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On LQ optimization problem subject to fractional order irregular singular systems

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In this paper we discuss the linear quadratic (LQ) optimization problem subject to fractional order irregular singular systems. The aim of this paper is to find the control-state pairs satisfying the dynamic constraint of the form a fractional order irregular singular systems such that the LQ objective functional is minimized. The method of solving is to convert such LQ optimization into the standard fractional LQ optimization problem. Under some particularly conditions we find the solution of the problem under consideration.

Key words: linear quadratic optimization, fractional order, irregular singular system, Caputo fractional derivative, Mittag-Leffler function

1. Introduction

The LQ optimization problem subject to singular system constitutes an active research area in optimization and control field. Much more attention has been paid to study the following LQ optimization problem subject to singular system:

$$\min \ J(\omega, \xi) = \frac{1}{2}\int_0^1 \begin{bmatrix} \xi \\ \omega \end{bmatrix}^\top \begin{bmatrix} Q & O \\ O & R \end{bmatrix} \begin{bmatrix} \xi \\ \omega \end{bmatrix} \mathrm{d}t, \tag{1}$$

$$\text{s.t. } E\Delta_t \xi = A\xi + B\omega, \qquad \xi(0) = \xi_0, \tag{2}$$

where Δ_t denotes the derivative operator with respect to t, $\xi = \xi(t)$ denotes state, $\omega = \omega(t)$ denotes control; $E, A \in \mathbb{R}^{n\times n}$, with $\operatorname{rank}(E) < n$, $B \in \mathbb{R}^{n\times r}$; Q and R

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are positive definite matrices. It is well known that the problem to be solved in the optimization problem (1) and (2) is to find the control-state pairs $(\boldsymbol{\omega}, \boldsymbol{\xi})$ satisfying the dynamic constraint (2) such that the objective functional (1) is minimized, see [1], [2], [3], [4] and the literatures therein for exhaustively explanation.

Recently, several researchers have extend the study about this LQ optimization problem by replacing the fractional derivative operator Δ_t^α for Δ_t in (2) where $\alpha \in (k-1, k)$ with $k \in \mathbb{N}$ such that the equation (2) exactly is replaced by

$$E\Delta_t^\alpha \boldsymbol{\xi} = A\boldsymbol{\xi} + B\omega, \qquad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0 . \tag{3}$$

The equation (3) is called the fractional singular system of order α [5]. Optimization of objective functional (1) subject to fractional dynamic system (3) has reported by Chiranjeevi in [6] and [7] for which Δ_t^α is the fractional derivative in terms of Riemann–Liouville with $0 < \alpha < 1$.

In this paper we discuss the LQ optimization problem subject to irregular singular system of fractional order of the following form:

$$\min\ \mathcal{J}(\omega, \boldsymbol{\xi}) = \int_0^\infty \boldsymbol{\psi}^\top \boldsymbol{\psi}\, \mathrm{d}t \tag{4}$$

$$\text{s.t.} \quad \begin{cases} E\Delta_t^\alpha \boldsymbol{\xi} &= A\boldsymbol{\xi} + B\omega, \quad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \\ \qquad \boldsymbol{\psi} &= C\boldsymbol{\xi} + D\omega, \end{cases} \tag{5}$$

where Δ_t^α is the fractional derivative in terms of Caputo of order α with $0 < \alpha < 1$, $E, A \in \mathbb{R}^{m\times n}$, with $\operatorname{rank}(E) = p < \min\{m, n\}$, $B \in \mathbb{R}^{m\times r}$, $C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{q\times r}$ and $\boldsymbol{\psi} = \boldsymbol{\psi}(t)$ is the output. The irregular term arises from the size of the matrix E which is $m \times n$ [8]. The aim of this paper is to find the control-state pairs $(\omega, \boldsymbol{\xi})$ satisfying the fractional dynamic constraint (5) such that the objective functional (4) is minimized. The method of solving is to convert the LQ optimization (4) and (5) into the standard fractional LQ optimization problem as introduced in [9]. Indeed the LQ optimization problem (4) and (5) constitutes an extension of LQ optimization problem that proposed in [6], therefore the results of this paper constitutes a new contribution in the field of optimization subject for fractional singular dynamic system.

The rest of the paper is organized as follows. Section 2 considers some preliminaries about Caputo fractional derivative and fractional order differential equation system. Section 3 presents the conversion process the LQ optimization problem subject to fractional order irregular singular system into the standard fractional LQ optimization problem. The main result of this article is presented in the section 3 as well. A numerical example that illustrating the results is given in section 4. Section 5 concludes the paper.

2. Preliminaries

In this section we recall several used mathematical tools in sequel. Let be $\mathbf{x} : [0, \infty) \to \mathbb{R}^n$ is an integrable function. The formula of Caputo fractional derivative of order α with $\alpha \in (0, 1)$ is defined by:

$$\Delta_t^\alpha \mathbf{x}(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \Delta_\tau \mathbf{x}(\tau) \mathrm{d}\tau \tag{6}$$

where $\Gamma(.)$ is the Gamma function [9].

The one parameter Mittag-Leffler function for an arbitrary square matrix A is given by:

$$\mathcal{E}_\beta(A) = \sum_{j=0}^{\infty} \frac{A^j}{\Gamma(j\beta + 1)}, \tag{7}$$

where $\beta > 0$ [10]. It is easy to see that

$$\mathcal{E}_1(A) = \sum_{j=0}^{\infty} \frac{A^j}{\Gamma(j+1)} = \sum_{j=0}^{\infty} \frac{A^j}{j!} = \exp(A). \tag{8}$$

The two parameters Mittag–Leffler function for an arbitrary square matrix A was given:

$$\mathcal{E}_{\beta,\gamma}(A) = \sum_{j=0}^{\infty} \frac{A^j}{\Gamma(j\beta + \gamma)}, \tag{9}$$

where $\beta,\ \gamma > 0$. It is clear that $\mathcal{E}_{\beta,1}(A) = \mathcal{E}_\beta(A)$.

The Mittag-Leffler play important role in solve the system of the following fractional differential equation:

$$\Delta_t^\alpha \mathbf{x} = A\mathbf{x} + B\omega, \qquad \mathbf{x}(0) = \mathbf{x}_0, \quad 0 < \alpha < 1, \tag{10}$$

where D_t^α is Caputo fractional derivative. Using the Laplace transformation one can easily to prove the following theorem.

Theorem 1 [11] *The solution of system* (10) *is*

$$\mathbf{x}(t) = \mathcal{E}_\alpha(At^\alpha)\mathbf{x}_0 + t^\alpha \mathcal{E}_{\alpha,1+\alpha}(At^\alpha)\boldsymbol{\omega}.$$

3. Conversion and solution

Let us consider the LQ optimization problem (4) and (5). Without loss of generality, we assume that $m < n$. For shortanly we write the LQ optimization problem (4) and (5) as Ω. Let be define the admissible control-state pair set of problem Ω by:

$$\mathcal{A} \overset{\text{def}}{=} \{(\omega, \xi) \mid (\omega, \xi) \text{ is piecewise continuous, satisfies (5) and } \mathcal{J}(\omega, \xi) < \infty\}.$$

The problem under consideration is how the explicitly formulation of the control-state pairs $(\omega^*, \xi^*) \in \mathcal{A}$ for a given initial condition $\xi_0 \in \mathbb{R}^n$ such that

$$\mathcal{J}(\omega^*, \xi^*) = \min \mathcal{J}(\omega, \xi). \tag{11}$$

In order to convert the the LQ optimization problem (4) and (5) into the standard fractional LQ optimization problem, we need the following definitions that is an adaptation of Definition 1 in [8].

Definition 1 *The fractional order irregular singular system*

$$\begin{aligned} \bar{E}\Delta_t^\alpha \breve{\xi} &= \bar{A}\breve{\xi} + \bar{B}\omega, \qquad \breve{\xi}(0) = \breve{\xi}_0, \\ \psi &= \bar{C}\breve{\xi} + D\omega, \end{aligned}$$

is said to be restricted system equivalent (r.s.e.) to the system (5) if there exists two nonsingular matrices $M \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times n}$ such that $MEN = \bar{E}$, $MAN = \bar{A}$, $MB = \bar{B}$, $CN = \bar{C}$ and $\xi = N\breve{\xi}$.

Obviously, restricted system equivalence is an equivalent relationship and it is consistent with the Definition 1 in [8] for the standard singular systems.

Under the assumptions $m < n$ and $\operatorname{rank}(E) = p < m$, there exists the nonsingular matrices $M \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times n}$ such that

$$MEN = \begin{bmatrix} I_p & O \\ O & O \end{bmatrix}, \tag{12}$$

where I_p is an identity matrix of size $p \times p$ and O is a zero matrix. The result (12) is guaranteed by the Singular Value Decomposition(SVD) Theorem [12]. Accordingly, let

$$\begin{aligned} MAN &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \\ CN &= \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \qquad N^{-1}\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \end{aligned} \tag{13}$$

where $A_{11} \in \mathbb{R}^{p\times p}$, $B_1 \in \mathbb{R}^{p\times r}$, $C_1 \in \mathbb{R}^{q\times p}$ and $\boldsymbol{\xi}_1 \in \mathbb{R}^p$. Therefore, for a given initial state $\boldsymbol{\xi}_0 \in \mathbb{R}^n$, the fractional dynamical system (5) is r.s.e. to the following fractional system

$$\begin{cases} \Delta_t^\alpha \boldsymbol{\xi}_1 &= A_{11}\boldsymbol{\xi}_1 + A_{12}\boldsymbol{\xi}_2 + B_1\boldsymbol{\omega}, \\ \mathbf{0} &= A_{21}\boldsymbol{\xi}_1 + A_{22}\boldsymbol{\xi}_2 + B_2\boldsymbol{\omega}, \\ \boldsymbol{\psi} &= C_1\boldsymbol{\xi}_1 + C_2\boldsymbol{\xi}_2 + D\boldsymbol{\omega}, \end{cases} \tag{14}$$

with $\boldsymbol{\xi}_1(0) = \boldsymbol{\xi}_{10} = \begin{bmatrix} I_p & O \end{bmatrix} M\boldsymbol{\xi}_0$. One can see that if the system (5) is impulse controllable, the transformation (12) and (13) implies the system (14) is impulse controllable as well, see [13].

Using the expression (14), the functional objective (4) is equivalent to the following integral:

$$\mathcal{J}\left(\boldsymbol{\omega}, \hat{\boldsymbol{\xi}}\right) = \int_0^\infty \bar{\boldsymbol{\xi}}^\top Q \bar{\boldsymbol{\xi}} \, \mathrm{d}t,$$

where

$$\hat{\boldsymbol{\xi}} = \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix}, \qquad \bar{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\boldsymbol{\xi}} \\ \boldsymbol{\omega} \end{bmatrix}, \qquad Q = \begin{bmatrix} C_1^\top C_1 & C_1^\top C_2 & C_1^\top D \\ C_2^\top C_1 & C_2^\top C_2 & C_2^\top D \\ D^\top C_1 & D^\top C_2 & D^\top D \end{bmatrix}.$$

It is well known that the impulse controllability of the fractional singular system (14) is equivalent to $\operatorname{rank}\begin{bmatrix} A_{21} & A_{22} & B_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_{22} & B_2 \end{bmatrix}$. In fact, the matrix $\begin{bmatrix} A_{22} & B_2 \end{bmatrix}$ may have no full row rank. Let us denote $\operatorname{rank}\begin{bmatrix} A_{22} & B_2 \end{bmatrix} = s$, where $s \leqslant m - p \leqslant n - p$. It follows that there exists a nonsingular matrix $\Phi \in \mathbb{R}^{(m-p)\times(m-p)}$ such that

$$\Phi \begin{bmatrix} A_{22} & B_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{22} & \bar{B}_2 \\ O & O \end{bmatrix}$$

where $\begin{pmatrix} \bar{A}_{22} & \bar{B}_2 \end{pmatrix}$ has full row rank. By adapting the procedure in [8] we have the following transformation:

$$\bar{\boldsymbol{\xi}} = \begin{bmatrix} I_p & O \\ O & F \end{bmatrix} \left[\begin{array}{c|c} I_p & O \\ \hline -\bar{A}_{21} & O \\ O & I_{n-p+r-s} \end{array}\right] \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\upsilon} \end{bmatrix}, \tag{15}$$

for some $\boldsymbol{\upsilon} \in \mathbb{R}^{n-p+r-s}$ and for some nonsingular matrix

$$F \overset{\text{def}}{=} \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in \mathbb{R}^{(n-p+r)\times(n-p+r)},$$

where $\bar{A}_{21} = \begin{bmatrix} I_s & 0 \end{bmatrix} \Phi A_{21}$. Finally, using (15) the objective $\mathcal{J}(\omega, \hat{\xi})$ is equivalent to the following objective functional:

$$\mathcal{J}(\upsilon, \xi_1) = \int_0^\infty \begin{bmatrix} \xi_1 \\ \upsilon \end{bmatrix}^\top \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \upsilon \end{bmatrix} \mathrm{d}t, \tag{16}$$

and (14) is equivalent to

$$\begin{cases} \Delta_t^\alpha \xi_1 = \bar{A}\xi_1 + \bar{B}_1 \upsilon, \qquad \xi_1(0) = \xi_{10}, \\ \psi = \bar{C}\xi_1 + \bar{D}\upsilon, \end{cases} \tag{17}$$

where

$$\begin{aligned} \bar{A} &= A_{11} - (A_{12}F_{11} + B_1 F_{21})\bar{A}_{21}, \\ \bar{B}_1 &= A_{12}F_{12} + B_1 F_{22}, \\ \bar{C} &= C_1 - \begin{bmatrix} C_2 & D \end{bmatrix} F \begin{bmatrix} \bar{A}_{21} \\ O \end{bmatrix}, \\ \bar{D} &= \begin{bmatrix} C_2 & D \end{bmatrix} F \begin{bmatrix} O \\ I_{n-p+r-s} \end{bmatrix}, \end{aligned}$$

$$Q_{11} = \bar{C}^\top \bar{C}, \qquad Q_{12} = \bar{C}^\top \bar{D}, \qquad Q_{22} = \bar{D}^\top \bar{D}.$$

Now, one can see that the LQ optimization problem Ω is equivalent to a new LQ optimization problem that minimize the objective functional (16) subject to fractional dynamic system (17). Let us denote this LQ optimization problem as Ω_1, and define the set of admissible control-state pairs of problem Ω_1 by

$$\mathcal{A}_1 \stackrel{\text{def}}{=} \{(\upsilon, \xi_1) | (\upsilon, \xi_1) \text{ is piecewise continuous, satisfy (17) and } \mathcal{J}(\upsilon, \xi_1) < \infty\}.$$

One can see that the fractional dynamic system (17) is a standard fractional dynamic system with the state ξ_1, the control υ and the output ψ, so Ω_1 is a standard fractional LQ optimization problem.

It is well known that the solution of LQ optimization problem Ω_1 hinges on the behavior of input weighting matrix Q_{22} in equation (16), whether it is positive definite or positive semidefinite. Under a certain property, Q_{22} may be positive definite, see [2] for detail. In the case where Q_{22} is positive definite, one can use the theory in [14] and [11] regarding the standard fractional LQ optimization problem in which it is mentioned that Ω_1 has a unique optimal control-state pair if the the system (17) is controllable. It is well known that the system (17) is controllable if $\operatorname{rank}\Big(\Big[\ \bar{B}_1 \ \mid\ \bar{A}\bar{B}_1 \ \mid\ \ldots\ \mid\ \bar{A}^{p-1}\bar{B}_1\ \Big]\Big) = p$. Under this controllability condition the control that minimizes $\mathcal{J}(\boldsymbol{v}, \boldsymbol{\xi}_1)$ is

$$\boldsymbol{v}^* = -Q_{22}^{-1}\left(Q_{12}^\top + \bar{B}_1^\top P\right)\boldsymbol{\xi}_1^* \tag{18}$$

where the state $\boldsymbol{\xi}_1^*$ is the solution of the following fractional differential equation:

$$\Delta_t^\alpha \boldsymbol{\xi}_1 = \left(\bar{A} - \bar{B}_1 Q_{22}^{-1}\left(Q_{12}^\top + \bar{B}_1^\top P\right)\right)\boldsymbol{\xi}_1, \qquad \boldsymbol{\xi}_1(0) = \boldsymbol{\xi}_{10} \tag{19}$$

with P is the unique positive semidefinite solution of the following algebraic Riccati equation:

$$\bar{A}^\top P + P\bar{A} + Q_{11} - \left(P\bar{B}_1 + Q_{12}\right)Q_{22}^{-1}\left(P\bar{B}_1 + Q_{12}\right)^\top = O. \tag{20}$$

One can see that the completion process of the LQ optimization problem Ω_1 requires solving of the fractional differential equation (19). Using the Theorem 1 the solution of equation (19) is

$$\boldsymbol{\xi}_1(t) = \mathcal{E}_\alpha\left(\left(\bar{A} - \bar{B}_1 Q_{22}^{-1}\left(Q_{12}^\top + \bar{B}_1^\top P\right)\right)t^\alpha\right)\boldsymbol{\xi}_{10},$$

where the matrix P satisfies the equation (20). Thus, using the transformation (15) and (13), the optimal control-state pair $(\boldsymbol{\omega}^*, \boldsymbol{\xi}^*)$ of the LQ optimization problem Ω is given by

$$\begin{bmatrix} \boldsymbol{\xi}^* \\ \boldsymbol{\omega}^* \end{bmatrix} = \begin{bmatrix} N & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} I_p \\ -F_{11}\bar{A}_{21} - F_{12}Q_{22}^{-1}(Q_{12}^\top + \bar{B}_1^\top P) \\ \hdashline -F_{21}\bar{A}_{21} - F_{22}Q_{22}^{-1}(Q_{12}^\top + \bar{B}_1^\top P) \end{bmatrix} \boldsymbol{\xi}_1^*. \tag{21}$$

4. An example

Let us consider the LQ optimization problem (4) and (5) where the matrices E, A, B, C and D are given as follows:

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ -1 & 4 & 2 & 0 & 1 \\ -1 & 4 & 2 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -4 \\ 1 & 0 \\ 1 & 2 \\ 1 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 4 & 0 & 2 & 3 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -3 & 6 & -1 \\ 3 & 0 & 1 & -2 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

with the initial state is $\boldsymbol{\xi}_0 = \begin{bmatrix} 2 & 0 & 0 & 0 \end{bmatrix}^\top$.

By taking the matrices $M = I_4$ and $N = I_5$, we have

$$MEN = \begin{bmatrix} I_2 & O \\ O & O \end{bmatrix}.$$

It is easy to verify that

$$\operatorname{rank}\begin{bmatrix} A_{21} & A_{22} & B_2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_{22} & B_2 \end{bmatrix} = 1,$$

thus the irregular singular system (5) is impulse controllable. By choosing

$$\Phi = \begin{bmatrix} 0.5 & 0.5 \\ -0.7071 & 0.7071 \end{bmatrix},$$

and

$$F = \begin{bmatrix} 0.5 & 0 & -0.3162 & -0.3162 & -0.6325 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.9387 & -0.0613 & -0.1225 \\ 0 & 0 & -0.0613 & 0.9387 & -0.1225 \\ 0 & 0 & -0.1225 & -0.1225 & 0.7550 \end{bmatrix},$$

the problem Ω can be equivalently changed into the following standard fractional LQ optimization problem:

$$\min \int_0^\infty \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{v} \end{bmatrix}^\top \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\top & Q_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{v} \end{bmatrix} \mathrm{d}t$$

$$\text{s.t.} \begin{cases} \Delta_t^\alpha \boldsymbol{\xi}_1 &= \bar{A}\boldsymbol{\xi}_1 + \bar{B}_1\boldsymbol{v}, \quad \boldsymbol{\xi}_1(0) = \begin{bmatrix} 2 & 0 \end{bmatrix}^\top \\ \boldsymbol{\psi} &= \bar{C}\boldsymbol{\xi}_1 + \bar{D}\boldsymbol{v}, \end{cases}$$

where $\boldsymbol{\xi}_1 \in \mathbb{R}^2$, $\boldsymbol{v} \in \mathbb{R}^4$,

$$\bar{A} = \begin{bmatrix} -0.5 & -2 \\ 1 & 0 \end{bmatrix}, \qquad \bar{B}_1 = \begin{bmatrix} 0 & 0.1126 & 1.1126 & -3.7749 \\ -1 & 0.8775 & 0.8775 & -0.2450 \end{bmatrix},$$

$$\bar{C} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \\ -0.5 & 7 \\ 3.5 & -2 \end{bmatrix}, \qquad \bar{D} = \begin{bmatrix} 2 & 2.4487 & 1.4487 & 0.8974 \\ 1 & 0 & 0 & 0 \\ 6 & -0.1738 & 1.8262 & 2.6523 \\ -2 & 1.5613 & -0.4387 & -0.8775 \end{bmatrix},$$

$$Q_{11} = \begin{bmatrix} 13.5 & -10.5 \\ -10.5 & 69 \end{bmatrix}, \qquad Q_{12} = \begin{bmatrix} -9 & 5.5513 & -2.4487 & -4.3974 \\ 54 & 5.4554 & 19.4554 & 23.9108 \end{bmatrix},$$

and

$$Q_{22} = \begin{pmatrix} 45 & 0.7319 & 14.7319 & 19.4637 \\ 0.7319 & 8.4638 & 2.5449 & 0.3663 \\ 14.7319 & 2.5449 & 5.6261 & 6.5286 \\ 19.4637 & 0.3663 & 6.5286 & 8.6101 \end{pmatrix}.$$

The solution of this LQ optimization problem is

$$\boldsymbol{v} = L\boldsymbol{\xi}_1^*,$$

where

$$L = \begin{bmatrix} 2.5167 & 0.1854 \\ 0.4781 & -0.3534 \\ 0.8865 & 3.2861 \\ -6.9710 & -0.1284 \end{bmatrix}, \qquad P = \begin{bmatrix} 0.1780 & 0.0218 \\ 0.0218 & 0.0027 \end{bmatrix},$$

and $\boldsymbol{\xi}_1^*$ satisfies the following fractional differential equation:

$$\Delta_t^\alpha \boldsymbol{\xi}_1 = \begin{bmatrix} -27.8545 & -3.1009 \\ 0.6112 & -2.4194 \end{bmatrix} \boldsymbol{\xi}_1, \quad \boldsymbol{\xi}_1(0) = \begin{bmatrix} 2 & 0 \end{bmatrix}^\top.$$

Solution of this equation is

$$\begin{aligned}
\boldsymbol{\xi}_1(t) &= \mathcal{E}_\alpha\left(\begin{bmatrix} -27.8545 & -3.1009 \\ 0.6112 & -2.4194 \end{bmatrix} t^\alpha\right)\begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
&= \sum_{j=0}^{\infty} \begin{bmatrix} -27.8545 & -3.1009 \\ 0.6112 & -2.4194 \end{bmatrix}^j \begin{bmatrix} 2 \\ 0 \end{bmatrix} \frac{t^{j\alpha}}{\Gamma(j\alpha+1)} \\
&= \sum_{j=0}^{\infty} \begin{bmatrix} -0.9997\dfrac{(-27.7798t^\alpha)^j}{\Gamma(j\alpha+1)} & 0 \\ 0 & -0.9926\dfrac{(-2.4941t^\alpha)^j}{\Gamma(j\alpha+1)} \end{bmatrix} \begin{bmatrix} -2.0065 \\ -0.0487 \end{bmatrix} \\
&= \begin{bmatrix} (2.0059)\mathcal{E}_\alpha(-27.7798t^\alpha) \\ (0.0483)\mathcal{E}_\alpha(-2.4941t^\alpha) \end{bmatrix}.
\end{aligned}$$

Using (15) we have

$$\xi^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -3.4773 & -1.1538 \\ -2.5167 & -0.1854 \\ 1.2485 & 0.5173 \end{bmatrix} \begin{bmatrix} (2.0059)\mathcal{E}_\alpha(-27.7798t^\alpha) \\ (0.0483)\mathcal{E}_\alpha(-2.4941t^\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} (2.0059)\mathcal{E}_\alpha(-27.7798t^\alpha) \\ (0.0483)\mathcal{E}_\alpha(-2.4941t^\alpha) \\ (-6.9751)\mathcal{E}_\alpha(-27.7798t^\alpha) + (-0.0557)\mathcal{E}_\alpha(-2.4941t^\alpha) \\ (-5.0482)\mathcal{E}_\alpha(-27.7798t^\alpha) + (-0.0089)\mathcal{E}_\alpha(-2.4941t^\alpha) \\ 2.5044\mathcal{E}_\alpha(-27.7798t^\alpha) + (0.0250)\mathcal{E}_\alpha(-2.4941t^\alpha) \end{bmatrix}$$

and

$$\omega^* = \begin{bmatrix} -1.6570 & -3.1222 \\ 5.4301 & 0.4562 \end{bmatrix} \begin{bmatrix} (2.0059)\mathcal{E}_\alpha(-27.7798t^\alpha) \\ (0.0483)\mathcal{E}_\alpha(-2.4941t^\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} -3.3238E_\alpha(-27.7798t^\alpha) + (-0.1508)\mathcal{E}_\alpha(-2.4941t^\alpha) \\ (10.8922)\mathcal{E}_\alpha(-27.7798t^\alpha) + (0.0220)\mathcal{E}_\alpha(-2.4941t^\alpha) \end{bmatrix}.$$

The trajectories of state ξ^* is shown in Fig. 1 and the control ω^* is shown in Fig. 2.

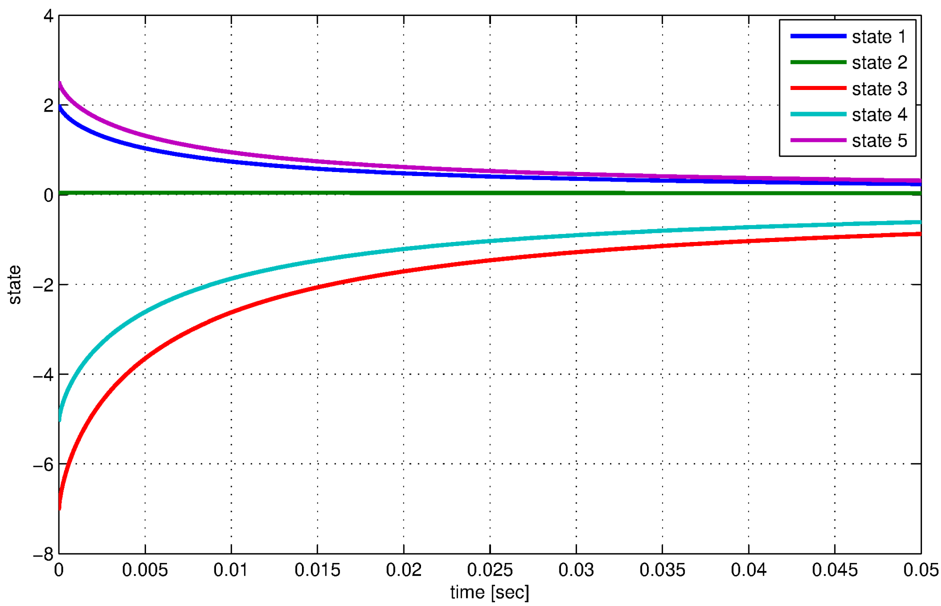

Figure 1: State trajectories for $\alpha = 0.9$

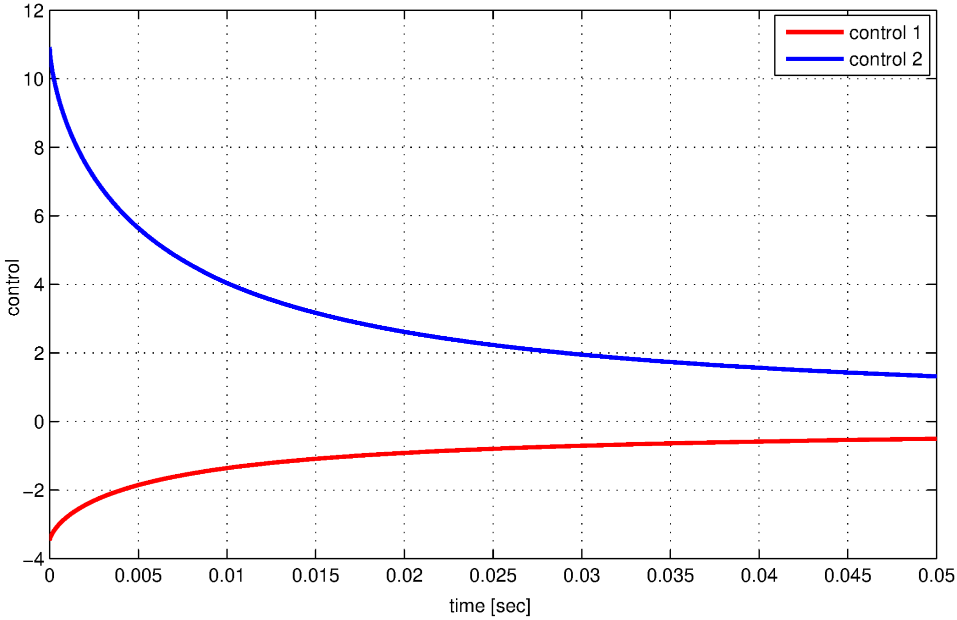

Figure 2: Control trajectory for $\alpha = 0.9$

5. Conclusion

We have find the explicitly formulation of control-state pairs that constitute the solution of the LQ optimization problem subject to fractional order irregular singular systems. An example that illustrating the result has been presented.

References

[1] G.R. Duan, *Analysis and design of descriptor linear systems*, Springer, 2010.

[2] Muhafzan: Use of semidefinite programming for solving the LQR problem subject to descriptor systems, *Int. J. Math. Computh. Sci.*, **20** (2010), 655–664.

[3] L. Yulianti, A. Nazra, Zulakmal, A. Bahar, and Muhafzan: On discounted LQR control problem for disturbanced singular system, *Archives of Control Sciences*, **29**(1) (2019), 147–156.

[4] X. Wang and B. Liu: Singular linear quadratic optimal control problem for stochastic nonregular descriptor systems, *Asian Journal of Control*, **20**(6) (2018), 1–11.

[5] A. Younus, I. Javaid, and A. Zehra: On controllability and observability of fractional continuous-time linear systems with regular pencils, *Bulletin of The Polish Academy of Sciences*, **65**(3) (2017), 297–304.

[6] T. Chiranjeevi, R.K. Biswas, and C. Sethi: Optimal control of fractional order singular system, *International Journal of Electrical Engineering & Education*, (2019), DOI: 0.1177/0020720919833031.

[7] T. Chiranjeevi and R.K. Biswas: Linear quadratic optimal control problem of fractional order Continuous-time singular system, *Procedia Computer Science*, **171** (2020), 1261–1268.

[8] Q. Fang, B. Zhang, and J. Feng: Singular LQ problem for irregular singular systems, *Journal of Applied Mathematics*, 2014 (853415) (2014), DOI: 10.1155/2014/853415.

[9] I. Matychyn and V. Onyshchenko: Optimal control of linear systems with fractional derivatives, *Fractional Calculus & Applied Analysis*, **21**(1) (2018), 134–150.

[10] I. Matychyn and V. Onyshchenko: On time-optimal control of fractional-order systems, *Journal of Computational and Applied Mathematics*, **339** (2018), 245–257.

[11] Y. Li and Y.Q. Chen: Fractional order linear quadratic regulator, *Proceeding of IEEE/ASME International Conference on Mechtronic and Embedded Systems and Applications*, (2008), 363–368.

[12] V.C. Klema and A.J. Laub: The singular value decomposition: its computation and some applications, *IEEE Transactions on Automatic Control*, **25**(2) (1980), 164–176.

[13] Zulakmal, Narwen, B. Rudianto, A.I. Baqi, and Muhafzan: On the LQ optimization subject to descriptor system under disturbance, *Asian Journal of Scientific Research*, **11**(4) (2018), 540–543.

[14] O.M. Fuentes and R.M. Guerra: A novel Mittag–Leffler stable estimator for nonlinear fractional-order systems: a linear quadratic regulator, *Nonlinear Dynamics*, **94** (2018), 1973–1986.