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On the role of measurement repetitions in the light of the theory of reliability of observation systems

Assuming correlation only within the results of measurement repetitions for each quantity observed in a network, equivalence has been proved for two forms of parametric adjustment model, which differ in the approach to measurement repetitions and are called the one-stage and the two-stage model respectively. As a complement to the known criterion of imperceptibility of disturbances in observations, the criterion of imperceptibility of correlation between the components of the observation vector has been formulated, which applies to each of the modules of the two-stage model. Assuming the structure of the observation error, being slightly developed as compared to the standard structure, the cases of meeting of each of the above criteria in those modules have been presented. Then, the relationship which combines measures of internal reliability for both the adjustment models under question has been given.

INTRODUCTION

Publications, which partially or entirely concern issues of reliability of networks, or, to say it more generally — reliability of observation systems (see e.g. Caspary 1988, Prószyński, Kwaśniak 2002) assume final results of measurements of every observed quantity, as values, which are the entries to the model of adjustment. Those publications do not consider the previous stage of determining those values based on the results of measurement repetitions. To say this strictly, the usage of such a model for specifying the measures of the internal reliability of a network, instead of a model with original results of measurements, neglects the stage of the process of development of measurement results, which is very important for diagnostics of gross errors. Therefore it can be expected, that measures of reliability determined for a model, which does not contain measurement repetitions, will not fully characterise reliability features of the complete model. We will make an attempt to specify values of measures for the complete model, as well as to estimate the level and the nature of influence of measurement repetitions upon those measures. Considerations will be performed with respect to the structure of an error of a single observation, which is slightly developed as compared to the structure, which has been assumed in existing methodology of reliability analyses.

1. *Equivalence of one- and two-stage models in the case of lack of disturbances in observations*

Assuming that only random errors in observations occur, we will consider two following forms of an adjustment model, which differ in the way of treatment of measurement repetitions for quantities observed in a network (see schematic approach, Fig. 1):

— one-stage model (I)

$$\mathbf{A}\boldsymbol{x} + \varepsilon = \boldsymbol{l}\,;\ \varepsilon \sim (\mathbf{0}, \mathbf{C}) \tag{1}$$

LS (least squares) estimation: $\hat{\boldsymbol{x}}$, $\mathbf{C}_{\hat{x}}$

— two-stage model (II)

modules IIa

$$\boldsymbol{1}_i y_i + \varepsilon_{[a],i} = \boldsymbol{l}_i \qquad \varepsilon_{[a],i} \sim (\mathbf{0}, \mathbf{C}_{[a],i}) \quad ; \qquad i = 1,...,n \tag{2}$$

LS estimation: $\hat{y}_i$, $\sigma_{\hat{y},i}$; $i = 1,...,n$

module IIb

$$\mathbf{A}_{[b]}\boldsymbol{x} + \varepsilon_{[b]} = \hat{\boldsymbol{y}}\ ; \qquad \varepsilon_{[b]} \sim (\mathbf{0}, \mathbf{C}_{[b]}) \tag{3}$$

LS estimation: $\hat{\boldsymbol{x}}$, $\mathbf{C}_{\hat{x}}$

where: i — indicator of a quantity observed in a network; r_i — number of repetitions of the measurements of the i-th observed quantity; $\boldsymbol{x}(u\times1)$; $\hat{\boldsymbol{y}}\ (n\times1)$; $\boldsymbol{l}_i\,(r_i\times1)$; $\Sigma r_i = w$; $\boldsymbol{l}^T = [\boldsymbol{l}_1^T,..., \boldsymbol{l}_i^T,..., \boldsymbol{l}_n^T]$; $\mathbf{A}(w\times u)$, $r(\mathbf{A}) = u$; $\mathbf{A}_{[b]}\,(n\times u)$, $r(\mathbf{A}_{[b]}) = u$; $\boldsymbol{1}_i\,(r_i\times1)$; $\mathbf{C}\,(w\times w)$ d.o.; $\mathbf{C}_{[a],i}\,(r_i\times r_i)$ d.o.; $\mathbf{C}_{[b]}\,(n\times n)$ d.o.;

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{1}_1\boldsymbol{a}_1 \\ \cdots \\ \boldsymbol{1}_n\boldsymbol{a}_n \end{bmatrix} \qquad \mathbf{A}_{[b]} = \begin{bmatrix} \boldsymbol{a}_1 \\ \cdots \\ \boldsymbol{a}_n \end{bmatrix} \qquad \mathbf{1}_i = \begin{bmatrix} 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix}$$

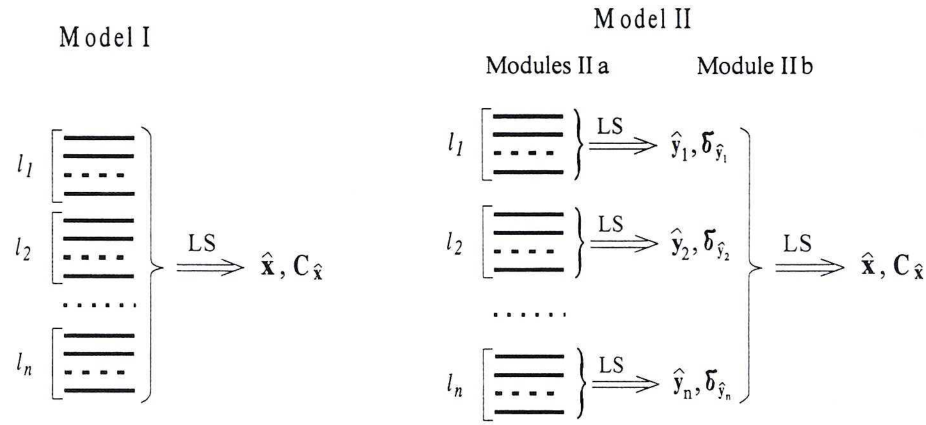

Fig.1. Adjustment models under consideration: one-stage model (I), two-stage model (II); repetitions of measurements of a given quantity are marked with square brackets

Assuming lack of correlation between the results of measurements of any two observed quantities in the network, i.e. l_i, l_j $(i \neq j;\ i,j = 1,...,n)$ we will obtain

$$\mathbf{C} = \text{diag}\ \{\mathbf{C}_{[a],i}\}\ ;\ \mathbf{C}_{[b]} = \text{diag}\ \{\sigma^2_{\hat{y},i}\} \qquad i = 1,...,n \tag{4}$$

Now we will find an estimator $\hat{x}$ for each model and its covariance matrix $\mathbf{C}_{\hat{x}}$.

Model (I)

$$\hat{x} = (\mathbf{A}^T\mathbf{C}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{C}^{-1}\boldsymbol{l}$$

where, according to (4) $\mathbf{C} = \text{diag}\ \{\mathbf{C}_{[a],i}\}$ $i = 1,...,n$
and, after transformations

$$\hat{x} = \left(\sum_1^n \boldsymbol{a}_i^T\ \boldsymbol{1}_i^T\ C_{a,i}^{-1}\ \boldsymbol{1}_i\ \boldsymbol{a}_i\right)^{-1} \left(\sum_1^n \boldsymbol{a}_i^T\ \boldsymbol{1}_i^T\ C_{[a],\ i}^{-1}\ \boldsymbol{l}_i\right)$$

$$\mathbf{C}_{\hat{x}} = (\mathbf{A}^T\mathbf{C}^{-1}\mathbf{A})^{-1} = \left(\sum_1^n \boldsymbol{a}_i^T\ \boldsymbol{1}_i^T\ \mathbf{C}_{[a],\ i}^{-1}\ \boldsymbol{l}_i\ \boldsymbol{a}_i\right)^{-1}$$

Model (II)

– modules IIa;

$$\hat{y}_i = (\boldsymbol{1}_i^T\,\mathbf{C}_{[a],i}^{-1}\,\boldsymbol{1}_i)^{-1}\ \boldsymbol{1}_i^T\ \mathbf{C}_{[a],i}^{-1}\ \boldsymbol{l}_i \qquad i = 1,...,n$$

$$\sigma^2_{\hat{y},i} = (\boldsymbol{1}_i^T\ \mathbf{C}_{[a],i}^{-1}\ \boldsymbol{1}_i)^{-1}$$

– module IIb

$$\hat{x} = (\mathbf{A}_{[b]}^T \mathbf{C}_{[b]}^{-1} \mathbf{A}_{[b]})^{-1} \mathbf{A}_{[b]}^T \mathbf{C}_{[b]}^{-1} \hat{y}$$

where, according to (1), $\mathbf{C}_{[b]} = \text{diag}\,\{\boldsymbol{1}_i^T \mathbf{C}_{[a],i}^{-1} \boldsymbol{1}_i\}$ $\quad i = 1,\ldots,n,$
and, after transformations

$$\hat{x} = \left(\sum_1^n \boldsymbol{a}_i^T \boldsymbol{1}_i^T \boldsymbol{C}_{a,i}^{-1} \boldsymbol{1}_i \mathbf{a}_i\right)^{-1} \left(\sum_1^n \boldsymbol{a}_i^T \boldsymbol{1}_i^T \boldsymbol{C}_{[a],\,i}^{-1} \boldsymbol{l}_i\right)$$

$$\mathbf{C}_{\hat{x}} = (\mathbf{A}_{[b]}^T \mathbf{C}_{[b]}^{-1} \mathbf{A}_{[b]})^{-1} = \left(\sum_1^n \boldsymbol{a}_i^T \boldsymbol{1}_i^T \mathbf{C}_{[a],\,i}^{-1} \boldsymbol{1}_i \boldsymbol{a}_i\right)^{-1}$$

Equivalence of models I and II (the latter is a two-stage decomposition of the model I), has been proved, allowing for correlation only within measurement repetitions for each of observed quantities. Since, if this assumption was not fulfilled, the equivalence would not have been achieved and we could also observe that such an assumption is the necessary condition of decomposability of the model I into modules IIa and the module IIb.

Equivalence of both models, with respect to $\hat{x}$ and $\mathbf{C}_{\hat{x}}$ should also imply consistency with respect to estimation of random errors $\boldsymbol{\varepsilon}$. We will prove it for the sub-vector $\hat{\boldsymbol{\varepsilon}}_i$ $(i = 1,\ldots,n)$:

$$\begin{aligned}\hat{\boldsymbol{\varepsilon}}_i &= \boldsymbol{1}_i \boldsymbol{a}_i \hat{x} - \boldsymbol{l}_i = \boldsymbol{1}_i \boldsymbol{a}_i \hat{x} - \boldsymbol{l}_i + \boldsymbol{1}_i \hat{y}_i - \boldsymbol{1}_i \hat{y}_i = \\ &= \boldsymbol{1}_i \hat{y}_i - \boldsymbol{l}_i + \boldsymbol{1}_i(\boldsymbol{a}_i \hat{x} - \hat{y}_i) = \hat{\boldsymbol{\varepsilon}}_{[a],\,i} + \boldsymbol{1}_i \hat{\boldsymbol{\varepsilon}}_{[b],\,i}\end{aligned}$$

2. *Criteria of imperceptibility of the error vector and imperceptibility of correlation between its components in the module IIa*

The first criterion is immediately obtained basing on the well known definition of the space of imperceptible disturbances; we will pay more attention to the second criterion, as a new one.

a) the criterion of imperceptibility of the error vector

Basing on (Prószyński 2000) we can immediately write

$$\varepsilon_i \in \mathcal{M}(\boldsymbol{1}_i) \Rightarrow \varepsilon_i = k\boldsymbol{1}_i \qquad k \neq 0 \tag{5}$$

where $\mathcal{M}(\boldsymbol{1}_i)$ means the space generated by the column vector of the coefficient matrix of the model.

This criterion is met by every error vector of identical components.

b) the criterion of imperceptibility of correlation between the components of the error vector

Let us consider the following, standardised form of the module IIa $_i$, written — for the reason of simplicity — without the lower indices i and $[a]$,

$$\boldsymbol{1}_{(q)} y + \boldsymbol{\varepsilon}_{(q)} = \boldsymbol{l}_{(q)}\,; \qquad \boldsymbol{\varepsilon}_{(q)} - (\boldsymbol{0}, \mathbf{C}_{(q)}) \tag{6}$$

where:

$$\boldsymbol{1}_{(q)} = \Sigma^{-1}\boldsymbol{1},\ \boldsymbol{\varepsilon}_{(q)} = \Sigma^{-1}\boldsymbol{\varepsilon},\ \boldsymbol{l}_{(q)} = \Sigma^{-1}\boldsymbol{l},$$

$$\mathbf{C}_{(q)} = \Sigma^{-1}\mathbf{C}\Sigma^{-1},\ \Sigma = (\mathrm{diag}\ \mathbf{C})^{1/2}$$

The matrix $\mathbf{C}_{(q)}$ is the correlation matrix for the components of the error vector $\boldsymbol{\varepsilon}$. Let us examine, for which matrices $\mathbf{C}_{(q)} \neq \mathbf{I}$ the following equations hold

$$\mathbf{R}_{\mathrm{C}(q)} = \mathbf{R}_{\mathrm{I}} \tag{7}$$

and

$$\mathbf{T}_{\mathrm{C}(q)} = \mathbf{T}_{\mathrm{I}}, \tag{8}$$

where: $\mathbf{R}_{\mathrm{C}(q)}$ — the matrix which transforms the vector of standardised observations $\boldsymbol{l}_{(q)}$ onto the vector of standardised corrections $\boldsymbol{v}_{(q)} = -\boldsymbol{\varepsilon}_{(q)}$ (which is the matrix of reliability in the standardised system); $\mathbf{T}_{\mathrm{C}(q)}$ — the matrix which transforms the vector of standardised observations $\boldsymbol{l}_{(q)}$ onto the unknowns $\hat{y}$.

The condition (7) may be written as:

$$\boldsymbol{1}_{(q)} (\boldsymbol{1}^T_{(q)} \mathbf{C}^{-1}_{(q)} \boldsymbol{1}_{(q)})^{-1} \boldsymbol{1}^T_{(q)} \mathbf{C}^{-1}_{(q)} = \boldsymbol{1}_{(q)} (\boldsymbol{1}^T_{(q)} \boldsymbol{1}_{(q)})^{-1} \boldsymbol{1}^T_{(q)}$$

After postmultiplication of both sides of the above equation by $\boldsymbol{1}_{(q)}$, we will immediately obtain the consistence of both sides $\boldsymbol{1}_{(q)} = \boldsymbol{1}_{(q)}$.

As a result of premultiplication by $\boldsymbol{1}^T_{(q)} \mathbf{C}^{-1}_{(q)}$ we will obtain

$$\boldsymbol{1}^T_{(q)} \mathbf{C}^{-1}_{(q)} = \boldsymbol{1}^T_{(q)} \mathbf{C}^{-1}_{(q)} \cdot \boldsymbol{1}_{(q)} (\boldsymbol{1}^T_{(q)} \boldsymbol{1}_{(q)})^{-1} \boldsymbol{1}^T_{(q)} \tag{9}$$

Both sides are consistent for $\mathbf{C}_{(q)}$ such, that:

$$\boldsymbol{1}^T_{(q)} \mathbf{C}^{-1}_{(q)} \in \mathcal{M}\ (\boldsymbol{1}^T_{(q)}),$$

or, equivalently

$$\boldsymbol{1}^T_{(q)} \mathbf{C}_{(q)} \in \mathcal{M}\ (\boldsymbol{1}^T_{(q)}) \tag{10}$$

The condition (8) may be written as

$$(\boldsymbol{1}^T_{(q)} \mathbf{C}^{-1}_{(q)} \boldsymbol{1}_{(q)})^{-1} \boldsymbol{1}^T_{(q)} \mathbf{C}^{-1}_{(q)} = (\boldsymbol{1}^T_{(q)} \boldsymbol{1}_{(q)})^{-1} \boldsymbol{1}^T_{(q)}$$

As a result of premultiplication by $\boldsymbol{1}_{(q)}^T\, \mathbf{C}_{(q)}^{-1}\, \boldsymbol{1}_{(q)}$ we will obtain the identical equation as (9), and thus finally, the condition identical with (10). We will call it the condition of imperceptibility of correlation between the measurement repetitions for l_i in the module IIa $_i$.

To get a more detailed form of that condition, we will present it as

$$\boldsymbol{1}_{(q)}^T\, \mathbf{C}_{(q)} = k\,\boldsymbol{1}_{(q)}^T \qquad k \neq 0$$

what finally leads to

$$\sum_{j=1}^{r} \{\mathbf{C}_{(q)}\}_{j1} = \sum_{j=1}^{r} \{\mathbf{C}_{(q)}\}_{j2} = \ldots = \sum_{j=1}^{r} \{\mathbf{C}_{(q)}\}_{jr} \tag{11}$$

i.e. to the requirement of identical sums of elements in columns (and due to symmetry — also in rows) of the matrix $\mathbf{C}_{(q)}$.

Among matrices $\mathbf{C}_{(q)}$ which meet this criterion, there are the matrices of identical values of all non-diagonal elements. We will call such a case of correlation as the uniform correlation.

In the case when repetitions are the observations of equal accuracy, it may be easily proved that the requirement (11) applies directly to the initial matrix $\mathbf{C}$, i.e.

$$\sum_{j=1}^{r} \{\mathbf{C}\}_{j1} = \sum_{j=1}^{r} \{\mathbf{C}\}_{j2} = \ldots = \sum_{j=1}^{r} \{\mathbf{C}\}_{jr} \tag{12}$$

It is left for the reader to check, that the correlation between measurement repetitions, as given in the examples of correlation matrices presented below

$$\mathbf{C}_{(q)} = \begin{bmatrix} 1 & 0.3 & 0.2 & 0.1 \\ 0.3 & 1 & 0.1 & 0.2 \\ 0.2 & 0.1 & 1 & 0.3 \\ 0.1 & 0.2 & 0.3 & 1 \end{bmatrix} \qquad \mathbf{C}_{(q)} = \begin{bmatrix} 1 & 0.3 & 0.3 & 0.3 \\ 0.3 & 1 & 0.3 & 0.3 \\ 0.3 & 0.3 & 1 & 0.3 \\ 0.3 & 0.3 & 0.3 & 1 \end{bmatrix}$$

is imperceptible in the model IIa, in the sense of equations (7) and (8).

3. *A model of observation errors in the case of disturbances*

The following structure of the vector of errors for the r_i — repetitions of measurement of the quantity l_i ($i = 1,\ldots,n$) will be assumed for the analysis of reliability of observation systems

$$\boldsymbol{e}_i = \boldsymbol{\varepsilon}_i + \boldsymbol{c}_i + \boldsymbol{g}_i \tag{13}$$

where: $\boldsymbol{\varepsilon}_i$ — vector of random errors, $E(\boldsymbol{\varepsilon}_i) = \mathbf{0}$, $\boldsymbol{c}_i$ — vector of the constant error, $\boldsymbol{c}_i = c_i \boldsymbol{1}_i$; c_i — error made in each r_i-th repetition of measurement, $\boldsymbol{g}_i$ — vector of gross errors (made in one or in several repetitions of measurement).

In order to ensure the decomposability of the model I, we will assume, according to (4), that $E(\boldsymbol{\varepsilon}_i \boldsymbol{\varepsilon}_j^T) = \mathbf{0}$; $i, j = 1,...,n, j \neq i$. Considering the deterministic nature of vectors $\boldsymbol{c}_i$ and $\boldsymbol{g}_i$ we will obtain $\mathbf{C}_{e_i} = \mathbf{C}_{\varepsilon_i}$.

For the needs of further considerations let us assume that the vector ε_i has the following structure (the lower indices i will be neglected in this notation)

$$\varepsilon = \mathbf{K}\boldsymbol{\delta} \tag{14}$$

where: $\boldsymbol{\delta}$ $(w \times 1)$ — vector of the elementary, mutually non-correlated random errors; $E(\delta_j) = 0$, $\mathrm{Var}(\delta_j) = \sigma_j^2$, $j = 1,...,w$, $w \geq r$; $\mathbf{K}$ $(r \times w)$ — matrix of coefficients, rank $(\mathbf{K}) = r$. Now we will write

$$\mathbf{C}_e = \mathbf{C}_\varepsilon = \mathbf{K}\mathbf{C}_\delta \mathbf{K}^T \tag{15}$$

4. *Imperceptibility of disturbances in measurement repetitions and imperceptibility of correlation between measurement repetitions in the module IIa*

Let us check, which of the vectors being the components of the error vector $\boldsymbol{e}$ (see formula (13)) are imperceptible in the module IIa.

We will immediately notice, that such a vector is the vector of constant error $\boldsymbol{c}$, since $\boldsymbol{c} = c\boldsymbol{1}_{(r)}$.

Now we will show, that it is also possible, at least from theoretical point of view, that there may occur the imperceptible component of the vector $\boldsymbol{e}$ of the random nature. Since if the following form of the structure of the vector ε is assumed

$$\varepsilon = \boldsymbol{\delta}_{(r)} + \delta_{r+1} \boldsymbol{1}_{(r)}$$

where δ_{r+1} is a random error, which burdens every of the r repetitions of measurement, what corresponds to the matrix $\mathbf{K}$ in the formula (14)

$$\mathbf{K} = [\mathbf{I}_{(r)} \quad \boldsymbol{1}_{(r)}], \tag{16}$$

we will obtain the random vector $\delta_{r+1} \boldsymbol{1}_{(r)}$, which is imperceptible in the module IIa.

It turns out from the specifics of repeating the measurement of the quantity l_i, that most probably this creates a system of equally accurate observations, with the same values of correlation coefficient for each pair of observations. It is the case of uniform correlation, where the covariance matrix of the observation vector has the form:

$$\mathbf{C}_e = \begin{bmatrix} a & b & \cdot & b \\ b & a & \cdot & b \\ \cdot & \cdot & \cdot & b \\ b & b & b & a \end{bmatrix}$$

where: $a = V(\varepsilon_j)$, $b = \mathrm{cov}(\varepsilon_j, \varepsilon_k)$, $j, k = 1, 2,...,r, j \neq k$.

As it turns out from the discussion in section 3 such correlation is imperceptible in the module IIa and, therefore, the measures of internal reliability can be calculated basing on the formulas for uncorrelated observations.

Analysis of the case $\mathbf{K} = [\mathbf{I}_{(r)} \quad \mathbf{\mathit{1}}_{(r)}]$ under the assumption that $\sigma_1 = \sigma_2 = ... = \sigma_r = \sigma$, leads to a similar conclusion. Since, using the formula (15) we obtain

$$\mathbf{C}_e = \mathbf{C}_\varepsilon = \sigma^2 \mathbf{I}_{(r)} + \sigma^2_{r+1} \mathbf{1}_{(r \times r)},$$

what means uniform correlation, i.e. correlation which is imperceptible in the module IIa.

As might be expected, the correlation of components of the vector $\boldsymbol{\varepsilon}$, generated by the random vector $\delta_{r+1} \mathbf{\mathit{1}}_{(r)}$ being imperceptible in the module IIa, is also imperceptible in this module.

5. *Relations between global measures of reliability for the one- and two-stage models*

In order to simplify relationships searched for we will assume that:

- measurement repetitions are considered as non-correlated observations (see considerations concerning imperceptibility of correlation in section 4);
- numbers of repetitions in each module are the same ($r_1 = r_2 = ... = r_n = r$);
- the observation system (e.g. the network) is a uniform structure with respect to reliability (the average global measure ≡ the local measure).

The expression which specifies the average global measure for a one-stage model will have the form:

$$f = 1 - \frac{u}{r \cdot n} \tag{17}$$

We may transform it to the form:

$$f = 1 - \frac{1}{r} - \frac{u}{r \cdot n} + \frac{1}{r} = 1 - \frac{1}{r} + \frac{1}{r}(1 - \frac{u}{n})$$

where: $1 - \frac{1}{r} = f_{[a]}$ is the reliability measure in each of IIa modules of the two-stage model,

$1 - \frac{u}{n} = f_{[b]}$ is the reliability measure in the module IIb of the two-stage model,

thus

$$f = f_{[a]} + \frac{1}{r} f_{[b]},$$

or, showing components of a single observation error, which correspond to those measures (i.e. those components the level of control of which in the model is determined by the given measure)

$$f(g) = f_{[a]}(g) + \frac{1}{r} f_{[b]}(c) \tag{18}$$

and, more precisely

$$f(g) = f_{[a]}(g) + \frac{1}{r} f_{[b]}(c + \delta_{r+1}) \tag{19}$$

Now, let us have a more detailed look at those simple relationships, using Fig. 2. We may state, that – with the one-stage model (i.e. the model I) – in the case of lack of the constant error c (or $c + \delta_{r+1}$) we will obtain the satisfactory level of controllability of a single observation l_i for the values of r starting from 2 ($f = 0.63$), even for a network of low level of the internal reliability ($f_{[b]} = 0.25$). This is the effect of measurement repetitions, which increase the total number of observations, which control the given observation l_i. However, considering the possibility of occurrence of the error c, attention should be paid to the appropriate level of controllability of observations with respect to error g as well as the error c. As a result, the two-stage model with the appropriate number of measurement repetitions ($r = 2$ or, more preferably, $r = 3$) is definitely preferred.

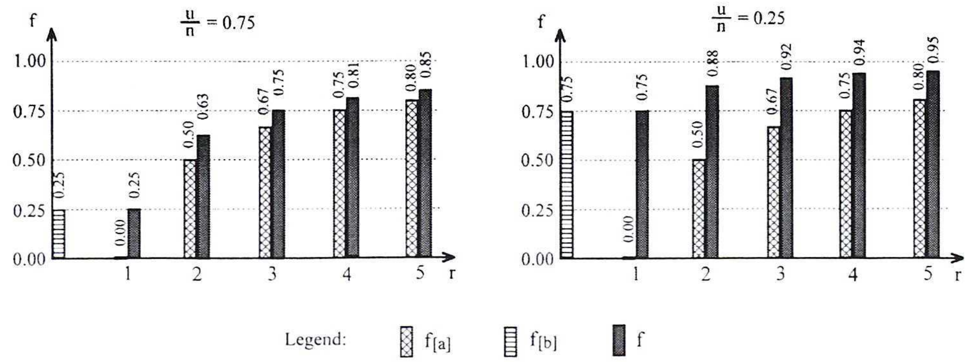

Fig. 2. Relationships between internal reliability measures for the one-stage (f) and two- stage ($f_{[a]}, f_{[b]}$) models

Using such a model, we have lower detectability of the error g in every module IIa, but — on the other hand — such modules are not sensitive to the constant error c, which may occur at the same time. The error c is perceptible in the module IIb and with sufficiently high level of the measure $f_{[b]}(c)$ it may be the quantity detectable in the model. Due to the fact, that the components of the equation (18) (or (19)) relate to errors of different nature and consequently, this equation may be used for orientation purposes only, it would be recommended to represent the internal reliability level for the two-stage model in the form $f_{[II]} = (f_{[a]\,\min}; f_{[b]\,\min})$, e.g. $f_{[II]} = (0.67; 0.70)$. The criterion $f_{[\cdot]\,\min} > 0.5$ should be obligatory for each component.

CONCLUSIONS

Considering the structure of an observation error, assumed in this paper, as characteristic for practical purposes, we will state, that – from the point of view of the reliability theory – the two-stage model is more advantageous than the one-stage model. As a result of transfer of the constant error c to the module IIb, due to its imperceptibility in the module IIa, the possibility of superposition of errors c and g, what might occur in the one-stage model, is considerably reduced. It is an additional argument which justifies the solution, which has been practically used for many years, but has been mainly the result of consideration of economy of calculations.

The approach to reliability analysis, presented in this paper, forms the basis for some general guidelines, which widen the existing methodology of reliability analysis for observation systems. The guidelines are as follows:

– if possible, the detailed structure of an observation error should be taken into consideration (which includes the regular random part and gross errors); the level of details of the analysis is obviously limited by the possibility to determine the nature and specifics of influence of the factors, which may effect the accuracy of measurements, as well as by the intention not to complicate the model form, both in the functional as well as the stochastic layer;

– one should examine the usefulness and possibility of decomposing the process of development of measurement results, with a view to specific features of an observation error. In the case of such decomposition an appropriate level of internal reliability should be ensured for each module. The occurrence of correlation between the repetitions of measurement of various quantities excludes the possibility to use the two-stage model, which is an advantageous model in respect of the efficiency of outlier detection.

For many practical situations, the detailed analyses concerning the structure of the observation error may be considered the subject of an academic discussion, since it is often difficult to identify the error structure and to estimate its stochastic parameters. Imperceptibility of correlation between measurement repetitions, which occurs in the case of highly probable uniform correlation (due to specific nature of repetitions), is of great help. In such a case there is no need to generate the covariance matrix for the observation vector. At the same time, it is a strong justification for practical use of the arithmetic mean of results of repetitions, although they are actually intercorrelated.

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O roli powtórzeń obserwacji w świetle teorii niezawodności układów obserwacyjnych

S t r e s z c z e n i e

W publikacjach poświęconych problematyce niezawodności układów obserwacyjnych, jako wielkości zasilające model wyrównawczy przyjmuje się ostateczne wyniki pomiaru każdej z wielkości obserwowanych, nie zajmując się uprzednim etapem ustalania tych wartości z reguły poprzez uśrednianie wyników powtórzeń obserwacji. Patrząc z punktu widzenia teorii niezawodności można oczekiwać, że miary niezawodności wyznaczone dla modelu nie uwzględniającego faktu powtarzania obserwacji nie będą w pełni charakteryzowały własności niezawodnościowych modelu pełnego. W niniejszym opracowaniu podjęto próbę ustalenia wartości miar dla modelu pełnego, jak też oszacowania wielkości i charakteru wpływu jaki na ich kształtowanie mają powtórzenia obserwacji.

Zakładając skorelowanie jedynie pomiędzy wynikami powtórzeń pomiaru każdej z wielkości obserwowanych w sieci, wykazano równoważność dwu, różniących się sposobem traktowania powtórzeń pomiaru, postaci pełnego modelu wyrównawczego, nazwanych modelem jednoetapowym i modelem dwuetapowym. W uzupełnieniu do znanego kryterium niedostrzegalności zaburzeń w obserwacjach sformułowano kryterium niedostrzegalności skorelowania składowych wektora obserwacji, stosujące się do każdego z modułów uśredniania wyników powtórzeń pomiaru w modelu dwuetapowym.

Kryterium to sprowadza się ostatecznie do postaci

$$\sum_{j=1}^{r}\{\mathbf{C}_{(q)}\}_{j1} = \sum_{j=1}^{r}\{\mathbf{C}_{(q)}\}_{j2} = \ldots = \sum_{j=1}^{r}\{\mathbf{C}_{(q)}\}_{jr}$$

oznaczającej wymóg identycznych sum wyrazów w kolumnach (a wobec symetrii — także i wierszach) macierzy korelacyjnej $\mathbf{C}_{(q)}$.

Przyjmując strukturę wektora błędów r_i – powtórzeń pomiaru wielkości l_i (i-ty moduł)

$$\boldsymbol{e}_i = \boldsymbol{\varepsilon}_i + \boldsymbol{c}_i + \boldsymbol{g}_i$$

gdzie: $\boldsymbol{\varepsilon}_i$ – wektor błędów przypadkowych, $E(\boldsymbol{\varepsilon}_i) = \boldsymbol{0}$, $\boldsymbol{c}_i$ – wektor błędu stałego, $\boldsymbol{c}_i = c_i \boldsymbol{1}_i$; c_i – błąd popełniony w każdym z r_i powtórzeń pomiaru; $\boldsymbol{g}_i$ – wektor błędów grubych (popełnionych w jednym bądź w kilku powtórzeniach pomiaru).

pokazano przypadki spełnienia każdego z ww. kryteriów.

Przy upraszczających założeniach wyprowadzono zależność

$$f(g) = f_{[a]}(g) + \frac{1}{r} f_{[b]}(c)$$

wiążącą wskaźniki niezawodności wewnętrznej: $f_{[a]}(g)$, $f_{[b]}(c)$ – dla modelu dwuetapowego oraz $f(g)$ – dla modelu jednoetapowego. Indeks dolny $[a]$ oznacza moduł uśredniania r wyników powtórzeń pomiaru i-tej wielkości obserwowanej, zaś $[b]$ – moduł wyrównawczy zasilany wynikami uzyskanymi ze wszystkich modułów $[a]$. W nawiasach zwykłych uwidoczniono składniki błędu pojedynczej obserwacji, stopień kontrolowalności których określa dana miara.

W zakończeniu podano wskazania natury ogólnej rozszerzające dotychczasową metodykę analiz niezawodności układów obserwacyjnych.

Витолдь Прушыньски

О роли повторений наблюдений в светле теории надёжности наблюдаемых систем

Резюме

В публикациях занимающихся вопросами надёжности наблюдаемых систем, как величины подкрепляющие уравнительную модель принимается конечные результаты измерения каждой с наблюдаемых величин, не занимаясь предыдпущим этапом опеределения этих величин, как правило путём усреднения результатов повторений наблюдений. С точки зрения теории надёжности можно ожидать, что меры надёжности определённые для модели, которая, не учитивает факта повторений наблюдений, не будут в нолном характеризовать признаков надёжности полной модели. В работе принята попытка определения мер для полной модели, а также оценки величины и характера влияния повторений наблюдений на их формирование.

Принимая, что корреляция происходит исключительно между результатами повторений измерения каждой из величин наблюдаемых в сети, указана эквивалентность двух, отличающихся способом подхода повторений измерений, видов полной уравнительной модели, называемых одноэтапной и двухэтапной меделей. В дополнении к известному критерии незаметности возмущий в наблюдениях был сформулирован критерий незаметности корреляции компонент вектора наблюдений подходящий для каждой из модули усреднения результатов повторения узмерений в двухэтапоной модели.

Этот критерий сводится в конечном счёте к формуле

$$\sum_{j=1}^{r}\{\mathbf{C}_{(q)}\}_{j1} = \sum_{j=1}^{r}\{\mathbf{C}_{(q)}\}_{j2} = \ldots = \sum_{j=1}^{r}\{\mathbf{C}_{(q)}\}_{jr}$$

обозначающей требование идентичных сум членов в столбцах (а в присутсвии симметрии, тоже и в строках) корреляционной матрицы $\mathbf{C}_{(q)}$.

Принимая структуру вектора ошибок r_i — повторений измерения величин l_i (i-та модул)

$$\boldsymbol{e}_i = \boldsymbol{\varepsilon}_i + \boldsymbol{c}_i + \boldsymbol{g}_i$$

где: $\boldsymbol{\varepsilon}_i$ — вектор случайных ошибок, $E(\boldsymbol{\varepsilon}_i) = \mathbf{0}$; $\boldsymbol{c}_i$ — вектор постоянной ошибки, $\boldsymbol{c}_i = c_i\boldsymbol{1}_i$; c_i — ошибка совершена в каждом из r_i повторений измерения; $\boldsymbol{g}_i$ — вектор промахов (совершеных в одним или в нескольких повторениях измерения);
указано случай удовлетворения каждого из вышеупомянутых критерий. Принимая упрощающие предпосылки была выведена зависимость

$$f(g) = f_{[a]}(g) + \frac{1}{r} f_{[b]}(c),$$

которая связывает указатели внутренней надёжности: $f_{[a]}(g)$, $f_{[b]}(c)$ — для двухэтапной модели и $f(g)$ — для одноэтапной модели. Нижний индекс $[a]$ обозначает модуль усреднения результатов повторений измерения i-той наблюдаемой величины а $[b]$ — уравнительный модуль, подкрепляемый результатами полученными со всех модулей $[a]$. В обычных скобках представлены составляющие ошибки отдельного наблюдения, степень возможности контролирования, которые определяет данная мера.

В заключении представлены рекомендации общего вида, распространяющие применяемую до сих пор методику анализа надёжности наблюдательских систем.