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Extremal values of differential equations with application to control systems

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Abstract. In the paper, maximal values $x_e(\tau)$ of the solutions $\mathrm{x}(\mathrm{t})$ of the linear differential equations excited by the Dirac delta function are determined. The analytical solutions of the equations and also the maximal positive values of these solutions are obtained. The analytical formulae enable the design of the system with prescribed properties. The complementary case to the earlier paper is presented. In an earlier paper it was assumed that the roots s_i are different, and now we consider the case when $s_1 = s_2 = \dots = s_n$.

Key words: extremal values; characteristic equation; transfer function; Dirac's impulse; multiple root.

1. Introduction

The extremal value of the state variable $x(t)$ has a fundamental role in the many branches of the industry. In the chemical industry, the overrising temperature or pressure can lead to the explosion. In the energy industry, the overvoltage waves can destroy the installation. In the economic systems, it is the determination of the maximal profit. The search for extremal values of the controlled quantity was the subject of many papers [1–5]. However, no analytic formulae for their calculation were found. This paper provides the original formulae which allow the determination of extremal values. For the first time, solutions of a certain class of transcendental equations are given in the form of analytical relationships. These formulae make it possible to estimate the accuracy of the performance of systems described by differential equations. In this way they fill the gap existing in the literature on the subject.

In paper [6] it was assumed that the roots s_i are different, and now we consider the case when $s_1 = s_2 = \dots = s_n$.

We consider the dynamic system which is described by the differential equation:

$$x^{(n)}(t) + a_1 x^{(n-1)}(t) + \dots + a_{n-1} x^{(1)}(t) + a_n x(t) = \\ = b_1 u^{(m)}(t) + b_2 u^{(m-1)}(t) + \dots + b_m u(t), \tag{1}$$

where
$a_i > 0$, $b_i \geq 0$, $i = 1, 2, \dots, n$ – are constant parameters,
$x(t)$ – is a dynamic error,
$u(t) = \delta(t)$ – is Dirac impulse of the external signal.

The characteristic equation of Eq. (1) is:

$$M(s) = (s - s_1)^n = 0. \tag{2}$$

We assume that the root s_1 of Eq. (2) is real and negative. We call it the pole:

$$0 > s_1. \tag{3}$$

The zeroes of the polynomial $L(s)$ given below, are real, different and negative:

$$L(s) = (s - z_1)(s - z_2) \dots (s - z_m), \quad m < n, \tag{4}$$

and

$$0 > s_1 > z_1 > z_2 > \dots > z_m. \tag{5}$$

We denote the transfer function:

$$G(s) = \frac{L(s)}{M(s)}, \tag{6}$$

and the solution of Eq. (1) in the operational form is:

$$X(s) = \frac{L(s)}{M(s)} \Delta(s). \tag{7}$$

Then the solution of Eq. (1) in the time domain is:

$$x(t) = e^{s_1 t} \sum_{k=1}^{n} A_k t^{k-1}, \tag{8}$$

where

$$A_k = \sum_{i=1}^{k} \frac{x^{(k-i)}(0)(-1)^i s_1^i}{i!(k-i)!}, \tag{9}$$

The initial conditions of Eq. (1) depend on Dirac impulse $\delta(t)$ and the poles and zeroes of the transfer function $G(s)$.

2. Statement of the problem

We want to determine the number of the extremal values $x(\tau)$ of the solution $x(t)$ and the analytic formulae for the extremal times τ.

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There are many papers connected with this problem [7–13]. In the literature there are no analytical formulae for the extremal error of the dynamic system. The extremum dynamic error gives information about the accuracy of the system. The time gives information about the velocity of the transient. The analytical formulae facilitate the design of the system with prescribed properties.

We consider the following four cases:

1. $L(s) = 1$,
2. $L(s) = (s - z_1)$,
3. $L(s) = (s - z_1)(s - z_2)$,
4. $L(s) = \prod_{i=1}^{n-1} (s - z_i)$, $i = 1, 2, \ldots, n - 1$.

3. Solution of the problem

3.1. Case 1. $m = 0$, $L(s) = 1$, $M(s) = (s - s_1)^n$.
The initial conditions of Eq. (1) are forced by $\delta(t)$ [14]

$$\left.\begin{array}{c} x^{(i)}(0) = 0, \ \ i = 0,1,\ldots,n-2 \\ x^{(n-1)}(0) = 1 \end{array}\right\}. \tag{10}$$

In this case the coefficients are:

$$A_k = 0 \quad \text{for} \quad k = 1,2,\ldots,n-1, \tag{11}$$

$$A_n = \frac{x^{(n-1)}(0)}{(n-1)!} = \frac{1}{(n-1)!}. \tag{12}$$

The solution of Eq. (1) is:

$$x(t) = e^{s_1 t} \frac{x^{(n-1)}(0)}{(n-1)!} t^{n-1}. \tag{13}$$

The derivative

$$x^{(1)}(t) = e^{s_1 t} t^{n-2} \left[\frac{x^{(n-1)}(0)}{(n-1)!} s_1 t + \frac{x^{(n-1)}(0)}{(n-2)!} \right]. \tag{14}$$

From the necessary condition

$$x^{(1)}(t) = 0, \tag{15}$$

we obtain:

Theorem 1. The times of the extremums are as follows: $\tau_1 = 0$ has multiplicity $n - 2$, and

$$\tau_2 = -\frac{n-1}{s_1}. \tag{16}$$

Remark 1. If solution $\tau(s_1)$ can be optimized with respect to s_1 then the optimal time τ_{2op} is equal to (see [15]):

$$\tau_2 = -\frac{n}{s_1}. \tag{17}$$

In Fig. 1 an illustrative example with $n = 4$, $s_1 = -1$ is shown:

$$\tau_2 = n - 1 = 3. \tag{18}$$

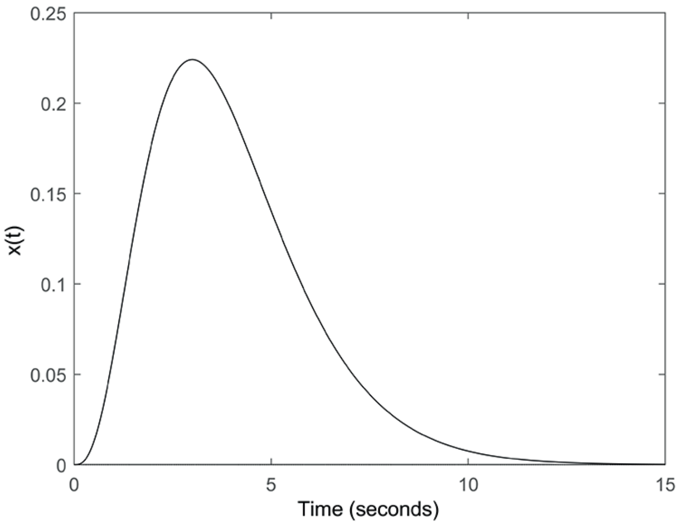

Fig. 1. Time response of the system for $n = 4$, $s_1 = -1$

From (16) we have: $\tau_1 = 0$, $\tau_2 = 3$. The extremal value is $x(\tau_2) = 0.2240418077$.

3.2. Case 2. We assume:

$$L(s) = s - z_1, \tag{19}$$

which means that: $m = 1$, $z_1 < s_1 < 0$.
The initial conditions of Eq. (1) in this case are:

$$\left.\begin{array}{c} x^{(i)}(0) = 0 \ \ \text{for } i = 0,1,2,\ldots,n-3 \\ x^{(n-2)}(0) = 1 \\ x^{(n-1)}(0) = ns_1 - z_1 \end{array}\right\}. \tag{20}$$

The solution of $x(t)$ is:

$$x(t) = -[(z_1 - s_1)t - (n-1)] \frac{t^{n-2}}{(n-1)!} e^{s_1 t}. \tag{21}$$

Theorem 2. The first derivative of $x(t)$ in this case is equal to:

$$\begin{aligned} \frac{dx(t)}{dt} = {} & [(z_1 - s_1)s_1 t^2 - (n-1)(2s_1 - z_1)t - \\ & - (n-1)(n-2)] t^{n-3} e^{s_1 t}. \end{aligned} \tag{22}$$

From the necessary condition $x^{(1)}(t) = 0$ we have:

$$\left.\begin{array}{c} \tau_1^{n-3} = 0 \\ \tau_2 = \frac{1}{2} \frac{(n-1)(2s_1 - z_1) \pm \sqrt{(n-1)^2(2s_1 - z_1)^2 + 4(n-1)(n-2)(z_1 - s_1)s_1}}{s_1(z_1 - s_1)} \end{array}\right\}. \tag{23}$$

It is easy to observe that the discriminant Δ of the Eq. (22) is positive:

$$\left.\begin{array}{c} \Delta = \\ (n-1)^2(2s_1 - z_1)^2 + 4(n-1)(n-2)(z_1 - s_1)s_1 > 0 \\ \text{and} \quad \sqrt{\Delta} > (n-1)(2s_1 - z_1) \end{array}\right\}, \tag{24}$$

and the Eq. (22) has only one solution $\tau_2 > 0$.

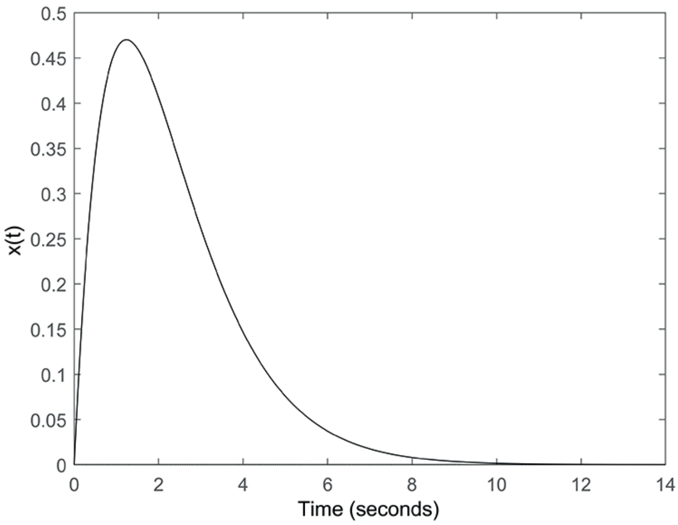

Fig. 2. Time response of the system for $n = 3$, $s_1 = -1$, $z_1 = -1.5$

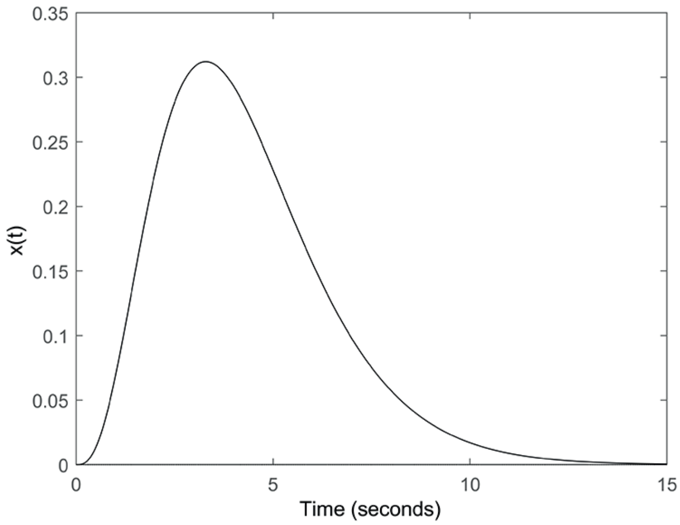

Fig. 4. Time response of the system for $n = 5$, $s_1 = -1$, $z_1 = -1.5$

Examples. In Fig. 2 the time response of the system is given for $n = 3$, $s_1 = -1$, $z_1 = -1.5$.

From (23) we have: $\tau_2 = 1.236067978$. The extremal value is $x(\tau_2) = 0.4700782295$.

In Fig. 3 the time response of the system is shown, for $n = 4$, $s_1 = -2$, $z_1 = -2.5$.

From (23) we have: $\tau_1 = 0$, $\tau_2 = 1.076033674$. The extremal value is: $x(\tau_2) = 0.07936506865$.

In Fig. 4 the time response of the system is shown, for $n = 5$, $s_1 = -1$, $z_1 = -1.5$.

From (23) we have: $\tau_1 = 0$, $\tau_2 = 3.291502624$. The extremal value is: $x(\tau_2) = 0.3120410227$.

3.3. Case 3.

$$L(s) = (s - z_1)(s - z_2), \tag{25}$$

which means that: $m = 2$, $z_2 < z_1 < s_1 < 0$.

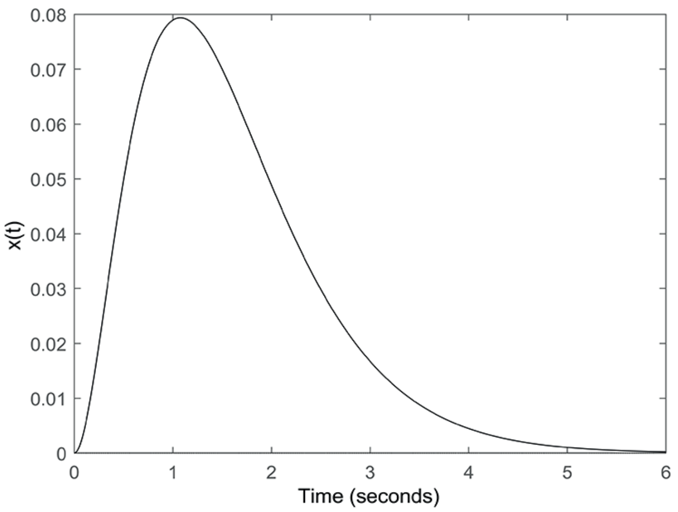

Fig. 3. Time response of the system for $n = 4$, $s_1 = -2$, $z_1 = -2.5$

The initial conditions of Eq. (1) in this case are:

$$\left.\begin{array}{c} x^{(i)}(0) = 0 \quad \text{for } i = 0,1,2,\dots,n-4 \\ x^{(n-3)}(0) = 1 \\ x^{(n-2)}(0) = ns_1 - (z_1 + z_2) \\ x^{(n-1)}(0) = z_1 z_2 - ns_1(z_1 + z_2) + \frac{1}{2}n(n+1)s_1^2 \end{array}\right\}. \tag{26}$$

Theorem 3. The solution $x(t)$ is:

$$x(t) = [(z_2 - s_1)(z_1 - s_1)t^2 - (n-1)(z_2 + {} + z_1 - 2s_1)t + (n-1)(n-2)]\,\frac{t^{n-3}}{(n-1)!}e^{s_1 t}. \tag{27}$$

The first derivative of $x(t)$ in this case is equal:

$$\begin{aligned} \frac{dx(t)}{dt} &= e^{s_1 t}\{(z_1 - s_1)(z_2 - s_1)s_1 t^3 + \\ &+ (n-1)[(z_1 - s_1)(z_2 - s_1) + (2s_1 - z_1 - z_2)s_1]t^2 + \\ &+ (n-1)(n-2)(3s_1 - z_1 - z_2)t + \\ &+ (n-1)(n-2)(n-3)\}t^{n-4}. \end{aligned} \tag{28}$$

From the necessary condition $x^{(1)}(t) = 0$ we have:

$$\tau_1 = 0, \tag{29}$$

with multiplicity $n - 4$, and τ_2 can be calculated from the equation:

$$\begin{aligned} &(z_1 - s_1)(z_2 - s_1)s_1\tau_2^3 + (n-1)[(z_1 - s_1)(z_2 - s_1) + \\ &+ (2s_1 - z_1 - z_2)s_1]\tau_2^2 + \\ &+ (n-1)(n-2)(3s_1 - z_1 - z_2)\tau_2 + \\ &+ (n-1)(n-2)(n-3) = 0. \end{aligned} \tag{30}$$

The analytical formula for τ_2 is rather complicated; however, it is possible to obtain it for three zeroes.

It can be shown that in the case $z_1, z_2, \ldots, z_{n-1}$ the time τ_2 is determined from the equation of degree n.

Examples. In Fig. 5 the time response of the system is shown, for $n = 3$, $s_1 = -1$, $z_1 = -1.5$, $z_2 = -2.5$.

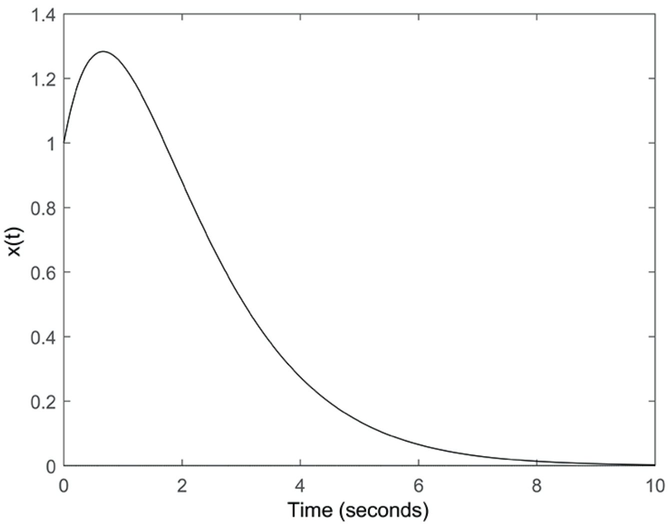

Fig. 5. Time response of the system for $n = 3$, $s_1 = -1$, $z_1 = -1.5$, $z_2 = -2.5$

From (30) we have: $\tau_2 = 0.666666665$. The extremal value is $x(\tau_2) = 1.283542798$.

In Fig. 6 the time response of the system is shown, for $n = 5$, $s_1 = -1$, $z_1 = -1.5$, $z_2 = -2.5$.

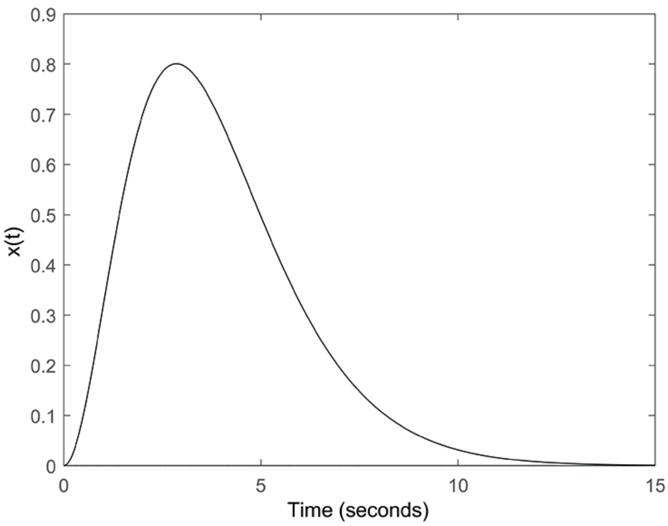

Fig. 6. Time response of the system for $n = 5$, $s_1 = -1$, $z_1 = -1.5$, $z_2 = -2.5$

From (29) and (30) we have $\tau_1 = 0$, $\tau_2 = 2.856475152$. The extremal value is $x(\tau_2) = 0.8005324665$.

3.4. Case 4. $m = n - 1$.

The equivalent form to Eq. (4), using Vieta's formulae is:

$$L(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_{n-1} s + b_n, \tag{31}$$

where

$$\left.\begin{aligned} \frac{b_2}{b_1} &= (-1)^{n-1} \sum_{i=1}^{n-1} z_i \\ \frac{b_3}{b_1} &= \sum_{\substack{i=1 \\ i \neq j}}^{n-1} z_i z_j \\ &\;\;\vdots \\ \frac{b_n}{b_1} &= (-1)^{n-1} \prod_{i=1}^{n-1} z_i \end{aligned}\right\}, \tag{32}$$

The transfer function is:

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{(s - s_1)^n} = \frac{b_1 \prod_{i=1}^{n-1}(s - z_i)}{(s - s_1)^n}. \tag{33}$$

Then the solution $x(t)$ is:

$$\begin{aligned} x(t) = \Big\{ & b_1 + \left[b_2 - \tbinom{n-1}{1} b_1 s_1\right] t + \left[b_3 - \tbinom{n-2}{1} b_2 s_1 + \right. \\ & \left. + \tbinom{n-1}{2} b_1 s_1^2\right] \frac{t^2}{2!} + \left[b_1 - \tbinom{n-3}{1} b_3 s_1 + \right. \\ & \left. + \tbinom{n-2}{2} b_2 s_1^2 - \tbinom{n-1}{3} b_1 s_1^3\right] \frac{t^3}{3!} + \cdots + \\ & + \left[b_n - b_{n-1} s_1 + \cdots + b_1 (-s_1)^{n-1}\right] \frac{t^{n-1}}{(n-1)!} \Big\} e^{s_1 t}. \end{aligned} \tag{34}$$

We assume that Eq. (31) has only real, negative roots. The sufficient conditions for this are:

Theorem 4. [16]. Let $L(s)$ be a polynomial of degree $n \geq 2$ with positive coefficients. If

$$\begin{cases} b_i^2 - 4 b_{i-1} b_{i+1} > 0, \quad i = 1, 2, \ldots, n-2 \\ \qquad b_i > 0 \end{cases}, \tag{35}$$

then all the roots of $L(s)$ are real, negative and distinct.

We obtain the derivative of $x(t)$ from the relation (34).

$$\begin{aligned} \frac{dx(t)}{dt} = \Big\{ & \left[b_2 - \tbinom{n-1}{1} b_1 s_1\right] s_1 + b_1 s_1 + s_1 \left[b_2 + \right. \\ & \left. + \tbinom{n-1}{1} b_1 s_1\right] t + \left[b_3 - \tbinom{n-2}{1} b_2 s_1 + \tbinom{n-1}{2} b_1 s_1^2\right] t + \\ & + \left[b_3 - \tbinom{n-2}{1} b_2 s_1 + \tbinom{n-1}{2} b_1 s_1^2\right] s_1 \frac{t^2}{2!} + 3\left[b_1 - \right. \\ & \left. - \tbinom{n-3}{1} b_3 s_1 + \tbinom{n-2}{2} b_2 s_1^2 - \tbinom{n-1}{3} b_1 s_1^3\right] \frac{t^2}{3!} + \\ & + \left[b_1 - \tbinom{n-3}{1} b_3 s_1 + \tbinom{n-2}{2} b_2 s_1^2 - \tbinom{n-1}{3} b_1 s_1^3\right] s_1 \frac{t^3}{3!} + \\ & + (n-1)\left[b_n - b_{n-1} s_1 + \cdots + b_1 (-s_1)^{n-1}\right] \frac{t^{n-2}}{(n-1)!} + \\ & + s_1 \left[b_n - b_{n-1} s_1 + \cdots + b_1 (-s_1)^{n-1}\right] \frac{t^{n-1}}{(n-1)!} \Big\} e^{s_1 t}, \end{aligned} \tag{36}$$

From the equation $x^{(1)}(t) = 0$ it is possible to determinate $\tau_2 > 0$ using numerical calculations.

Theorem 5. If the initial conditions of the stable system described by (1) are positive

$$\left.\begin{array}{c} x^{(n-1)}(0) = ns_1 - \sum_{i=1}^{n-1} z_i > 0 \\ x^{(n-2)}(0) = 1 \end{array}\right\}, \tag{37}$$

then they represent the sufficient conditions for the existence of the positive extremum of $x(t)$, where $x(t)$ is the solution of Eq. (1).

Proof. It is evident, because $x(0) \geq 0$, $x^{(n-1)}(0) > 0$ and

$$\lim_{t\to\infty} x^{(n-1)}(t) = 0$$

for the stable system.

4. Conclusions

If the dynamic system is controlled by Dirac impulse $\delta(t)$ then it is possible to obtain analytical formulae for the extremal τ_2 in general from the equation whose degree depends on the number of zeroes z_i. For $i = 0, 1, 2, \ldots, n-1$ it is possible to obtain the algebraic equation of degree appropriately $i = 1, 2, \ldots, n$. Contrary to the cases when $s_i \neq s_j$, for $i \neq j$, for the $m = n-1$ the extremum may exist [6].

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