unsensitive to disturbation of the heat exchange coefficient R_a. The sensitivity to disturbation of heat conduction coefficient a_w is bigger, but its value is resonable. Next, the sample time h is generally more senstive to parameters uncertainty than number of length divisions N.

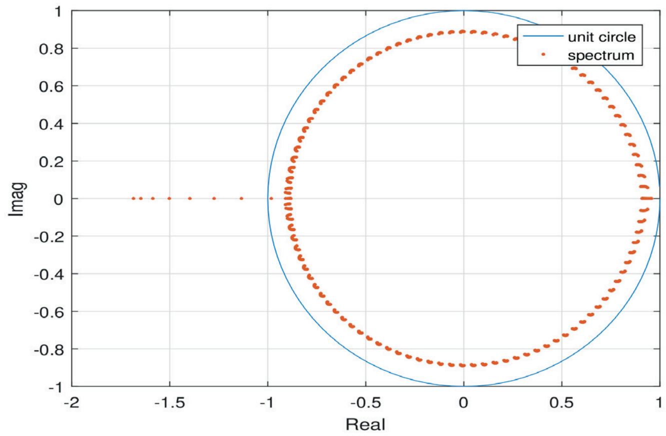

Fig. 10. Spectrum of the system (44) for for $N = 22$, $h = 5$, the system is unstable

Table 2
Sensitivity functions (60) for plant parameters from Table 1 and $\Delta a = \Delta R = 0.1$ (10%)

S_{Na_w} [%]	S_{NR_a} [%]	S_{ha_w} [%]	S_{hR_a} [%]
4.6567	0.0620	9.6401	0.4062

To complete the presentation of numerical results it is worth to compare the duration of calculations using practical implementation with delay (41) to calculation using the state space model without delay (44). During experiments the step response was assigned, all the parameters of model are given in Table 1, the number of tests was equal 1000, the duration of calculations was tested using MATLAB function *cputime*. Results are illustated by histograms 11, 12 and described in Table 3.

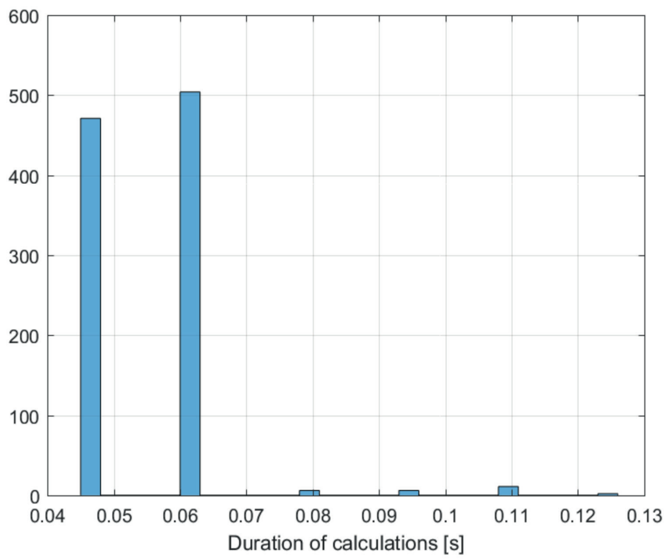

Fig. 11. Duration of calculations for practical implementation (41)

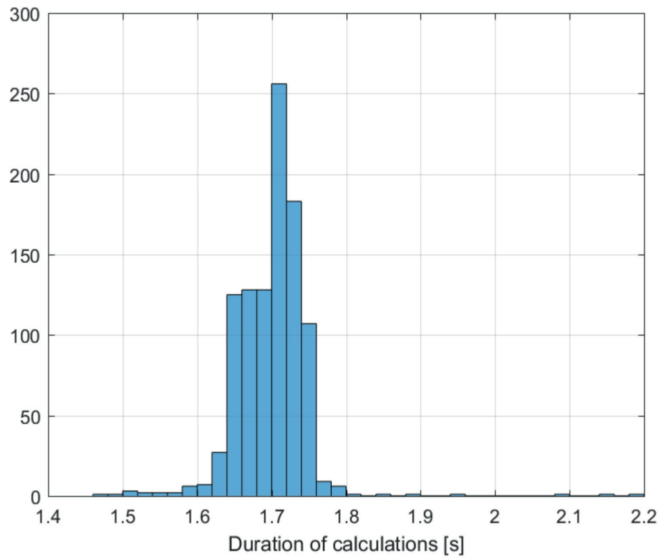

Fig. 12. Duration of calculations for state space model without delay (44)

Analysis Table 3 allows to conclude immidiately that computing of step response with the use of practical implementation (41) is circa 28 times faster than analogical calculation employing the state space model without delay (44) and MATLAB function *step*.

Table 3
Duration of calculations using the standard implementation (41) and state space model without delay (44)

model	min $[s]$	max $[s]$	mean $[s]$	std dev $[s]$
model (41)	0.0469	0.1250	0.0561	0.0106
model (44)	1.4688	2.1875	1.7014	0.0491

7. Final conclusions

The main final conclusion from this paper is that the discrete FO Grünwald-Letnikov operator allows to precisely describe partial derivatives along time and length. The proposed model is accurate in the sense of MSE cost function.The practical stability of the model as well as its numerical complexity were analyzed too.

Further investigation of the proposed model will cover its comparing to another similar models proposed previously. The analytical tests of complexity as well as convergence and accuracy seem also to be interesting.

Another issue to further analysis is the use of discrete versions of C and RL operators in the proposed model.

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