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Efficiency in vector ratio variational control problems involving geodesic quasiinvex multiple integral functionals

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In this paper, we introduce necessary and sufficient efficiency conditions associated with a class of multiobjective fractional variational control problems governed by geodesic quasiinvex multiple integral functionals and mixed constraints containing m-flow type PDEs. Using the new notion of (*normal*) *geodesic efficient solution*, under (ρ, b)-*geodesic quasiinvexity* assumptions, we establish sufficient efficiency conditions for a feasible solution.

Key words: multiobjective fractional control problem; geodesic efficient solution; (ρ, b)-geodesic quasiinvexity

1. Introduction and preliminaries

Due to important applications in various branches of pure and applied science, the concept of *convexity* has attracted several researchers over the years. As a consequence, depending on need and some useful details, it has been generalized using some interesting and novel techniques and ideas (see Hanson [5], Weir and Mond [25], Jeyakumar [7], Noor and Noor [12], Tang and Yang [16], Antczak [2], Mititelu and Treanţă [11] and Treanţă and Arana-Jiménez [21, 22]). As well, the notion of convexity has been generalized and extended on manifolds by Udrişte [24], Rapcsák [15], Pini [14], Barani and Pouryayevali [4] and Agarwal [1]. In 1985, Martin [8] introduced the concept of KT-invexity, later used for the study

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of some optimal control problems (see, for instance, Arana-Jiménez et al. [3] and Oliveira et al. [13]).

In this paper, motivated by the ongoing research in this area and taking into account Mititelu et al. [9–11] and Treanţă [17–20, 23] (as a natural continuation of these works), we introduce necessary and sufficient conditions of efficiency for a new class of multiobjective (vector) fractional variational control problems governed by geodesic quasiinvex multiple integral functionals and mixed constraints. The mathematical framework used in this paper involves geometric tools as geodesic invex sets and geodesic quasiinvex multiple integral functionals. Using the new notion of (*normal*) *geodesic efficient solution*, under (ρ, b)-*geodesic quasiinvexity* assumptions, sufficient efficiency conditions for a feasible solution are derived.

The paper is structured as follows: in Section 2, using the new notions of (*normal*) *geodesic optimal solution* and (*normal*) *geodesic efficient solution*, we formulate and prove necessary conditions of efficiency in scalar, vector and vector ratio control problems governed by multiple integral cost functionals and m-flow type PDEs constraints. Section 3 introduces the new concepts of (ρ, b)-*geodesic invexity* and (ρ, b)-*geodesic quasiinvexity* associated with multiple integral functionals. As well, under (ρ, b)-geodesic quasiinvexity assumptions, we formulate and prove sufficient efficiency conditions in vector and vector ratio control problems. Finally, in Section 4, we conclude the paper.

In order to provide a good understanding and for the completeness of our exposure, we set the following notations and working hypotheses:

- three Riemannian manifolds: a n-dimensional complete Riemannian manifold $(M, \mathbf{g})$, a k-dimensional complete Riemannian manifold (U, s) and a Riemannian manifold $(T, \mathbf{h})$ of dimension m;

- consider $t = (t^\alpha)$, $\alpha = \overline{1, m}$, $u = (u^j)$, $j = \overline{1, k}$, and $x = (x^i)$, $i = \overline{1, n}$, the local coordinates on $(T, \mathbf{h})$, (U, s) and $(M, \mathbf{g})$, respectively;

- let $\Omega \subset T$ be a relative compact subset in T and $t = (t^\alpha) \in \Omega \subset T$ a *multi-parameter of evolution* or a *multi-time*;

- consider the following continuously differentiable functions

$$\begin{aligned}
&f = \left(f_1, \ldots, f_p\right) = (f_r) : \Omega \times M \times U := \mathcal{P} \to \mathbb{R}^p,\\
&g = \left(g_1, \ldots, g_p\right) = (g_r) : \mathcal{P} \to \mathbb{R}^p, \quad r = \overline{1, p},\\
&\mathbf{X} = \left(X^i_\alpha\right) : \mathcal{P} \to \mathbb{R}^{nm}, \qquad Y = \left(Y_1, \ldots, Y_q\right) = \left(Y_\beta\right) : \mathcal{P} \to \mathbb{R}^q,\\
&\qquad\qquad\qquad\qquad\qquad\qquad i = \overline{1, n}, \quad \alpha = \overline{1, m}, \quad \beta = \overline{1, q};
\end{aligned}$$

- denote by $\mathcal{X}$ the space of piecewise smooth *state functions* $x\colon \Omega \subset T \to M$ endowed with the distance $\mathsf{d}\left(x, x^0\right) = \mathsf{d}\left(x(\cdot), x^0(\cdot)\right) := \sup_{t\in\Omega} d_{\mathbf{g}}\left(x(t), x^0(t)\right)$, where $d_{\mathbf{g}}\left(x(t), x^0(t)\right)$ is geodesic distance in $(M, \mathbf{g})$, and by $\mathcal{U}$ the space of piecewise continuous *control functions* $u\colon \Omega \subset T \to U$ equipped with the distance $\mathsf{d}\left(u, u^0\right) = \mathsf{d}\left(u(\cdot), u^0(\cdot)\right) := \sup_{t\in\Omega} d_s\left(u(t), u^0(t)\right)$, where $d_s\left(u(t), u^0(t)\right)$ is geodesic distance in (U, s). Hence, $(\mathcal{X}, \mathsf{d})$ and $(\mathcal{U}, \mathsf{d})$ become metric spaces.

Taking into account the previous mathematical context, we consider the following multidimensional vector fractional control problem

$$(VFCP) \quad \min_{(x,u)} \left(\frac{\displaystyle\int_\Omega f_1\left(t, x(t), u(t)\right) \mathrm{d}v}{\displaystyle\int_\Omega g_1\left(t, x(t), u(t)\right) \mathrm{d}v}, \ldots, \frac{\displaystyle\int_\Omega f_p\left(t, x(t), u(t)\right) \mathrm{d}v}{\displaystyle\int_\Omega g_p\left(t, x(t), u(t)\right) \mathrm{d}v} \right),$$

subject to

$$\frac{\partial x^i}{\partial t^\alpha}(t) = X^i_\alpha\left(t, x(t), u(t)\right), \qquad i = \overline{1, n}, \;\; \alpha = \overline{1, m}, \;\; t \in \Omega, \tag{1}$$

$$Y\left(t, x(t), u(t)\right) \leqslant 0, \qquad t \in \Omega, \tag{2}$$

$$x(t)\big|_{\partial\Omega} = \varphi(t) = given, \tag{3}$$

a first-order PDEs (partial differential equations) constrained vector ratio control problem.

Working hypotheses:

- $dv := \sqrt{\det \mathbf{h}}\, dt^1 \wedge dt^2 \wedge \ldots \wedge dt^m$ represents the volume m-form on $T \supset \Omega$;
- the continuously differentiable functions $X_\alpha = \left(X^i_\alpha\right) : \mathcal{P} \to \mathbb{R}^n$, $i = \overline{1, n}$, $\alpha = \overline{1, m}$ satisfy the closeness conditions $D_\zeta X^i_\alpha = D_\alpha X^i_\zeta$, α, $\zeta = \overline{1, m}$, $\alpha \neq \zeta$, $i = \overline{1, n}$, where D_ζ is the total derivative operator;
- the following convention for equalities and inequalities will be used throughout the paper:

$$u = v \Leftrightarrow u_j = v_j, \quad u \leqslant v \Leftrightarrow u_j \leqslant v_j, \quad u < v \Leftrightarrow u_j < v_j,$$
$$u \preceq v \Leftrightarrow u \leqslant v, \; u \neq v, \quad j = \overline{1, p},$$

  for any two p-tuples $u = \left(u_1, \ldots, u_p\right)$, $v = \left(v_1, \ldots, v_p\right)$ in $\mathbb{R}^p$.

Further, consider the complete Riemannian manifold $(M \times U, w)$. In accordance to Barani and Pouryayevali [4], following Udrişte [24] and Mititelu et al. [10], we introduce the following definitions.

Definition 1 *Let* $\eta\colon (M \times U)^2 \to T(M \times U)$ *be a function such that for every* $(x(t), u(t))$, $\left(x^0(t), u^0(t)\right) \in M \times U$, *we have* $\eta\left(x(t), u(t), x^0(t), u^0(t)\right) \in T_{\left(x^0(t), u^0(t)\right)}(M \times U)$. *A non-empty subset* $\mathtt{M} \times \mathtt{U}$ *of* $M \times U$ *is said to be geodesic invex with respect to* η, *or* η*-geodesic invex, if, for every* $(x(t), u(t))$, $\left(x^0(t), u^0(t)\right) \in \mathtt{M} \times \mathtt{U}$, *there exists a unique geodesic* $\gamma_{(x(t),u(t)),(x^0(t),u^0(t))}\colon [0,1] \to M \times U$ *such that*

$$\begin{aligned}
&\gamma_{(x(t),u(t)),(x^0(t),u^0(t))}(0) = \left(x^0(t), u^0(t)\right),\\
&\dot{\gamma}_{(x(t),u(t)),(x^0(t),u^0(t))}(0) = \eta\left(x(t), u(t), x^0(t), u^0(t)\right),\\
&\gamma_{(x(t),u(t)),(x^0(t),u^0(t))}(\tau) \in \mathtt{M} \times \mathtt{U}, \quad \forall \tau \in [0,1].
\end{aligned}$$

In Definition 1, the notation $T(M \times U)$ represents the tangent bundle of $M \times U$ and $T_{(x^0(t),u^0(t))}(M \times U)$ is the set of all tangent vectors at $\left(x^0(t), u^0(t)\right) \in M \times U$.

Definition 2 *Let* $\left(x^0(\cdot), u^0(\cdot)\right)$ *and* $(x(\cdot), u(\cdot))$ *be in* $\mathcal{X} \times \mathcal{U}$. *A function* $\phi\colon \Omega \times [0,1] \to M \times U$, $\phi = \phi(t, \tau)$, *is called* geodesic deformation *of the pairs* $\left(x^0(\cdot), u^0(\cdot)\right)$ *and* $(x(\cdot), u(\cdot))$ *if it fulfills the following properties:*

i) the function $\phi_t\colon [0,1] \to M \times U$, $\tau \to \phi_t(\tau) = \phi(t, \tau)$, $t \in \Omega$, *is a geodesic;*

ii) $\phi(t, 0) = \left(x^0(t), u^0(t)\right)$, $\phi(t, 1) = (x(t), u(t))$, $t \in \Omega$.

Denote by $\mathtt{X}$ the space of piecewise smooth state functions $x\colon \Omega \subset T \to \mathtt{M}$ equipped with the distance $\mathsf{d}\left(x, x^0\right) = \mathsf{d}\left(x(\cdot), x^0(\cdot)\right) := \sup_{t \in \Omega} d_{\mathbf{g}}\left(x(t), x^0(t)\right)$, where $d_{\mathbf{g}}\left(x(t), x^0(t)\right)$ is geodesic distance in $(M, \mathbf{g})$, and by $\mathfrak{U}$ the space of piecewise continuous control functions $u\colon \Omega \subset T \to \mathtt{U}$ endowed with the distance $\mathsf{d}\left(u, u^0\right) = \mathsf{d}\left(u(\cdot), u^0(\cdot)\right) := \sup_{t \in \Omega} d_s\left(u(t), u^0(t)\right)$, where $d_s\left(u(t), u^0(t)\right)$ is geodesic distance in (U, s).

For each $(x^0, u^0) \in \mathtt{X} \times \mathfrak{U}$, denote by $(x, u) \in \mathtt{X} \times \mathfrak{U}$ an arbitrary geodesic perturbation. Considering the previous two definitions, we formulate the following central definition.

Definition 3 *The subset* $\mathtt{X} \times \mathfrak{U} \subset \mathcal{X} \times \mathcal{U}$ *is called* η*-geodesic invex if, for every* $(x(\cdot), u(\cdot))$, $\left(x^0(\cdot), u^0(\cdot)\right) \in \mathtt{X} \times \mathfrak{U}$, *there exists a unique geodesic deformation*

$\phi = \phi(t, \tau)$, $t \in \Omega$, $\tau \in [0, 1]$, *of the pairs* $\left(x^0(\cdot), u^0(\cdot)\right)$ *and* $(x(\cdot), u(\cdot))$, *with* $\phi(\cdot, \tau)$, $\tau \in [0, 1]$, *included in* $\mathtt{X} \times \mathfrak{U}$, *such that the vector valued function*

$$T_{\left(x^0(t), u^0(t)\right)}(M \times U) \ni \eta\left(x(t), u(t), x^0(t), u^0(t)\right) = \frac{\partial \phi}{\partial \tau}(t, \tau)\Big|_{\tau=0}$$
$$:= (\eta(t), \xi(t)) := (\eta_1(t), \eta_2(t), ..., \eta_n(t), \xi_1(t), \xi_2(t), ..., \xi_k(t))$$

is of C^1*-class and* $\eta|_{\partial\Omega} = 0$.

In the sequel, let $\mathtt{X} \times \mathfrak{U}$ be an open η-geodesic invex subset of $\mathcal{X} \times \mathcal{U}$. Define the set $\mathfrak{D}$ of all *feasible solutions* (*domain*) for $(VFCP)$ as follows

$$\mathfrak{D} := \{(x, u) | x = x(\cdot) \in \mathtt{X}, \quad u = u(\cdot) \in \mathfrak{U} \text{ satisfying } (1.1),\ (1.2),\ (1.3)\}.$$

2. Necessary efficiency conditions in scalar, vector and vector ratio control problems with multiple integral cost functionals

In this section, using the new notions of (*normal*) *geodesic optimal solution* and (*normal*) *geodesic efficient solution*, we formulate and prove necessary conditions of efficiency in scalar, vector and vector ratio control problems governed by multiple integral cost functionals and m-flow type PDEs constraints.

Scalar case. Let us consider the following multidimensional scalar control problem

$$(SCP) \quad \min_{(x,u)} \int_\Omega X(t, x(t), u(t))\, \mathrm{d}v \quad \text{subject to} \quad (x, u) \in \mathfrak{D}.$$

A feasible solution $(x^0, u^0) \in \mathfrak{D}$ in the aforementioned multidimensional scalar control problem (SCP) is called *geodesic optimal solution* if there exists no other feasible solution $(x, u) \in \mathfrak{D}$ such that $\int_\Omega X(t, x(t), u(t))\, \mathrm{d}v < \int_\Omega X\left(t, x^0(t), u^0(t)\right) \mathrm{d}v$. The necessary conditions of geodesic optimality, for a feasible solution $(x^0, u^0) \in \mathfrak{D}$ in (SCP), are formulated in the following result.

Theorem 1 *If* $(x^0, u^0) \in \mathfrak{D}$ *is a geodesic optimal solution in* (SCP), *then there exist a scalar* $\theta \in \mathbb{R}$ *and the piecewise smooth functions* $\mu(t) = (\mu^\beta(t)) \in \mathbb{R}^q$,

$\lambda(t) = (\lambda_i^\alpha(t)) \in \mathbb{R}^{nm}$ *satisfying the following conditions*

$$\theta \frac{\partial X}{\partial x^i}\left(t, x^0(t), u^0(t)\right) + \lambda_i^\alpha(t) \frac{\partial X_\alpha^i}{\partial x^i}\left(t, x^0(t), u^0(t)\right) + \mu^\beta(t) \frac{\partial Y_\beta}{\partial x^i}\left(t, x^0(t), u^0(t)\right) + \frac{\partial \lambda_i^\alpha}{\partial t^\alpha}(t) = 0, \quad i = \overline{1, n},$$

$$\theta \frac{\partial X}{\partial u^j}\left(t, x^0(t), u^0(t)\right) + \lambda_i^\alpha(t) \frac{\partial X_\alpha^i}{\partial u^j}\left(t, x^0(t), u^0(t)\right) + \mu^\beta(t) \frac{\partial Y_\beta}{\partial u^j}\left(t, x^0(t), u^0(t)\right) = 0, \quad j = \overline{1, k},$$

$$\mu^\beta(t) Y_\beta\left(t, x^0(t), u^0(t)\right) = 0 \quad (\text{no summation}), \quad (\theta, \mu(t)) \geq 0,$$

for all $t \in \Omega$, *except at discontinuities.*

Proof. The proof follows in the same manner as in Theorem 1, Mititelu [9]. □

Definition 4 *The geodesic optimal solution* $(x^0, u^0) \in \mathfrak{D}$ *in* (SCP) *is said to be a* normal geodesic optimal solution *if* $\theta > 0$. *Without loss of generality, we can assume that* $\theta = 1$.

Vector case. Further, we consider the following multidimensional vector control problem

$$(VCP) \quad \min_{(x,u)} \left\{ I(x,u) = \int_\Omega f\left(t, x(t), u(t)\right) \mathrm{d}v \right\} \quad \text{subject to} \quad (x,u) \in \mathfrak{D},$$

where

$$\int_\Omega f\left(t, x(t), u(t)\right) \mathrm{d}v := \left(\int_\Omega f_1\left(t, x(t), u(t)\right) \mathrm{d}v, \ \ldots, \ \int_\Omega f_p\left(t, x(t), u(t)\right) \mathrm{d}v \right) := \left(I_1(x,u), \ \ldots, I_p(x,u) \right).$$

Definition 5 *A feasible solution* $(x^0, u^0) \in \mathfrak{D}$ *in the multidimensional vector control problem* (VCP) *is called geodesic efficient solution if there exists no other feasible solution* $(x,u) \in \mathfrak{D}$ *such that* $I(x,u) \leq I(x^0, u^0)$.

Theorem 2 *If* $(x^0, u^0) \in \mathfrak{D}$ *is a geodesic efficient solution of the problem* (VCP), *then there exist a vector* $\theta = (\theta^r) \in \mathbb{R}^p$ *and the piecewise smooth functions* $\mu(t) = (\mu^\beta(t)) \in \mathbb{R}^q$, $\lambda(t) = (\lambda_i^\alpha(t)) \in \mathbb{R}^{nm}$ *fulfilling the following conditions*

$$\theta^r \frac{\partial f_r}{\partial x^i}\left(t, x^0(t), u^0(t)\right) + \lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial x^i}\left(t, x^0(t), u^0(t)\right) + \mu^\beta(t)\frac{\partial Y_\beta}{\partial x^i}\left(t, x^0(t), u^0(t)\right) + \frac{\partial \lambda_i^\alpha}{\partial t^\alpha}(t) = 0, \quad i = \overline{1,n},$$

$$\theta^r \frac{\partial f_r}{\partial u^j}\left(t, x^0(t), u^0(t)\right) + \lambda_i^\alpha(t)\frac{\partial X_\alpha^i}{\partial u^j}\left(t, x^0(t), u^0(t)\right) + \mu^\beta(t)\frac{\partial Y_\beta}{\partial u^j}\left(t, x^0(t), u^0(t)\right) = 0, \quad j = \overline{1,k},$$

$$\mu^\beta(t) Y_\beta\left(t, x^0(t), u^0(t)\right) = 0 \quad (\textit{no summation}), \quad (\theta, \mu(t)) \geq 0,$$

for all $t \in \Omega$, *except at discontinuities.*

Proof. Let $(x^0, u^0) \in \mathfrak{D}$ be a geodesic efficient solution in (VCP). Consequently, the inequality $I(x,u) \leq I(x^0, u^0)$, $\forall (x,u) \in \mathfrak{D}$, is not true. Then there exists $r \in \{1, 2, \ldots p\}$ and a neighborhood N_r in $\mathfrak{D}$ of the point (x^0, u^0) such that $I_r(x,u) > I_r(x^0, u^0)$, $\forall (x,u) \in \mathfrak{D}$. Thus, (x^0, u^0) is a geodesic optimal solution to the following multidimensional scalar control problem

$$(SCP)_r \quad \min_{(x,u)} \left\{ I_r(x,u) = \int_\Omega f_r\left(t, x(t), u(t)\right) \mathrm{d}v \right\} \quad \text{subject to} \quad (x,u) \in \mathfrak{D}.$$

Applying Theorem 1, there exist multipliers θ_r, $\mu_r(t)$ and $\lambda_r(t)$, with $\mu_r(t) = (\mu_r^\beta(t))$, $\lambda_r(t) = (\lambda_{i,r}^\alpha(t))$ piecewise smooth functions, satisfying the following conditions (no summation over r)

$$\theta_r \frac{\partial f_r}{\partial x^i}\left(t, x^0(t), u^0(t)\right) + \lambda_{i,r}^\alpha(t)\frac{\partial X_\alpha^i}{\partial x^i}\left(t, x^0(t), u^0(t)\right) + \mu_r^\beta(t)\frac{\partial Y_\beta}{\partial x^i}\left(t, x^0(t), u^0(t)\right) + \frac{\partial \lambda_{i,r}^\alpha}{\partial t^\alpha}(t) = 0, \quad i = \overline{1,n},$$

$$\theta_r \frac{\partial f_r}{\partial u^j}\left(t, x^0(t), u^0(t)\right) + \lambda_{i,r}^\alpha(t)\frac{\partial X_\alpha^i}{\partial u^j}\left(t, x^0(t), u^0(t)\right) + \mu_r^\beta(t)\frac{\partial Y_\beta}{\partial u^j}\left(t, x^0(t), u^0(t)\right) = 0, \quad j = \overline{1,k},$$

$$\mu_r^\beta(t) Y_\beta\left(t, x^0(t), u^0(t)\right) = 0 \quad \text{(no summation)}, \quad r = \overline{1,p}; \quad (\theta_r, \mu_r(t)) \geq 0,$$

for all $t \in \Omega$, except at discontinuities.

Making the following notations: $S = \sum_{r=1}^{p} \theta_r$, with $\theta^r = \dfrac{\theta_r}{S}$ when $I_r(x,u) \geqslant I_r(x^0,u^0)$, and $\theta^r = 0$ when $I_r(x,u) < I_r(x^0,u^0)$, $\lambda_i^{\alpha}(t) = \dfrac{\lambda_{i,r}^{\alpha}(t)}{S}$, $\mu^{\beta}(t) = \dfrac{\mu_r^{\beta}(t)}{S}$, the proof is complete. □

Definition 6 *A feasible solution* $(x^0,u^0) \in \mathfrak{D}$ *in* (VCP) *is said to be normal geodesic efficient solution if the necessary efficiency conditions for* (VCP)*, formulated in Theorem 2, hold for* $\theta \geq 0$ *and* $e^t\theta = 1$*, where* $e^t = (1,\ldots,1) \in \mathbb{R}^p$.

<u>Vector ratio case.</u> Now, let us consider the multidimensional vector fractional control problem

$$(VFCP) \quad \min_{(x,u)} \left\{ J(x,u) = \left(\frac{\int\limits_{\Omega} f_1\left(t,x,u\right) \mathrm{d}v}{\int\limits_{\Omega} g_1\left(t,x,u\right) \mathrm{d}v}, \ \ldots, \ \frac{\int\limits_{\Omega} f_p\left(t,x,u\right) \mathrm{d}v}{\int\limits_{\Omega} g_p\left(t,x,u\right) \mathrm{d}v} \right) \right\}$$

$$\text{subject to} \ \ (x,u) \in \mathfrak{D},$$

where it is assumed that $\int\limits_{\Omega} g_r(t,x,u)\mathrm{d}v > 0$, $r = \overline{1,p}$.

Definition 7 *A feasible solution* $(x^0,u^0) \in \mathfrak{D}$ *in* $(VFCP)$ *is said to be* geodesic efficient solution *if there is no other* $(x,u) \in \mathfrak{D}$ *such that* $J(x,u) \preceq J(x^0,u^0)$.

Let $(x^0,u^0) \in \mathfrak{D}$ be a geodesic efficient solution of the control problem $(VFCP)$ and consider the following scalar fractional control problem

$$(SFCP)_r \quad \min_{(x,u)} \left\{ J_r(x,u) = \frac{\int\limits_{\Omega} f_r\left(t,x(t),u(t)\right) \mathrm{d}v}{\int\limits_{\Omega} g_r\left(t,x(t),u(t)\right) \mathrm{d}v} \right\} \quad \text{subject to} \quad (x,u) \in \mathfrak{D}$$

and

$$\frac{\int\limits_{\Omega} f_j\left(t,x(t),u(t)\right) \mathrm{d}v}{\int\limits_{\Omega} g_j\left(t,x(t),u(t)\right) \mathrm{d}v} \leqslant \frac{\int\limits_{\Omega} f_j\left(t,x^0(t),u^0(t)\right) \mathrm{d}v}{\int\limits_{\Omega} g_j\left(t,x^0(t),u^0(t)\right) \mathrm{d}v}, \quad j = \overline{1,p}, \quad j \neq r.$$

Denote

$$R_r^0 := \min_{(x,u)} \frac{\int_\Omega f_r\,(t, x(t), u(t))\,\mathrm{d}v}{\int_\Omega g_r\,(t, x(t), u(t))\,\mathrm{d}v} = \frac{\int_\Omega f_r\left(t, x^0(t), u^0(t)\right)\mathrm{d}v}{\int_\Omega g_r\left(t, x^0(t), u^0(t)\right)\mathrm{d}v}, \quad r \in \{1, \ldots, p\}$$

and rewrite the control problem $(SFCP)_r$ as follows

$$\min_{(x,u)} \left\{ J_r(x,u) = \frac{\int_\Omega f_r\,(t, x(t), u(t))\,\mathrm{d}v}{\int_\Omega g_r\,(t, x(t), u(t))\,\mathrm{d}v} \right\} [= R_r^0] \quad \text{subject to} \quad (x,u) \in \mathfrak{D}$$

and

$$\int_\Omega \left[f_j\,(t, x(t), u(t)) - R_j^0 g_j\,(t, x(t), u(t)) \right] \mathrm{d}v \leqslant 0, \quad j = \overline{1, p}, \;\; j \neq r,$$

or, equivalently (Jagannathan [6]),

$$\min_{(x,u)} \int_\Omega \left[f_r\,(t, x(t), u(t)) - R_r^0 g_r\,(t, x(t), u(t)) \right] \mathrm{d}v \quad \text{subject to} \quad (x,u) \in \mathfrak{D}$$

and

$$\int_\Omega \left[f_j\,(t, x(t), u(t)) - R_j^0 g_j\,(t, x(t), u(t)) \right] \mathrm{d}v \leqslant 0, \quad j = \overline{1, p}, \;\; j \neq r.$$

Theorem 3 *If $(x^0, u^0) \in \mathfrak{D}$ is a geodesic efficient solution of the control problem $(VFCP)$, then there exist a scalar vector $\theta = (\theta^r) \in \mathbb{R}^p$ and the piecewise smooth functions $\mu(t) = \left(\mu^\beta(t)\right) \in \mathbb{R}^q$, $\lambda(t) = (\lambda_i^\alpha(t)) \in \mathbb{R}^{nm}$ satisfying the following conditions*

$$\theta^r \left[\frac{\partial f_r}{\partial x^i}\left(t, x^0(t), u^0(t)\right) - R_r^0 \frac{\partial g_r}{\partial x^i}\left(t, x^0(t), u^0(t)\right) \right] + \lambda_i^\alpha(t) \frac{\partial X_\alpha^i}{\partial x^i}\left(t, x^0(t), u^0(t)\right)$$
$$+ \mu^\beta(t) \frac{\partial Y_\beta}{\partial x^i}\left(t, x^0(t), u^0(t)\right) + \frac{\partial \lambda_i^\alpha}{\partial t^\alpha}(t) = 0, \quad i = \overline{1, n}$$

$$\theta^r \left[\frac{\partial f_r}{\partial u^j} \left(t, x^0(t), u^0(t) \right) - R_r^0 \frac{\partial g_r}{\partial u^j} \left(t, x^0(t), u^0(t) \right) \right] + \lambda_i^\alpha(t) \frac{\partial X_\alpha^i}{\partial u^j} \left(t, x^0(t), u^0(t) \right)$$

$$+ \mu^\beta(t) \frac{\partial Y_\beta}{\partial u^j} \left(t, x^0(t), u^0(t) \right) = 0, \quad j = \overline{1, k},$$

$$\mu^\beta(t) Y_\beta \left(t, x^0(t), u^0(t) \right) = 0 \quad (\textit{no summation}), \quad (\theta, \mu(t)) \geq 0,$$

for all $t \in \Omega$, except at discontinuities.

Proof. Let $(x^0, u^0) \in \mathfrak{D}$ be a geodesic efficient solution in $(VFCP)$. Applying the procedure used in Theorem 2, the point $(x^0, u^0) \in \mathfrak{D}$ is a geodesic optimal solution of the scalar control problem $(SFCP)_r$, $r \in \{1, \ldots, p\}$. The proof follows in the same manner as in Theorem 2 by considering $f_r\,(t, x(t), u(t)) - R_r^0 g_r\,(t, x(t), u(t))$ instead of $f_r\,(t, x(t), u(t))$, $r = \overline{1, p}$. □

Definition 8 *The feasible solution $(x^0, u^0) \in \mathfrak{D}$ is a* normal geodesic efficient solution *in $(VFCP)$ if the necessary efficiency conditions for $(VFCP)$, formulated in Theorem 3, hold for $\theta \geq 0$ and $e^t\theta = 1$, where $e^t = (1, \ldots, 1) \in \mathbb{R}^p$.*

3. Sufficient efficiency conditions in vector and vector ratio control problems with multiple integral cost functionals

In this section, in accordance with the geometric objects defined in Section 1, we introduce the new concepts of (ρ, b)-*geodesic invexity* and (ρ, b)-*geodesic quasiinvexity*. Moreover, under (ρ, b)-geodesic quasiinvexity assumptions, we formulate and prove sufficient efficiency conditions in (VCP) and $(VFCP)$.

Let us consider $\rho \in \mathbb{R}^p$, $x \in \mathcal{X}$, $u \in \mathcal{U}$ and the following vector functional

$$H \colon \mathcal{X} \times \mathcal{U} \to \mathbb{R}^p, \quad H(x, u) = \int_\Omega h\,(t, x(t), x_\alpha(t), u(t))\, \mathrm{d}v,$$

where $h \colon J^1(T, M) \times U \to \mathbb{R}^p$, $h = (h_r)$, $r = \overline{1, p}$ (see $J^1(T, M)$ as the first-order jet bundle determined by T and M). Also, we consider the following functional $b \colon \mathcal{X} \times \mathcal{U} \times \mathcal{X} \times \mathcal{U} \to [0, \infty)$ and the distance function $d\left((x, u), (x^0, u^0)\right)$ on $\mathcal{X} \times \mathcal{U}$, defined as $d\left((x, u), (x^0, u^0)\right) := \sup\limits_{t \in \Omega} d_w\left((x(t), u(t)), (x^0(t), u^0(t))\right)$, where $d_w\left((x(t), u(t)), (x^0(t), u^0(t))\right)$ is geodesic distance in $(M \times U, w)$.

Definition 9 *Let $(M \times U, w)$ be a complete Riemannian manifold and $\mathtt{X} \times \mathfrak{U}$ an open η-geodesic invex subset of $\mathcal{X} \times \mathcal{U}$.*

(i) *If*

$$\eta\colon M\times U\times M\times U\to\mathbb{R}^n,\quad \eta=\left(\eta_i\left(x(t),u(t),x^0(t),u^0(t)\right)\right),\quad i=\overline{1,n},$$

is of C^1*-class with* $\eta|_{\partial\Omega}=0$*, and*

$$\xi\colon M\times U\times M\times U\to\mathbb{R}^k,\quad \xi=\left(\xi_j\left(x(t),u(t),x^0(t),u^0(t)\right)\right),\quad j=\overline{1,k},$$

is of C^0*-class with* $\xi|_{\partial\Omega}=0$*, such that for any* $(x,u)\in\mathtt{X}\times\mathfrak{U}$*:*

$$H(x,u)-H\left(x^0,u^0\right)\geqslant b\left(x,u,x^0,u^0\right)\int_\Omega\Big[h_x\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)\eta$$
$$+h_{x_\alpha}\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)D_\alpha\eta\Big]\mathrm{d}v$$
$$+b\left(x,u,x^0,u^0\right)\left\{\int_\Omega\Big[h_u\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)\xi\Big]\mathrm{d}v+\rho d^2\left((x,u),(x^0,u^0)\right)\right\},$$

then H *is called* (ρ,b)*-geodesic invex at* $\left(x^0,u^0\right)\in\mathtt{X}\times\mathfrak{U}$ *with respect to* η *and* ξ*;*

(i′) *in the above inequality, with* $(x,u)\neq\left(x^0,u^0\right)$*, if we replace* $\geqslant$ *with* $>$*, we say that* H *is strictly* (ρ,b)*-geodesic invex at* $\left(x^0,u^0\right)\in\mathtt{X}\times\mathfrak{U}$ *with respect to* η *and* ξ*;*

(i″) *in the above inequality, with* $(x,u)\neq\left(x^0,u^0\right)$*, if we replace* $\geqslant$ *with* $\geq$*, we say that* H *is strongly* (ρ,b)*-geodesic invex at* $\left(x^0,u^0\right)\in\mathtt{X}\times\mathfrak{U}$ *with respect to* η *and* ξ*;*

(ii) *If* $\eta\colon(M\times U)^2\to\mathbb{R}^n$*,* $\eta=\eta\left(x(t),u(t),x^0(t),u^0(t)\right)$ *is of* C^1*-class with* $\eta|_{\partial\Omega}=0$*, and* $\xi\colon(M\times U)^2\to\mathbb{R}^k$*,* $\xi=\xi\left(x(t),u(t),x^0(t),u^0(t)\right)$ *is of* C^0*-class with* $\xi|_{\partial\Omega}=0$*, such that for any* $(x,u)\in\mathtt{X}\times\mathfrak{U}$*:*

$$H(x,u)\leqslant H\left(x^0,u^0\right)\implies b\left(x,u,x^0,u^0\right)\int_\Omega\Big[h_x\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)\eta$$
$$+h_{x_\alpha}\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)D_\alpha\eta\Big]\mathrm{d}v+b\left(x,u,x^0,u^0\right)\int_\Omega h_u\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)\xi\,\mathrm{d}v$$
$$\leqslant-\rho b\left(x,u,x^0,u^0\right)d^2\left((x,u),(x^0,u^0)\right),$$

then H *is called* (ρ,b)*-geodesic quasiinvex at* $\left(x^0,u^0\right)\in\mathtt{X}\times\mathfrak{U}$ *with respect to* η *and* ξ*;*

(ii′) if, in the same hypotheses, with $(x,u) \neq \left(x^0, u^0\right)$, we have

$$H(x,u) \leqslant H\left(x^0,u^0\right) \implies b\left(x,u,x^0,u^0\right)\int_\Omega \Big[h_x\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)\eta + h_{x_\alpha}\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)D_\alpha\eta\Big]\mathrm{d}v + b\left(x,u,x^0,u^0\right)\int_\Omega h_u\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)\xi\,\mathrm{d}v < -\rho b\left(x,u,x^0,u^0\right)d^2\left((x,u),(x^0,u^0)\right),$$

then H is called strictly (ρ,b)-geodesic quasiinvex at $\left(x^0,u^0\right) \in \mathtt{X}\times\mathfrak{U}$ with respect to η and ξ;

(iii) If $\eta\colon (M\times U)^2 \to \mathbb{R}^n$, $\eta = \eta\left(x(t),u(t),x^0(t),u^0(t)\right)$ is of C^1-class with $\eta|_{\partial\Omega} = 0$, and $\xi\colon (M\times U)^2 \to \mathbb{R}^k$, $\xi = \xi\left(x(t),u(t),x^0(t),u^0(t)\right)$ is of C^0-class with $\xi|_{\partial\Omega} = 0$, such that for any $(x,u) \in \mathtt{X}\times\mathfrak{U}$:

$$H(x,u) = H\left(x^0,u^0\right) \implies b\left(x,u,x^0,u^0\right)\int_\Omega \Big[h_x\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)\eta + h_{x_\alpha}\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)D_\alpha\eta\Big]\mathrm{d}v + b\left(x,u,x^0,u^0\right)\int_\Omega h_u\left(t,x^0(t),x^0_\alpha(t),u^0(t)\right)\xi\,\mathrm{d}v = -\rho b\left(x,u,x^0,u^0\right)d^2\left((x,u),(x^0,u^0)\right),$$

then H is called monotonic (ρ,b)-geodesic quasiinvex at $\left(x^0,u^0\right) \in \mathtt{X}\times\mathfrak{U}$ with respect to η and ξ.

Remark 1 *In Definition 9, the existence of the functions η (of C^1-class with $\eta|_{\partial\Omega} = 0$) and ξ (of C^0-class with $\xi|_{\partial\Omega} = 0$) is ensured by the fact that $\mathtt{X}\times\mathfrak{U}$ is an open η-geodesic invex subset of $\mathcal{X}\times\mathcal{U}$ (see Definitions 1–3). As well, Definition 9 emphasizes the η's rupture in two parts (see Definition 3).*

Examples.

1. The following functional

$$H(x,u) = \int_{[0,1]^m} [x(t)+u(t)]\ln[x(t)+u(t)]\,\mathrm{d}v, \quad (x,u) \in \mathtt{X}\times\mathfrak{U}$$

is, as it can be verified, $(\rho, 1)$-geodesic quasiinvex at $\left(x^0, u^0\right) \in \mathtt{X} \times \mathfrak{U}$, for $\rho \leqslant 0$ and any distance function d, with respect to

$$\eta = \xi = \begin{cases} \left(H(x,u) - H\left(x^0, u^0\right)\right)\left[1 + \ln\left(x^0(t) + u^0(t)\right)\right], & t \in Int(\Omega) \\ 0, & t \in \partial\Omega, \end{cases}$$

where

$$\mathtt{X} = \left\{x\colon [0,1]^m \to \mathbb{R}_+,\ \ x(\cdot)\ \text{ of }\ C^0\text{-class}\right\},$$
$$\mathfrak{U} = \left\{u\colon [0,1]^m \to \mathbb{R}_+,\ \ u(\cdot)\ \text{ of }\ C^0\text{-class}\right\}.$$

2. In the same hypotheses as in Example 3, for a fixed $\alpha \in \{1, 2, \ldots, m\}$ and $x(\cdot)$ of C^1-class, the functional

$$H(x,u) = \int\limits_{[0,1]^m} [x_\alpha(t) + u(t)] \ln [x_\alpha(t) + u(t)] \, \mathrm{d}v, \quad (x,u) \in \mathtt{X} \times \mathfrak{U},$$

is $(\rho, 1)$-geodesic quasiinvex at $\left(x^0, u^0\right) \in \mathtt{X} \times \mathfrak{U}$, for $\rho \leqslant 0$ and any distance function d, with respect to

$$\eta = \begin{cases} \left(H\left(x^0, u^0\right) - H(x,u)\right) D_\alpha \left[1 + \ln\left(x^0_\alpha(t) + u^0(t)\right)\right], & t \in Int(\Omega), \\ 0, & t \in \partial\Omega \end{cases}$$

and

$$\xi = \begin{cases} \left(H(x,u) - H\left(x^0, u^0\right)\right)\left[1 + \ln\left(x^0_\alpha(t) + u^0(t)\right)\right], & t \in Int(\Omega), \\ 0, & t \in \partial\Omega. \end{cases}$$

Further, the aforementioned definition of (ρ, b)-geodesic quasiinvexity helps us to formulate and prove the results included in this section.

The next result establishes sufficient conditions of efficiency in (VCP).

Theorem 4 *Let $(x^0, u^0) \in \mathfrak{D}$ be a feasible solution in (VCP), $\theta = (\theta^r)$ a vector and $\mu(t) = (\mu^\beta(t))$, $\lambda(t) = \left(\lambda^\alpha_i(t)\right)$ two piecewise smooth functions, all satisfying the conditions formulated in Theorem 2. Assume that:*

a) each functional $F_r(x,u) = \displaystyle\int\limits_\Omega f_r\left(t, x(t), u(t)\right) \mathrm{d}v$, $r \in \{1, \ldots, p\}$, is (ρ^1_r, b)-geodesic quasiinvex at (x^0, u^0) with respect to η and ξ;

b) the functional $X(x,u)=\int\limits_\Omega \lambda_i^\alpha(t)\left[X_\alpha^i\left(t,x(t),u(t)\right)-\frac{\partial x^i}{\partial t^\alpha}(t)\right]\mathrm{d}v$ *is monotonic* (ρ^2,b)*-geodesic quasiinvex at* (x^0,u^0) *with respect to* η *and* ξ*;*

c) the functional $Y(x,u)=\int\limits_\Omega \mu^\beta(t)Y_\beta\left(t,x(t),u(t)\right)\mathrm{d}v$ *is* (ρ^3,b)*-geodesic quasiinvex at* (x^0,u^0) *with respect to* η *and* ξ*;*

d) at least one of the functionals given in points a), c) is strictly (ρ,b)*-geodesic quasiinvex at* (x^0,u^0) *with respect to* η *and* ξ*, where* $\rho=\rho_r^1$ *or* ρ^3*;*

e) $\theta^r\rho_r^1+\rho^2+\rho^3\geqslant 0$, ρ_r^1, ρ^2, $\rho^3\in\mathbb{R}$.

Then the point (x^0,u^0) *is a geodesic efficient solution in* (VCP).

Proof. By reductio ad absurdum, suppose that (x^0,u^0) is not a geodesic efficient solution in (VCP). For $r=\overline{1,p}$, define the following non-empty set

$$S=\Big\{(x,u)\in\mathfrak{D}\,|\; F_r(x,u)\leqslant F_r\left(x^0,u^0\right),\\ X(x,u)=X\left(x^0,u^0\right),\;\; Y(x,u)\leqslant Y\left(x^0,u^0\right)\Big\}.$$

By using the hypothesis a), for $(x,u)\in S$ and $r=\overline{1,p}$, it follows

$$F_r(x,u)\leqslant F_r(x^0,u^0)\implies$$

$$b\left(x,u,x^0,u^0\right)\int\limits_\Omega\left[(f_r)_x\left(t,x^0(t),u^0(t)\right)\eta(t)+(f_r)_u\left(t,x^0(t),u^0(t)\right)\xi(t)\right]\mathrm{d}v$$

$$\leqslant-\rho_r^1 b\left(x,u,x^0,u^0\right)d^2\left((x,u),(x^0,u^0)\right).$$

Multiplying by $\theta^r\geqslant 0$ the above inequality and making summation over $r=\overline{1,p}$, we obtain [see $(\theta^r f_r)_x:=\dfrac{\partial(\theta^r f_r)}{\partial x}$, $(\theta^r f_r)_u:=\dfrac{\partial(\theta^r f_r)}{\partial u}$]

$$b\left(x,u,x^0,u^0\right)\int\limits_\Omega\left[(\theta^r f_r)_x\left(t,x^0(t),u^0(t)\right)\eta(t)+(\theta^r f_r)_u\left(t,x^0(t),u^0(t)\right)\xi(t)\right]\mathrm{d}v$$

$$\leqslant-(\theta^r\rho_r^1)b\left(x,u,x^0,u^0\right)d^2\left((x,u),(x^0,u^0)\right). \tag{4}$$

For $(x,u) \in S$, the equality $X(x,u) = X(x^0,u^0)$ holds and, according to b), we get

$$b\left(x,u,x^0,u^0\right)\int_{\Omega}\left[\lambda_i^{\alpha}(t)(X_{\alpha}^i)_x\left(t,x^0(t),u^0(t)\right)\eta(t) - \lambda^{\alpha}(t)D_{\alpha}\eta(t)\right]\mathrm{d}v$$
$$+\, b\left(x,u,x^0,u^0\right)\int_{\Omega}\lambda_i^{\alpha}(t)(X_{\alpha}^i)_u\left(t,x^0(t),u^0(t)\right)\xi(t)\mathrm{d}v$$
$$= -\rho^2 b\left(x,u,x^0,u^0\right)d^2\left((x,u),(x^0,u^0)\right). \tag{5}$$

As well, the inequality $Y(x,u) \leqslant Y(x^0,u^0)$, $(x,u) \in S$, gives (see c))

$$b\left(x,u,x^0,u^0\right)\int_{\Omega}\Big[\mu^{\beta}(t)(Y_{\beta})_x\left(t,x^0(t),u^0(t)\right)\eta(t)$$
$$+\,\mu^{\beta}(t)(Y_{\beta})_u\left(t,x^0(t),u^0(t)\right)\xi(t)\Big]\mathrm{d}v$$
$$\leqslant -\rho^3 b\left(x,u,x^0,u^0\right)d^2\left((x,u),(x^0,u^0)\right). \tag{6}$$

Making the sum (4) + (5) + (6), side by side, of the previous relations and taking into account the assumption d) (implying $b\left(x,u,x^0,u^0\right) > 0$), we have

$$\int_{\Omega}\eta(t)\left[(\theta^r f_r)_x\left(t,x^0(t),u^0(t)\right) + \lambda_i^{\alpha}(t)(X_{\alpha}^i)_x\left(t,x^0(t),u^0(t)\right)\right]\mathrm{d}v$$
$$+\int_{\Omega}\left[\mu^{\beta}(t)(Y_{\beta})_x\left(t,x^0(t),u^0(t)\right)\eta(t) - \lambda^{\alpha}(t)D_{\alpha}\eta(t)\right]\mathrm{d}v$$
$$+\int_{\Omega}\xi(t)\left[(\theta^r f_r)_u\left(t,x^0(t),u^0(t)\right) + \lambda_i^{\alpha}(t)(X_{\alpha}^i)_u\left(t,x^0(t),u^0(t)\right)\right]\mathrm{d}v$$
$$+\int_{\Omega}\mu^{\beta}(t)(Y_{\beta})_u\left(t,x^0(t),u^0(t)\right)\xi(t)\mathrm{d}v < -\left(\theta^r\rho_r^1+\rho^2+\rho^3\right)d^2\left((x,u),(x^0,u^0)\right).$$

Taking into account the necessary optimality conditions formulated in Theorem 2, it follows

$$-\int_{\Omega}\eta(t)(\lambda^{\alpha}(t))_t\mathrm{d}v - \int_{\Omega}\left[\lambda^{\alpha}(t)D_{\alpha}\eta(t)\right]\mathrm{d}v + 0$$
$$< -(\theta^r\rho_r^1 + \rho^2 + \rho^3)d^2\left((x,u),(x^0,u^0)\right).$$

By direct computation, we find

$$\int_{\Omega} \eta(t) D_\alpha \lambda^\alpha(t) \mathrm{d}v = \int_{\Omega} D_\alpha \left[\eta(t)\lambda^\alpha(t)\right] \mathrm{d}v - \int_{\Omega} \left[\lambda^\alpha(t) D_\alpha \eta(t)\right] \mathrm{d}v,$$

but, applying the condition $\eta(t)|_{\partial\Omega} = 0$ and the flow-divergence formula, we get

$$\int_{\Omega} D_\alpha \left[\eta(t)\lambda^\alpha(t)\right] \mathrm{d}v = \int_{\partial\Omega} \left[\eta(t)\lambda^\alpha(t)\right] \vec{n} \mathrm{d}\sigma = 0,$$

where $\vec{n} = (n_\alpha)$, $\alpha = \overline{1, m}$, is the normal unit vector to the hypersurface $\partial\Omega$. It follows that $\int_{\Omega} \eta(t) D_\alpha \lambda^\alpha(t) \mathrm{d}v = -\int_{\Omega} \left[\lambda^\alpha(t) D_\alpha \eta(t)\right] \mathrm{d}v$ and further $-\int_{\Omega} \eta(t)(\lambda^\alpha(t))_t \mathrm{d}v - \int_{\Omega} \left[\lambda^\alpha(t) D_\alpha \eta(t)\right] \mathrm{d}v = 0$. As a result, we obtain $0 < -(\theta^r \rho_r^1 + \rho^2 + \rho^3) d^2\left((x, u), (x^0, u^0)\right)$ and applying the hypothesis e) and $d\left((x, u), (x^0, u^0)\right) \geqslant 0$, we get a contradiction. Thus, the point (x^0, u^0) is a geodesic efficient solution in (VCP) and the proof is now complete. □

Now, taking into account the aforementioned theorem, the next result is obvious.

Corollary 1 *Let $(x^0, u^0) \in \mathfrak{D}$ be a feasible solution in (VCP), $\theta = (\theta^r)$ a scalar vector and $\mu(t) = (\mu^\beta(t))$, $\lambda(t) = (\lambda_i^\alpha(t))$ two piecewise smooth functions, all satisfying the conditions formulated in Theorem 2. Also, assume that:*

a) each functional $F_r(x, u) = \int_{\Omega} f_r\left(t, x(t), u(t)\right) \mathrm{d}v$, $r \in \{1, \ldots, p\}$, is (ρ_r^1, b)-geodesic quasiinvex at (x^0, u^0) with respect to η and ξ;

b′) the functional

$$X(x, u) = \int_{\Omega} \left[\lambda_i^\alpha(t) \left(X_\alpha^i\left(t, x(t), u(t)\right) - \frac{\partial x^i}{\partial t^\alpha}(t)\right) + \mu^\beta(t) Y_\beta\left(t, x(t), u(t)\right)\right] \mathrm{d}v$$

is $(\overline{\rho}^2, b)$-geodesic quasiinvex at (x^0, u^0) with respect to η and ξ;

d′) at least one of the functionals given in a), b′) is strictly (ρ, b)-geodesic quasiinvex at (x^0, u^0) with respect to η and ξ, where $\rho = \rho_r^1$ or $\overline{\rho}^2$;

e′) $\theta^r \rho_r^1 + \overline{\rho}^2 \geqslant 0 \quad (\rho_r^1, \overline{\rho}^2 \in \mathbb{R})$.

Then the point (x^0, u^0) *is a geodesic efficient solution in* (VCP).

In the following, we shall set sufficient conditions of efficiency in the multidimensional multiobjective fractional control problem $(VFCP)$.

Theorem 5 *Let* $(x^0, u^0) \in \mathfrak{D}$ *be a feasible solution in* $(VFCP)$*,* $\theta = (\theta^r)$ *a scalar vector and* $\mu(t) = (\mu^\beta(t))$*,* $\lambda(t) = (\lambda_i^\alpha(t))$ *two piecewise smooth functions, all satisfying the conditions formulated in Theorem 3. Also, assume that:*

a) each functional $F_r(x, u) - R_r^0 G_r(x, u)$*,* $r \in \{1, \dots, p\}$*, is* (ρ_r^1, b)*-geodesic quasiinvex at* (x^0, u^0) *with respect to* η *and* ξ*;*

b) the functional $X(x, u) = \displaystyle\int_\Omega \lambda_i^\alpha(t) \left[X_\alpha^i(t, x(t), u(t)) - \frac{\partial x^i}{\partial t^\alpha}(t) \right] \mathrm{d}v$ *is monotonic* (ρ^2, b)*-geodesic quasiinvex at* (x^0, u^0) *with respect to* η *and* ξ*;*

c) the functional $Y(x, u) = \displaystyle\int_\Omega \mu^\beta(t) Y_\beta(t, x(t), u(t))\, \mathrm{d}v$ *is* (ρ^3, b)*-geodesic quasiinvex at* (x^0, u^0) *with respect to* η *and* ξ*;*

d) at least one of the functionals given in a), c) is strictly (ρ, b)*-geodesic quasiinvex at* (x^0, u^0) *with respect to* η *and* ξ*, where* $\rho = \rho_r^1$ *or* ρ^3*;*

e) $\theta^r \rho_r^1 + \rho^2 + \rho^3 \geqslant 0 \ (\rho_r^1, \rho^2, \rho^3 \in \mathbb{R})$.

Then the point (x^0, u^0) *is a geodesic efficient solution in* $(VFCP)$.

Proof. The proof is similar to that of Theorem 4 by considering the functions $f_r(t, x(t), u(t)) - R_r^0 g_r(t, x(t), u(t)), r = \overline{1, p}$ instead of $f_r(t, x(t), u(t)), r = \overline{1, p}$. □

Further, in accordance to the previous theorem, the next result is obvious.

Corollary 2 *Let* $(x^0, u^0) \in \mathfrak{D}$ *be a feasible solution in* $(VFCP)$*,* $\theta = (\theta^r)$ *a scalar vector and* $\mu(t) = (\mu^\beta(t))$*,* $\lambda(t) = (\lambda_i^\alpha(t))$ *two piecewise smooth functions, all satisfying the conditions formulated in Theorem 3. Also, assume that:*

a) each functional $F_r(x, u) - R_r^0 G_r(x, u)$*,* $r \in \{1, \dots, p\}$*, is* (ρ_r^1, b)*-geodesic quasiinvex at* (x^0, u^0) *with respect to* η *and* ξ*;*

b′) the functional

$$X(x,u)=\int_{\Omega}\left[\lambda_i^{\alpha}(t)\left(X_{\alpha}^{i}(t,x(t),u(t))-\frac{\partial x^i}{\partial t^{\alpha}}(t)\right)+\mu^{\beta}(t)Y_{\beta}(t,x(t),u(t))\right]\mathrm{d}v$$

is $(\overline{\rho}^2, b)$-geodesic quasiinvex at (x^0, u^0) with respect to η and ξ;

d′) at least one of the functionals given in a), b′) is strictly (ρ, b)-geodesic quasiinvex at (x^0, u^0) with respect to η and ξ, where $\rho = \rho_r^1$ or $\overline{\rho}^2$;

e′) $\theta^r \rho_r^1 + \overline{\rho}^2 \geqslant 0$ $(\rho_r^1, \overline{\rho}^2 \in \mathbb{R})$.

Then the point (x^0, u^0) is a geodesic efficient solution in $(VFCP)$.

4. Conclusion

In this paper, we have considered a multiobjective variational control problem of minimizing a vector of multiple integral cost functionals quotients subject to mixed constraints involving m-flow type PDEs. Using the new notions of *geodesic efficient solution* and *normal geodesic efficient solution*, we have formulated and proved necessary efficiency conditions for (VCP) and $(VFCP)$. As well, using our original concept of (ρ, b)-*geodesic quasiinvexity*, sufficient conditions of efficiency for a feasible solution in (VCP) and $(VFCP)$ have been derived.

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