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A proposition of a new damping function as a component of the objective function in the adjustment resistant to gross errors

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Abstract: The aim of the work is to develop and test an algorithm of adjustment of geodetic observations, resistant to gross errors (method of robust estimations), with the use of the damping function, proposed by the author. Detailed formulae of the damping function as a component of the objective function in a modified classic least squares method were derived. The selection criteria for the controlling parameters of the damping functions have also been provided. The effectiveness of the algorithm has been verified with two numerical examples. The results have been analysed with reference to the methods of resistant compensation, which apply other damping functions, e.g. Hampel's function.

Keywords: Outlying observations, resistant adjustment, damping function

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1. Introduction

Deviating observations or gross errors are meant to denote a measurement result containing an error rebounding from the Gaussian probabilistic model (Kadaj, 1995). The theory of adjustment sometimes distinguishes the concept of gross error and outlying error (Kadaj, 1978). However, this is a question of convention and is not so important when the object of our consideration has already been defined: protecting "good" observations, i.e. those which fall within the acceptable limits, from adverse effect of "contaminated" observations, i.e. those from outside of the acceptable interval. It is generally accepted that an outlying observation is such that contains a gross error (Wiśniewski, 2005).

In the commonly applied least squares method (LSQ), gross errors are dispersed. The method is classified as a neutral estimation (Wiśniewski, 2005). An assumption of the methods of resistant adjustment is that an observation containing a gross error should be "ignored" in the computation process. The idea of robust (resistant) estimation was thought up by Huber (1964). It was then developed by Hampel (1971) and other researchers (e.g. Xu, 2005). A similar subject scope is dealt with by a number of other papers (e.g. Baarda, 1968; Krarup et al., 1980; Kadaj, 1980, 1984, 1988; Kamiński and Wiśniewski, 1992; Wiśniewski, 1987). A detailed description of selected methods of resistant adjustment can also be found in the studies conducted by Kwaśniak and Prószyński (2002) as well as Wiśniewski (2005).

The subject of this study is a modified least squares method, which is included in the class of M-estimations. It is the component of the objective function that is modified: original weights are replaced with a weight function, whose element is the so called damping function (e.g. Wiśniewski, 2005). Figure 1 shows a graphical illustration of some examples of damping functions.

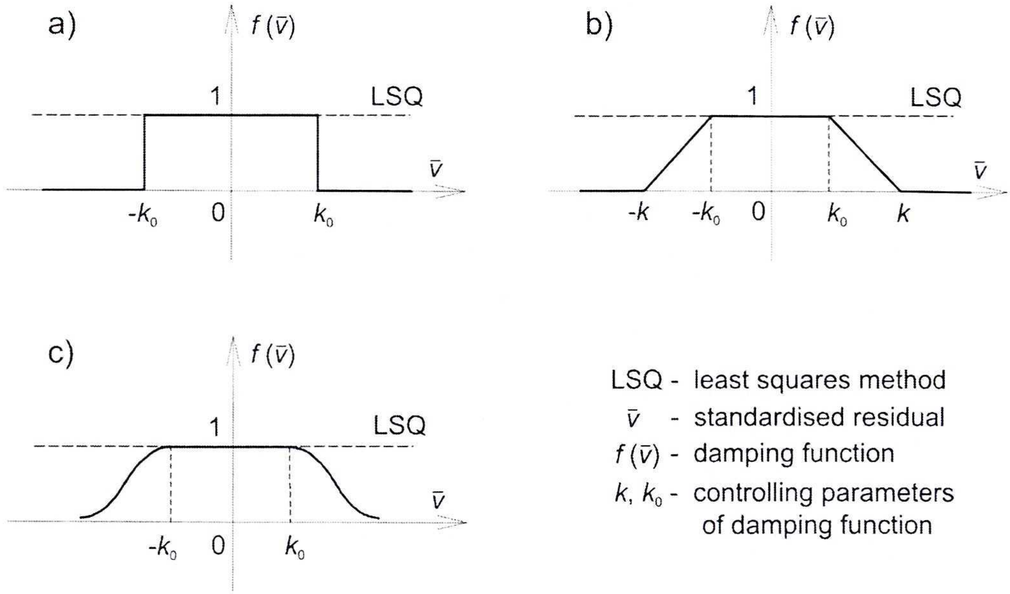


Fig. 1. Examples of damping functions: a) Huber's function, b) Hampel's function, c) Danish function

Huber's function (Fig. 1a) radically rejects all the observations (their weights are assigned the zero value), whose corrections (estimated with classic LSQ) are outside the established acceptable interval $\langle -k_0; k_0 \rangle$. In Hampel's function (Fig. 1b) two additional acceptable intervals are adopted, $\langle -k; -k_0)$, $(k_0; k \rangle$, in which observations are damped linearly. In Danish function (Fig. 1c), the observations, whose corrections are outside the basic acceptable interval, are damped exponentially.

The shape of the diagram of the proposed damping function (a quadratic damping function QDF) is as shown in Figure 2. The function is, in a sense, a modified Hampel's function (Fig. 1b). The linear course of the diagram in additional intervals $\langle -k; -k_0), (k_0; k \rangle$ are replaced with fragments of a parabola. The assumed objective of the projected function was as follows: the observations that only slightly go beyond the limits of the acceptable interval $\langle -k_0; k_0 \rangle$ should be damped "softly" (like in the Danish function), while those that definitely deviate from it should be damped more radically. It is also necessary to define the limit k, outside of which the observation is assigned the weight equal to zero.

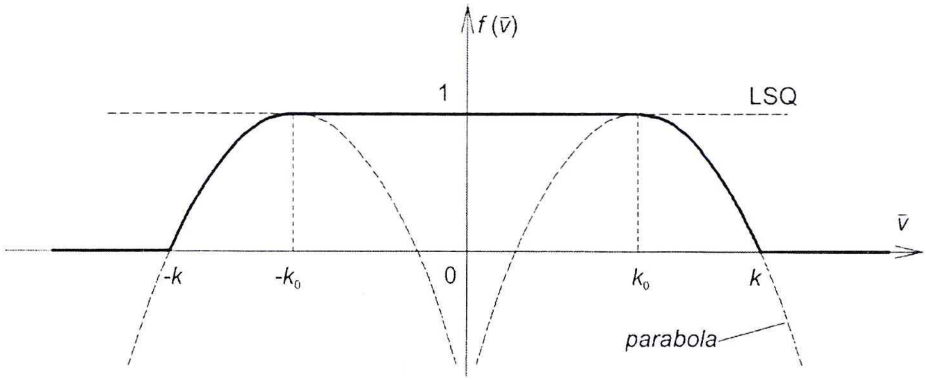


Fig. 2. A diagram of the proposed quadratic damping function QDF

2. Deriving formulae for the damping function

Below there is the procedure of deriving formulae for the damping function as a function of variable values of standardised corrections $\bar{v}$ and controlling parameters k_0 and k. Detailed formulae have been derived for a function increasing in the interval $\langle -k; -k_0)$ and decreasing in the interval $(k_0; k\rangle$.

First let us quote the formulae for the quadratic function, which are well known from algebra. The general form of a quadratic trinomial in the system of $(0, \bar{v}, f(\bar{v}))$ is as follows:

$$f(\bar{v}) = a\bar{v}^2 + b\bar{v} + c; \quad a \neq 0; \quad b, c \in R \tag{1}$$

whereas its standard form (Fig. 3) is given as

$$f(\bar{v}) = a(\bar{v} - p)^2 + q \tag{2}$$

where

$$p = -\frac{b}{2a}; \quad q = -\frac{\Delta}{4a} \tag{3}$$

$$\Delta = b^2 - 4ac \tag{4}$$

(Δ is the discriminant of a quadratic trinomial).

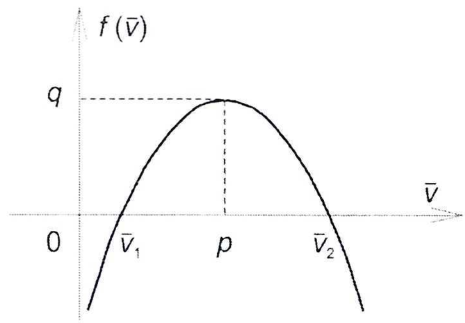


Fig. 3. An example of a quadratic function in the canonical notation

The roots of a quadratic trinomial (in cases similar to the damping function – cf. Fig. 3) are calculated from the following formulae:

$$\bar{v}_1 = \frac{-b - \sqrt{\Delta}}{2a}; \quad \bar{v}_2 = \frac{-b + \sqrt{\Delta}}{2a}; \quad \Delta > 0 \tag{5}$$

The damping function in the interval $\langle -k; -k_0)$

Here the parameters assume the following values (cf. Fig. 2)

$$q = 1; \quad p = -k_0 \tag{6}$$

Considering (4), the x-intercept (5) at the point $-k$ can be expressed as

$$\bar{v}_1 = -k = \frac{-b - \sqrt{\Delta}}{2a} = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \tag{7}$$

The other root is of no interest to us, as it is outside the considered interval $\langle -k; -k_0)$.

Based on (3), (6) and (4) one can also note that

$$k_0 = \frac{b}{2a} \tag{8}$$

and

$$-\frac{\Delta}{4a} = -\frac{b^2 - 4ac}{4a} = 1 \tag{9}$$

Thereby a system of 3 equations (7), (8), (9) with three unknowns (a, b, c) have been obtained

$$\begin{cases} k = \dfrac{b + \sqrt{b^2 - 4ac}}{2a} \\ k_0 = \dfrac{b}{2a} \\ -\dfrac{b^2 - 4ac}{4a} = 1 \end{cases} \tag{10}$$

By solving the system (10) one obtains the coefficients of the quadratic trinomial as the functions of controlling parameters k_0 and k

$$a = -\frac{1}{(k_0 - k)^2} \tag{11}$$

$$b = -\frac{2k_0}{(k_0 - k)^2} \tag{12}$$

$$c = 1 - \frac{k_0}{(k_0 - k)^2} \tag{13}$$

The system (10) is also satisfied for $a = 0$, but according to the assumption in (1), that solution is rejected. Finally, the formula (2) for a damping function within the interval $\langle -k; -k_0)$ adopts the following form

$$f_1(\bar{v}) = 1 - \frac{\bar{v}^2 + 2k_0\bar{v} + k_0^2}{(k_0 - k)^2} \tag{14}$$

In order to find the specific formula for the damping function within the interval $(k_0; k\rangle$, one adopts the same assumptions as previously (6)

$$q = 1; \quad p = k_0 \tag{15}$$

Following the same steps (formulae (7) to (13)) leads finally to the parabola equation with the apex at k_0 (Fig. 2)

$$f_2(\bar{v}) = 1 - \frac{\bar{v}^2 - 2k_0\bar{v} + k_0^2}{(k - k_0)^2} \tag{16}$$

Comparing (14) and (16), it is easy to establish a general formula for the damping function for $|\bar{v}_i| \in (k; k_0\rangle$

$$f(\bar{v}) = 1 - \frac{\bar{v}^2 - 2k_0|\bar{v}| + k_0^2}{(k - k_0)^2} \tag{17}$$

A damping function for an algorithm of adjustment resistant to outlying observations (for any $\bar{v}_i$) can be expressed as follows

$$f(\bar{v}) = \begin{cases} 1 & \text{for} \quad \bar{v} \in \langle -k_0; k_0 \rangle \\ 1 - \dfrac{\bar{v}^2 - 2k_0 |\bar{v}| + k_0^2}{(k - k_0)^2} & \text{for} \quad |\bar{v}| \in (k_0; k\rangle \\ 0 & \text{for} \quad |\bar{v}| > k \end{cases} \tag{18}$$

This notation directly results in the formula of a weight function

$$p(\bar{v}) = f(\bar{v}) \cdot p = \begin{cases} p & \text{for} \quad \bar{v} \in \langle -k_0; k_0 \rangle \\ \left(1 - \dfrac{\bar{v}^2 - 2k_0 |\bar{v}| + k_0^2}{(k - k_0)^2}\right) \cdot p & \text{for} \quad |\bar{v}| \in (k_0; k\rangle \\ 0 & \text{for} \quad |\bar{v}| > k \end{cases} \tag{19}$$

where p are the weights of observations, established from their *a priori* mean square errors $\left(p_i = 1/m_i^2\right)$.

3. Algorithm of adjustment resistant to outlying observations

Algorithms of adjustment resistant to outlying observations (gross errors) require a certain modification of the least squares method. The problem can be formulated by defining a functional model

$$\mathbf{V} = \mathbf{AX} - \mathbf{L} \tag{20}$$

and adopting the adjustment criterion

$$\mathbf{V}^{\mathrm{T}} \left[\mathbf{F}\left(\bar{\mathbf{V}}\right) \cdot \mathbf{P}\right] \mathbf{V} = \min \tag{21}$$

The variables **A**, **P**, **L**, **X** and **V** denote matrices in a classic adjustment according to the least squares method (**A** – design matrix, **P** – weight matrix, **L** – observation vector, **X** – vector of unknown parameters – estimated increments to approximated unknowns, **V** – vector of residuals). The expression $\left[\mathbf{F}\left(\bar{\mathbf{V}}\right) \cdot \mathbf{P}\right]$ denotes an equivalent weight matrix, whereas $\mathbf{F}\left(\bar{\mathbf{V}}\right)$ is a diagonal damping matrix in the following form

$$\mathbf{F}\left(\bar{\mathbf{V}}\right) = \begin{bmatrix} f(\bar{v}_1) & 0 & \cdots & 0 \\ 0 & f(\bar{v}_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\bar{v}_n) \end{bmatrix} \tag{22}$$

Hence, the equivalent weight matrix can be written as

$$\left[\mathbf{F}\left(\bar{\mathbf{V}}\right)\cdot\mathbf{P}\right] = \begin{bmatrix} p(\bar{v}_1) & 0 & \cdots & 0 \\ 0 & p(\bar{v}_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(\bar{v}_n) \end{bmatrix} \tag{23}$$

The values of standardised residuals $\bar{v}_i$ are calculated from residuals v_i estimated with the classic least squares method (the initial step of adjustment)

$$\bar{v}_i = \frac{v_i}{m_{v_i}} \tag{24}$$

Information about mean square errors of residuals is contained in a covariance matrix of residuals $\mathbf{Q_V}$ (square roots of the diagonal elements of the matrix will be needed)

$$\mathbf{Q_V} = \mathbf{P}^{-1} - \mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{PA}\right)^{-1}\mathbf{A}^{\mathrm{T}} \tag{25}$$

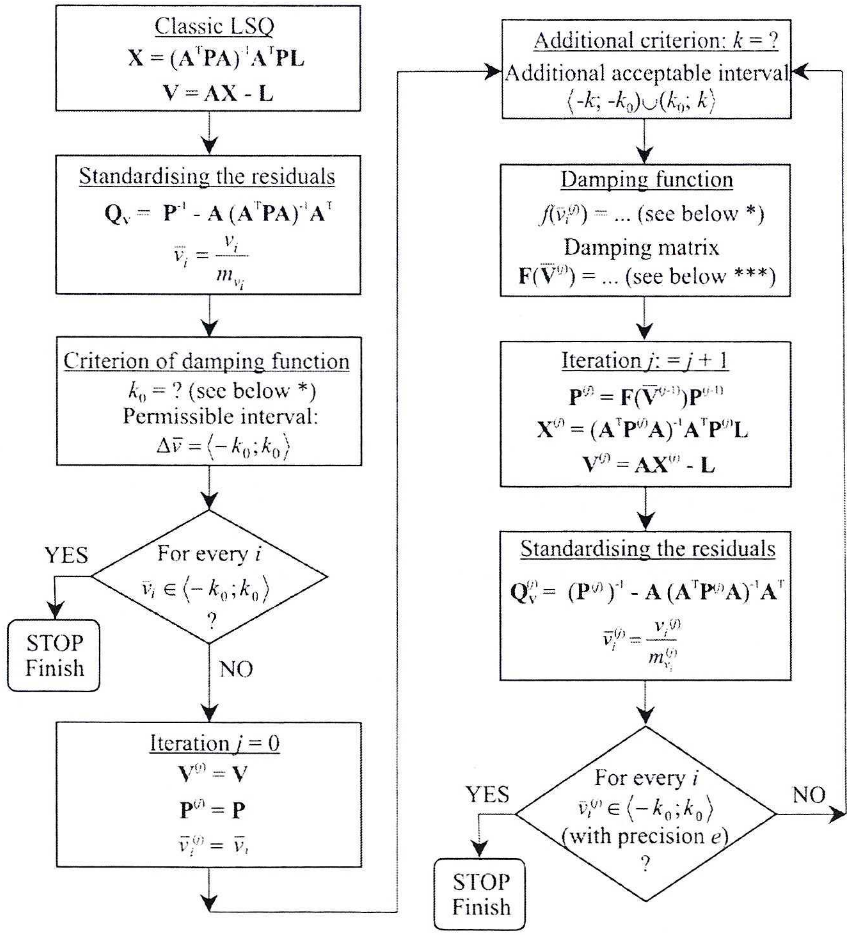
To simplify it, the mean square error of the unit weight (the value of the standard deviation estimator) in (25) has been adopted as $m_0 = 1$.

Standardised residuals, determined from (24) will be elements of the vector $\bar{\mathbf{V}}$, which in turn is an argument of a matrix function $\mathbf{F}\left(\bar{\mathbf{V}}\right)$ – cf. (22). The algorithm of adjustment resistant to outlying observations with the use of the proposed damping function is presented in the block diagram in Figure 4.

The controlling parameter found in the damping function (18) is chosen similarly as in other known damping functions (Wiśniewski, 2005). The first step consists in adopting a level of probability γ for the determined interval of standardised corrections $\Delta\bar{v} = \langle -k_0; k_0 \rangle$. Then, assuming the normal distribution for observation errors, the value of parameter k_0 is determined

$$P\left(\bar{v} \in \langle -k_0; k_0 \rangle\right) = 2\int_0^{k_0} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-\bar{v}^2}{2}\right) d\bar{v} = 2\phi(k_0) = \gamma \tag{26}$$

The values of function $\phi(k_0)$ are listed in tables, hence it is easy to find the value of the argument k_0 for the given value of the function. An additional criterion of the damping function (parameter k – cf. Fig. 2) is adopted empirically. A too low value of the parameter can result in a risk of failure to spot an outlying observation, while a too high one – can result in a very slow convergence of the iterative process. It is recommended that the initial value of parameter k should be taken from the interval $\langle 4; 6 \rangle$, like for the Hampel's function.



*) $k_0 = f(\gamma)$ - for normal distribution (γ - assumed probability)

$$
**)\quad f(\bar{v}_i^{(j)}) = \begin{cases} 1 & \text{for } \bar{v}_i^{(j)} \in \langle -k_0; k_0 \rangle \\ 1 - \dfrac{(v_i^{(j)})^2 - 2k_0|v_i^{(j)}| + k_0^2}{(k - k_0)^2} & \text{for } |\bar{v}_i^{(j)}| \in (-k_0; k\rangle \\ 0 & \text{for } |\bar{v}_i^{(j)}| > k \end{cases}
$$

***) $\mathbf{F}(\overline{\mathbf{V}}^{(j)}) = diag\{ f(\bar{v}_1^{(j)}), f(\bar{v}_2^{(j)}), \ldots, f(\bar{v}_n^{(j)})\}$

Fig. 4. The algorithm of resistant adjustment with the use of the proposed damping function

4. Numerical examples

The process of adjustment (by the parametric method) of a set of observations of one quantity (to make calculations simpler) as an example to illustrate the application of theoretical algorithmic assumptions (Fig. 4) is presented. The task can obviously be performed more easily with the use of the direct observation adjustment, but the parametric method is more common (universal) in the algorithms of adjustment of geodetic observations.

Example 1

A set of observations contains equally precise results of the measurement of a certain length x, performed 4 times: $d_i = \{100.006; 100.003; 99.997; 100.054\}$, of which one distinctly differs (deviates) from the others. The approximate value of the unknown quantity has been adopted as $x_0 = 100.000$ m while the mean square error of the measurement: $m = 0.005$ m.

The adjustment will be performed by applying the proposed quadratic damping function (QDF) and then, for comparison, the procedure will be repeated with a different damping function – Hampel's function (cf. Fig. 1).

The method of resistant adjustment with the use of QDF

The initial stage – classic least squares method (LSQ)

$$\begin{cases} d_1 + v_1 = x_0 + \delta x \\ d_2 + v_2 = x_0 + \delta x \\ d_3 + v_3 = x_0 + \delta x \\ d_4 + v_4 = x_0 + \delta x \end{cases} \rightarrow \mathbf{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} 6 \text{ mm} \\ 3 \text{ mm} \\ -3 \text{ mm} \\ 54 \text{ mm} \end{bmatrix}; \quad \mathbf{X} = [\delta x]$$

$$\mathbf{P} = diag\,\{0.04; 0.04; 0.04; 0.04\}$$

$$\mathbf{X} = \delta x = \left(\mathbf{A}^{\mathrm{T}}\mathbf{PA}\right)^{-1} \mathbf{A}^{\mathrm{T}}\mathbf{PL} = 15 \text{ mm}$$

$$\mathbf{V} = \mathbf{AX} - \mathbf{L} = \begin{bmatrix} 9 \text{ mm} \\ 12 \text{ mm} \\ 18 \text{ mm} \\ -39 \text{ mm} \end{bmatrix}$$

Standardising the residuals:

$$\mathbf{Q_V} = \mathbf{P}^{-1} - \mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 18.75 & -6.25 & -6.25 & -6.25 \\ -6.25 & 18.75 & -6.25 & -6.25 \\ -6.25 & -6.25 & 18.75 & -6.25 \\ -6.25 & -6.25 & -6.25 & 18.75 \end{bmatrix} \rightarrow m_{v_i} = \sqrt{18.25}$$

$$\left\{\begin{array}{l} \bar{v}_1 = \dfrac{v_1}{m_{v_1}} = \dfrac{9}{\sqrt{18.25}} = 2.08 \\ \bar{v}_2 = \dfrac{v_2}{m_{v_2}} = \dfrac{12}{\sqrt{18.25}} = 2.77 \\ \bar{v}_3 = \dfrac{v_3}{m_{v_3}} = \dfrac{18}{\sqrt{18.25}} = 4.16 \\ \bar{v}_4 = \dfrac{v_4}{m_{v_4}} = \dfrac{-39}{\sqrt{18.25}} = -9.01 \end{array}\right. \rightarrow \bar{\mathbf{V}} = \begin{bmatrix} 2.08 \\ 2.77 \\ 4.16 \\ -9.01 \end{bmatrix}$$

The level of probability is adopted as $\gamma = 0.95$, hence (normal distribution tables) the criterion of the damping function is determined as $k_0 \cong 2$. Therefore, the acceptable interval is $\Delta\bar{v} = \langle -2; 2 \rangle$. None of the four standardised residuals lies within the interval. Let us assume an additional criterion of a damping function: $k = 6$. An additional acceptable interval is $\langle -6; -2) \cup (2; 6 \rangle$.

Iteration $j = 0$

$$\mathbf{V}^{(0)} = \mathbf{V}; \quad \mathbf{P}^{(0)} = \mathbf{P}; \quad \bar{v}_i^{(0)} = \bar{v}_i$$

$$\left\{\begin{array}{lll} \left|\bar{v}_1^{(0)}\right| \in (2; 6 \rangle & \rightarrow & f\left(\bar{v}_1^{(0)}\right) = 1 - \dfrac{\left(\bar{v}_1^{(0)}\right)^2 - 2k_0\left|\bar{v}_1^{(0)}\right| + k_0^2}{(k - k_0)^2} = 1.00 \\ \left|\bar{v}_2^{(0)}\right| \in (2; 6 \rangle & \rightarrow & f\left(\bar{v}_2^{(0)}\right) = 1 - \dfrac{\left(\bar{v}_2^{(0)}\right)^2 - 2k_0\left|\bar{v}_2^{(0)}\right| + k_0^2}{(k - k_0)^2} = 0.96 \\ \left|\bar{v}_3^{(0)}\right| \in (2; 6 \rangle & \rightarrow & f\left(\bar{v}_3^{(0)}\right) = 1 - \dfrac{\left(\bar{v}_3^{(0)}\right)^2 - 2k_0\left|\bar{v}_3^{(0)}\right| + k_0^2}{(k - k_0)^2} = 0.71 \\ \left|\bar{v}_4^{(0)}\right| > 6 & \rightarrow & f\left(\bar{v}_4^{(0)}\right) = 0.0001\ (\cong 0) \end{array}\right.$$

$$\mathbf{F}\left(\bar{\mathbf{V}}^{(0)}\right) = diag\{1.00; 0.96; 0.71; 0.0001\}$$

The residual $\bar{v}_4^{(0)}$ is outside each of the acceptable intervals, hence the damping function should be assigned the value of zero. However, in order to be able to continue the calculations in the next iteration, the value close to zero (e.g. 0.0001) is adopted. This will make an equivalent element in the weight matrix $\mathbf{P}^{(1)}$, numerically equal to zero.

Iteration $j = 1$

$$\mathbf{P}^{(1)} = \mathbf{F}\left(\bar{\mathbf{V}}^{(0)}\right) \cdot \mathbf{P}^{(0)} = diag\left\{0.040; 0.038; 0.028; 4 \cdot 10^{-6}\right\}$$

$$\mathbf{X}^{(1)} = \delta x^{(1)} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}^{(1)}\mathbf{A}\right)^{-1} \mathbf{A}^{\mathrm{T}}\mathbf{P}^{(1)}\mathbf{L} = 2.53 \text{ mm}$$

$$\mathbf{V}^{(1)} = \mathbf{A}\mathbf{X}^{(1)} - \mathbf{L} = \begin{bmatrix} -3.47 \text{ mm} \\ -0.47 \text{ mm} \\ 5.53 \text{ mm} \\ -51.47 \text{ mm} \end{bmatrix}$$

$$\mathbf{Q}_{\mathbf{V}}^{(1)} = \left(\mathbf{P}^{(1)}\right)^{-1} - \mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}^{(1)}\mathbf{A}\right)^{-1} \mathbf{A}^{\mathrm{T}} =$$

$$= \begin{bmatrix} 15.64 & -9.36 & -9.36 & -9.36 \\ -9.36 & 16.68 & -9.36 & -9.36 \\ -9.36 & -9.36 & 25.85 & -9.36 \\ -9.36 & -9.36 & -9.36 & 2.5 \cdot 10^5 \end{bmatrix} \rightarrow \begin{cases} m_{v_1} = \sqrt{15.64} \\ m_{v_2} = \sqrt{16.68} \\ m_{v_3} = \sqrt{25.85} \\ m_{v_4} = \sqrt{2.5 \cdot 10^5} \end{cases}$$

$$\begin{cases} \bar{v}_1^{(1)} = \dfrac{v_1^{(1)}}{m_{v_1}^{(1)}} = \dfrac{-3.47}{\sqrt{15.64}} = -0.88 \\ \bar{v}_2^{(1)} = \dfrac{v_2^{(1)}}{m_{v_2}^{(1)}} = \dfrac{-0.47}{\sqrt{16.68}} = -0.12 \\ \bar{v}_3^{(1)} = \dfrac{v_3^{(1)}}{m_{v_3}^{(1)}} = \dfrac{5.53}{\sqrt{25.85}} = 1.09 \\ \bar{v}_4^{(1)} = \dfrac{v_4^{(1)}}{m_{v_4}^{(1)}} = \dfrac{-51.47}{\sqrt{2.5 \cdot 10^5}} = -0.10 \end{cases} \rightarrow \bar{\mathbf{V}} = \begin{bmatrix} -0.88 \\ -0.12 \\ 1.09 \\ -0.10 \end{bmatrix}$$

$$\mathbf{F}\left(\bar{\mathbf{V}}^{(1)}\right) = diag\{1; 1; 1; 1\}$$

As all the standardised residuals $\bar{v}_i^{(1)}$ lie within the interval $\Delta\bar{v} = \langle -2; 2 \rangle$, the matrix weight function $\mathbf{F}\left(\bar{\mathbf{V}}^{(1)}\right)$ becomes a unit matrix and subsequent iterations would have no effect. The values calculated in this iteration ($j = 1$) are regarded as final. Further calculations (observations adjustment, unknowns, precision evaluation, etc.) are conducted according to the classic LSQ.

The method of resistant adjustment with the use of Hampel's function

The initial stage (LSQ) together with standardisation of residuals and determination of parameters k_0 and k will be identical as in the previous method (quadratic damping function). Calculations will be continued starting with iteration $j = 0$. The formula for Hampel's damping function – cf. e.g. (Wiśniewski, 2005) will be applied

$$f(\bar{v}) = \frac{|\bar{v}| - k}{k_0 - k} \tag{27}$$

Iteration $j = 0$

$$\mathbf{V}^{(0)} = \mathbf{V}; \quad \mathbf{P}^{(0)} = \mathbf{P}; \quad \bar{v}_i^{(0)} = \bar{v}_i$$

$$\begin{cases} \left|\bar{v}_1^{(0)}\right| \in (2; 6\rangle & \rightarrow \quad f\left(\bar{v}_1^{(0)}\right) = \dfrac{\left|\bar{v}_1^{(0)}\right| - k}{k_0 - k} = 0.98 \\ \left|\bar{v}_2^{(0)}\right| \in (2; 6\rangle & \rightarrow \quad f\left(\bar{v}_2^{(0)}\right) = \dfrac{\left|\bar{v}_2^{(0)}\right| - k}{k_0 - k} = 0.81 \\ \left|\bar{v}_3^{(0)}\right| \in (2; 6\rangle & \rightarrow \quad f\left(\bar{v}_3^{(0)}\right) = \dfrac{\left|\bar{v}_3^{(0)}\right| - k}{k_0 - k} = 0.46 \\ \left|\bar{v}_4^{(0)}\right| > 6 & \rightarrow \quad f\left(\bar{v}_4^{(0)}\right) = 0.0001 \ (\cong 0) \end{cases}$$

$$\mathbf{F}\left(\bar{\mathbf{V}}^{(0)}\right) = diag\{0.98; 0.81; 0.46; 0.0001\}$$

Iteration $j = 1$

$$\mathbf{P}^{(1)} = \mathbf{F}\left(\bar{\mathbf{V}}^{(0)}\right) \cdot \mathbf{P}^{(0)} = diag\left\{0.039; 0.032; 0.018; 4 \cdot 10^{-6}\right\}$$

$$\mathbf{X}^{(1)} = \delta x^{(1)} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}^{(1)}\mathbf{A}\right)^{-1} \mathbf{A}^{\mathrm{T}}\mathbf{P}^{(1)}\mathbf{L} = 3.08 \text{ mm}$$

$$\mathbf{V}^{(1)} = \mathbf{AX}^{(1)} - \mathbf{L} = \begin{bmatrix} -2.92 \text{ mm} \\ 0.08 \text{ mm} \\ 6.08 \text{ mm} \\ -50.92 \text{ mm} \end{bmatrix}$$

$$\mathbf{Q}_{\mathbf{V}}^{(1)} = \left(\mathbf{P}^{(1)}\right)^{-1} - \mathbf{A}\left(\mathbf{A}^{\mathrm{T}}\mathbf{P}^{(1)}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}} =$$

$$= \begin{bmatrix} 14.40 & -11.11 & -11.11 & -11.11 \\ -11.11 & 19.75 & -11.11 & -11.11 \\ -11.11 & -11.11 & 43.24 & -11.11 \\ -11.11 & -11.11 & -11.11 & 2.5 \cdot 10^5 \end{bmatrix} \rightarrow \begin{cases} m_{v_1} = \sqrt{14.40} \\ m_{v_2} = \sqrt{19.75} \\ m_{v_3} = \sqrt{43.24} \\ m_{v_4} = \sqrt{2.5 \cdot 10^5} \end{cases}$$

$$\begin{cases} \bar{v}_1^{(1)} = \dfrac{v_1^{(1)}}{m_{v_1}^{(1)}} = \dfrac{-3.47}{\sqrt{15.64}} = -0.77 \\ \bar{v}_2^{(1)} = \dfrac{v_2^{(1)}}{m_{v_2}^{(1)}} = \dfrac{-0.47}{\sqrt{16.68}} = 0.02 \\ \bar{v}_3^{(1)} = \dfrac{v_3^{(1)}}{m_{v_3}^{(1)}} = \dfrac{5.53}{\sqrt{25.85}} = 0.92 \\ \bar{v}_4^{(1)} = \dfrac{v_4^{(1)}}{m_{v_4}^{(1)}} = \dfrac{-51.47}{\sqrt{2.5 \cdot 10^5}} = -0.10 \end{cases} \rightarrow \bar{\mathbf{V}} = \begin{bmatrix} -0.77 \\ 0.02 \\ 0.92 \\ -0.10 \end{bmatrix}$$

$$\mathbf{F}\left(\bar{\mathbf{V}}^{(1)}\right) = diag\{1; 1; 1; 1\}$$

Like in the previous method (QDF), the calculation cycle ends with the first iteration.

Classic adjustment according to the LSQ (without outlying observations)

In order to compare the two methods (QDF and Hampel's function) to find the one which produces more reliable results, adjustment according to the classic LSQ will also be performed. In this case the set of observations does not contain gross errors ($d_i = \{100.006; 100.003; 99.997; 100.002\}$). The outlying observation d_4 was replaced with the expected value (the arithmetic mean) of the other three observations.

$$\begin{cases} d_1 + v_1 = x_0 + \delta x \\ d_2 + v_2 = x_0 + \delta x \\ d_3 + v_3 = x_0 + \delta x \\ d_4 + v_4 = x_0 + \delta x \end{cases} \rightarrow \mathbf{A} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}; \quad \mathbf{L} = \begin{bmatrix} 6 \text{ mm} \\ 3 \text{ mm} \\ -3 \text{ mm} \\ 2 \text{ mm} \end{bmatrix}; \quad \mathbf{X} = [\delta x]$$

$$\mathbf{P} = diag\,\{0.04; 0.04; 0.04; 0.04\}$$

$$\mathbf{X} = \delta x = \left(\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{L} = 2.0 \text{ mm}$$

$$\mathbf{V} = \mathbf{A}\mathbf{X} - \mathbf{L} = \begin{bmatrix} -4.0 \text{ mm} \\ -1.0 \text{ mm} \\ 5.0 \text{ mm} \\ 0.0 \text{ mm} \end{bmatrix}$$

The major data from the adjustment of observations with the use of each of the three methods are presented in Table 1.

Table 1. Data from the adjustment with the use of various methods (example 1)

No.	Observations d_i [m]			Adjustment weights p_i [m^{-2}]			Residuals v_i [mm]			Unknown δx [mm]		
	QDF	Hampel	LSQ	QDF	Hampel	LSQ	QDF	Hampel	LSQ	QDF	Hampel	LSQ
1	100.006			0.040	0.039	0.040	−3.5	−2.9	−4.0	2.5	3.1	2.0
2	100.003			0.038	0.032	0.040	−0.5	0.1	−1.0			
3	99.997			0.028	0.018	0.040	5.5	6.1	5.0			
4	100.054	100.054	100.002	0.000	0.000	0.040	−51.5	−50.9	0.0			

The results of the adjustment (residuals, unknowns) obtained with the LSQ can be regarded as neutral (expected). The results obtained with the QDF are closer to those values than the ones obtained with Hampel's method. It is worth noticing the values of adjustment weights: Hampel's function damps too strongly the effect of observations No 2 and 3 (not containing gross errors) on the adjustment results. The proposed QDF function distorts correct (non-outlying) observations to a lesser degree.

Example 2

The set of observations discussed in example 1 contained a result which deviated from the others. It was eliminated from the computation process (by adopting the weight value equal to zero) due to the application of resistant adjustment. The set of observations in example 2 also contained a deviating result, but it cannot be regarded beyond doubt as incorrect. The value of the measurement (d_4) was adopted in such a way that its standardised residual $\bar{v}_4^{(0)}$ was included in the additional acceptable intervals $\langle -6; -2) \cup (2; 6\rangle$. Let us check how the resistant adjustment algorithms (including the QDF) operate in such cases. The criterion of precision, at which standardised residuals should fall within the basic acceptable interval ($\Delta\bar{v} = \langle -2; 2\rangle$) is adopted as: $e = 0.1$. The starting data and the course of the computation process for both methods is shown in Tables 2 and 3.

Table 2. Adjustment with the use of Hampel's function (example 2)

Observation index i	1	2	3	4
Observations d_i [m]	100.006	100.003	99.997	100.034
Weights p_i [m^{-2}]	0.04	0.04	0.04	0.04
Observation $(-l_i)$ [mm]	6	3	−3	34
Residuals v_i [mm]	4	7	13	−24
Unknown δx [mm]	10			
Standardised residual $\bar{v}_i$	0.92	1.62	3.00	−5.54
Controlling parameters	$k_0 = 2$; $k = 6$			
Damping function $f(\bar{v}_i)$	1	1	0.75	0.11
Iteration $j = 1$				
Weights p_i [m^{-2}]	0.040	0.040	0.030	0.004
Residuals v_i [mm]	−2.33	0.67	6.67	−30.33
Unknown δx [mm]	3.67			
Standardised residuals $\bar{v}_i$	−0.58	0.17	1.35	−2.05
Controlling parameters	$k_0 = 2$; $k = 6$			
Damping function $f(\bar{v}_i)$	1	1	1	0.99

Table 3. Adjustment with the use of the QDF (example 2)

Observation index i	1	2	3	4
Standardised residuals $\bar{v}_i$	0.92	1.62	3.00	−5.54
Controlling parameters	$k_0 = 2$; $k = 6$			
Damping function $f(\bar{v}_i)$	1	1	0.94	0.22
Iteration $j = 1$				
Weights p_i [m^{-2}]	0.040	0.040	0.038	0.009
Residuals v_i [mm]	−1.68	1.32	7.32	−29.68
Unknown δx [mm]	4.32			
Standardised residuals $\bar{v}_i$	−0.41	0.32	1.69	−2.89
Controlling parameters	$k_0 = 2$; $k = 6$			
Damping function $f(\bar{v}_i)$	1	1	1	0.99
Iteration $j = 2$				
Weights p_i [m^{-2}]	0.040	0.040	0.038	0.008
Residuals v_i [mm]	−1.78	1.22	7.22	−29.78
Unknown δx [mm]	4.22			
Standardised residuals $\bar{v}_i$	−0.43	0.30	1.67	−2.82
Controlling parameters	$k_0 = 2$; $k = 4$			
Damping function $f(\bar{v}_i)$	1	1	1	0.83
- - - - - - - - (Iterations $j = 3 \div 8$) - - - - - - - -				
Iteration $j = 9$				
Weights p_i [m^{-2}]	0.040	0.040	0.038	0.004
Residuals v_i [mm]	−2.73	0.27	6.27	−30.73
Unknown δx [mm]	3.27			
Standardised residuals $\bar{v}_i$	−0.67	0.07	1.46	−2.10
Controlling parameters	$k_0 = 2$; $k = 3$			
Damping function $f(\bar{v}_i)$	1	1	1	0.99

Calculations by Hampel's method are interrupted after the first iteration, as at this stage all the standardised residuals belong to the interval $\langle -2; 2\rangle$ at the precision not exceeding the adopted value of $e = 0.1$.

Adjustment with the use of the proposed quadratic damping function (Table 3) can be started with the calculation of the damping function. The initial stage is identical as in the previous method (see Table 2).

The results obtained in iteration $j = 2$ indicate low convergence of the iterative process, hence a change the value of parameter k was decided. After iteration $j = 4$ (for the same reason as previously) a new value of parameter $k = 3$ was again assumed. To simplify the presentation of results, the iterations $j = 3 \div 8$ are not included in Table 3.

Calculations were interrupted after iteration $j = 9$, as the condition $\bar{v}_i \in \langle -2; 2\rangle$ was satisfied at the assumed precision $e = 0.1$. Further calculations will not cause any significant changes, which can be inferred from the values of the damping function: $f(\bar{v}_i) \cong 1$. It would be possible to shorten the iteration process by consecutive changes of the value of k. The value of the parameter should be determined empirically, by observing the course of the iteration process. Reducing the value of the parameter k too much may result in unnecessary damping correct observations (in this case – d_3).

To compare the effectiveness of the applied methods of damping the outlying observations, the results were put in Table 4. The table uses (taken from example 1) the adjustment results by the classic LSQ (the set of observations does not contain incorrect results).

Table 4. Data from the adjustment with the use of the three methods (example 2)

No.	Observations d_i [m]			Adjustment weights p_i [m^{-2}]			Residuals v_i [mm]			Unknown δx [mm]		
	QDF	Hampel	LSQ	QDF	Hampel	LSQ	QDF	Hampel	LSQ	QDF	Hampel	LSQ
1	100.006			0.040	0.040	0.040	−2.7	−2.3	−4.0	3.3	3.7	2.0
2	100.003			0.040	0.040	0.040	0.3	0.7	−1.0			
3	99.997			0.038	0.030	0.040	6.3	6.7	5.0			
4	100.034	100.034	100.002	0.004	0.004	0.040	−30.7	−30.3	0.0			

Like in example 1, it can be noted that by applying the proposed QDF there were obtained the results (observation corrections and the value of the unknown) which are much closer to the expected ones (LSQ). It is again worth noticing the weights (from the last iteration of each adjustment): the weight $p_3 = 0.030$ in Hampel's method is clearly understated, though observation d_3 is correct. This results in an adverse effect on the values of the other residuals, too. The QDF function allows for more reliable determination of the weights of correct observation (without gross errors).

5. Summary and conclusions

The aim of this study was to develop and test a new method of identification and elimination of outlying observations from a set of geodetic measurements (as a modification of the classic LSQ). The novelty of the method lies in the application of a new form of the damping function as a component of the objective function.

The theoretical part of the work consists of the proposing the form and deriving the detailed formula of the damping function. The practical part of the study was based on two numerical examples. The following tasks were performed (for each of the two examples):

- adjustment of a set of observations containing a gross error with the use of the proposed quadratic damping function (QDF),
- performing the adjustment of the same set of observations with the use of the known Hampel's function,
- performing the adjustment of a set of observations without gross errors with the use of the classic LSQ.

The results were analysed and the criteria of determination of so called controlling parameters of a damping function were given, assuming that the pattern of observation errors is consistent with the normal distribution.

The following conclusions can be drawn from the results:

- the proposed damping function (QDF) can be applied as an objective function in modifying the classic method of least squares,
- the QDF can be applied with a view to detecting (identifying) outlying observations and damping their negative effect on the adjustment result,
- a negative effect of the QDF on correct observations (excessive damping) is lower than in other known damping functions, e.g. Hampel's function,
- compared to Hampel's function, the QDF damps more mildly the observations which are outside the acceptable interval, yet close to its limits,
- the iterative adjustment process performed with the application of the QDF is a slowly convergent process, which positively affects the corrections of observations without gross errors,
- the differences between the results adjusted with the use of the QDF and then with the use of Hampel's function are not too big, but they can be significant in some geodetic tasks, e.g. in determination of displacement or distortion of objects.

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References

Baarda W., (1968): *A testing procedure for use in geodetic networks*, Publications on Geodesy, New Series, Vol. 2, No 5, Delft.

Hampel F.R., (1971): *A general quantitative definition on robustness*, Ann. Math. Statist., Vol. 42, pp. 1887-1896.

Huber P., (1964): *Robust estimation of a location parameter*, Ann. Math. Statist., Vol. 35, pp. 73-101.

Kadaj R., (1978): *Adjustment with outliers* (in Polish), Przegląd Geodezyjny, 8, Warszawa, pp. 252-253.

Kadaj R., (1980): *Explication of conception of non-standard method of estimation* (in Polish), Geodezja i Kartografia, z. 3/4, Warszawa, pp. 186-195.

Kadaj R., (1984): *Die Methode der besten Alternative: Ein Ausgleichungsprinzip für Beobachtungssysteme*, Zeitschrift für Vermessungswesen, H3, pp. 301-307.

Kadaj R., (1988): *Eine Klasse Schatzverfahren mit praktischen Anwendungen*, Zeitschrift für Vermessungswesen, H8, pp. 157-165.

Kadaj R., (1995): *Geodetic system Geonet – Functional description and service principles* (in Polish), Wyd. Algo-Res, Rzeszow.

Kamiński W., Wiśniewski Z., (1992): *Analysis of some selected methods of geodetic observations adjustment, resistant to gross errors* (in Polish), Geodezja i Kartografia, t. XLI, z. 3/4, Warszawa, pp. 173-195.

Krarup T., Juhl J., Kubik K., (1980): *Gotterdammerung over least squares adjustment*, 14th ISP Congress, Hamburg, pp. 369-378.

Prószyński W., Kwaśniak M., (2002): *Reliability of geodetic networks* (in Polish), Oficyna Wydawnicza Politechniki Warszawskiej.

Wiśniewski Z., (1987): *Alternative selection principle and the maximum likelihood method* (in Polish), Geodezja i Kartografia, Warszawa, pp. 123-138.

Wiśniewski Z., (2005): *Adjustment calculus in geodesy* (in Polish), Wyd. Uniwersytetu Warmińsko-Mazurskiego, Olsztyn.

Xu P., (2005): *Sign-constrained robust least squares, subjective breakdown point and the effect of weights of observations on robustness*, Journal of Geodesy, Vol. 75, pp. 146-159.

Propozycja nowej funkcji tłumienia jako składowej funkcji celu w metodzie wyrównania odpornego na błędy grube

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Streszczenie

Celem niniejszej pracy jest opracowanie i przetestowanie algorytmu wyrównania obserwacji geodezyjnych, odpornego na błędy grube (metoda estymacji mocnych), z zastosowaniem zaproponowanej przez autora nowej funkcji tłumienia. Wyprowadzono wzory szczegółowe funkcji tłumienia jako składowej funkcji celu w modyfikowanej klasycznej metodzie najmniejszych kwadratów. Podano również kryteria doboru parametrów sterujących funkcji tłumienia. Skuteczność działania przedstawionego algorytmu wyrównania zweryfikowano na dwóch przykładach numerycznych. Analizę otrzymanych wyników przeprowadzono w odniesieniu do metod wyrównania odpornego, wykorzystujących inne, znane funkcje tłumienia (np. funkcja Hampela).