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Variance analysis and properties of NW-E method

This paper presents a general concept of an adjustment method Edgeworth series applied. The author proposes to replace distributions with approximating functions represented by Edgeworth series, which would be a universal, probabilistic model of the adjustment problem. Special attention is paid to variance analysis. A covariance matrix C_x of the vector \hat{X} , its estimate \hat{C}_x , and an estimate of the variance coefficient σ_0 are derived. Also some further properties of the proposed method are presented.

INTRODUCTION

Development of methods that apply probabilistic models of measurement errors is one of the subjects of adjustment calculus. The adjustment problem is usually defined as a mathematical model of a network plus an assumed criterion of adjustment. The criterion stems from the applied method of estimation. The mathematical model, however, should describe geometric properties of a network itself but also should contain necessary information about functional and probabilistic properties of measurement errors.

Some papers [1, 2] propose to replace distributions with approximating functions represented by Edgeworth series. This series not only approximates distributions but also is a probabilistic model of the adjustment problem. This work includes new properties of NW-E method (e. g. variance analysis of adjustment results) but also a short, theoretical reminder of the methods principles.

1. *Theoretical assumptions*

The origin of Edgeworth series is a rather general assumption that a density function f^a of every distribution can be presented in the following form

$$f^\alpha(\varepsilon) = f^{RN}(\varepsilon) + r(\varepsilon),$$

where f^{RN} is a density function of normal distribution and $r(\varepsilon)$ is a residue (ε is a standardised random variable).

For our purposes Edgeworth series can be written as follows

$$f^E(\varepsilon) = f^{RN}(\varepsilon) - \frac{1}{3!}\gamma_1 \frac{d^3 f^{RN}(\varepsilon)}{d\varepsilon^3} + \frac{1}{4!}\gamma_2 \frac{d^4 f^{RN}(\varepsilon)}{d\varepsilon^4} + \frac{10}{6!}\gamma_1^2 \frac{d^6 f^{RN}(\varepsilon)}{d\varepsilon^6} + \\ + \frac{10}{5!}\gamma_1 \frac{d^5 f^{RN}(\varepsilon)}{d\varepsilon^5} - \frac{35}{7!}\gamma_1\gamma_2 \frac{d^7 f^{RN}(\varepsilon)}{d\varepsilon^7} - \frac{280}{9!}\gamma_1^3 \frac{d^9 f^{RN}(\varepsilon)}{d\varepsilon^9}.$$

where $f^E(\varepsilon)$ is a density function of Edgeworth series, and γ_1 and γ_2 are respectively asymmetry and excess coefficients.

The general adjustment problem, which contains non-natural distributions, can be presented in the following way [3]:

– functional model of network

$$\mathbf{x} = \mathbf{A}\mathbf{X} + \varepsilon$$

– probabilistic model

$$\mathbf{x} \sim \mathbf{R}^\alpha[\Theta^\alpha, \mathbf{X}]$$

– adjustment criterion which stems from the maximum likelihood method

$$\max_{\mathbf{X} \in X_0} L^{NW}(\mathbf{X}) = \max_{\mathbf{X} \in X_0} \prod_{i=1}^n f_x^\alpha(\mathbf{X}, x_i)$$

where $\alpha \in T_\alpha$, and $\mathbf{x} \in M_{(n,1)}$ – vector of measurement results or free terms, $\mathbf{A} \in M_{(n,u)}$ – matrix of coefficients, $\mathbf{X} \in M_{(u,1)}$ – vector of unknown parameters of the functional model, $\varepsilon \in M_{(n,1)}$ – vector of measurement errors, $M_{(a,b)}$ – set of matrices of dimension (a,b) , X_0 – set of possible solutions, $x_i = [x]_i$ – i -th measurement result, $\mathbf{R}^\alpha[\Theta^\alpha, \mathbf{X}] \in R$ is a distribution with parameters $\Theta^\alpha, \mathbf{X}$, belonging to a set of potential, probabilistic models of measurement errors $R = \{\mathbf{R}^\alpha : \alpha \in T_\alpha\}$ represented by the following density function

$$f_x^\alpha(\mathbf{x}; \mathbf{X}) = \prod_{i=1}^n f_{x_i}^\alpha(x_i; \mathbf{X}), \quad \alpha \in T_\alpha$$

(one assumes that variables x_i are independent for every $i = 1, \dots, n$).

For practical purposes the original target function $L^{NW}(\mathbf{x}; \mathbf{X})$ is replaced with a logarithmic function

$$l^\alpha(\mathbf{x}; \mathbf{X}) = \ln L^{NW}(\mathbf{x}; \mathbf{X}) = \sum_{i=1}^n \ln f_{x_i}^\alpha(x_i; \mathbf{X}).$$

Consequently the adjustment criterion turns into the following form

$$\max_{\mathbf{X} \in X_0} l^{NW}(\mathbf{x}; \mathbf{X}) = \max_{\mathbf{X} \in X_0} \left\{ \sum_{i=1}^n \ln f_{x_i}^{\alpha}(x_i; \mathbf{X}) \right\}, \quad \alpha \in T_{\alpha}.$$

To find such $\hat{\mathbf{X}} \in X_0$ that $\max l^{\alpha}(\mathbf{X}) = l^{\alpha}(\hat{\mathbf{X}})$, one of optimisation methods can be applied e. g. Newton method where an iterative formula is as follows

$$\begin{aligned} \mathbf{X}^{j+1} &= \mathbf{X}^j + \tau [\mathbf{Q}(\mathbf{X}^j)]^{-1} \mathbf{g}(\mathbf{X}^j) \\ \boldsymbol{\varepsilon}^{(j+1)} &= \mathbf{x} - \mathbf{A}\mathbf{X}^{(j+1)} \end{aligned}$$

where: τ is a coefficient of convergence improvement, $\mathbf{g}(\mathbf{X})$ is the gradient of the target function:

$$\mathbf{g}(\mathbf{X}) = \frac{\partial l(\mathbf{X})}{\partial \mathbf{X}}$$

and

$$\mathbf{Q}(\mathbf{X}) = \frac{\partial^2 l(\mathbf{X})}{\partial \mathbf{X} \partial \mathbf{X}^T} \text{ (Hessian of this function.)}$$

Replacing the density function with Edgeworth series i. e. assuming that

$$f^{\alpha}(\boldsymbol{\varepsilon}; \boldsymbol{\Theta}^{\alpha})_{\alpha \in T_{\alpha}} = f^E(\boldsymbol{\varepsilon}; \boldsymbol{\Theta}^E),$$

the earlier formulated adjustment problem turns into the following form:

– functional model of a geodetic network

$$\mathbf{x} = \mathbf{A}\mathbf{X} + \boldsymbol{\varepsilon}$$

– probabilistic model

$$\mathbf{x} \sim \mathbf{R}^E[\boldsymbol{\Theta}^E; \mathbf{X}]$$

– adjustment criterion following from application of the method of maximum likelihood

$$\max_{\mathbf{X} \in X_0} l^E(\mathbf{x}; \mathbf{X}) = \max_{\mathbf{X} \in X_0} \left\{ \sum_{i=1}^n \ln f_{x_i}^E(x_i; \mathbf{X}) \right\}.$$

The worked-out method, named *NW-E* [1], has a very important general property

$$NW - E \xrightarrow{\gamma_1=0, \gamma_2=0} NK$$

(*NK* the least squares -method).

It means that if there is no anomalies in distributions of measurement errors, the only method to be used is the least squares one.

2. Variance analysis

Presented in previous papers results of research into application of a new probabilistic model represented by Edgeworth series would not be complete if one neglected very important, from practical but also theoretical point of view, variance analysis. This paper is concern with a covariance matrix of the parameter vect;or $\mathbf{C}_{\hat{x}}$ also its estimate $\hat{\mathbf{C}}_{\hat{x}}$ and with an estimator of the variance coefficient $\hat{\sigma}_0$.

According to the properties of the maximum likelihood method the estimate of \mathbf{C}_x can be presented in the following form:

$$\mathbf{C}_{\hat{x}}^{\infty} = -E\{\mathbf{Q}(\hat{\mathbf{X}})\}^{-1} = (\mathbf{A}^T E\{\mathbf{D}(\hat{\mathbf{X}})\} \mathbf{A})^{-1} = (\mathbf{A}^T \mathbf{D}(\hat{\mathbf{X}}) \mathbf{A})^{-1} \quad (1)$$

where

$$\mathbf{Q}(\hat{\mathbf{X}}) = \frac{\partial^2}{\partial \mathbf{X} \partial \mathbf{X}^T} l^E(\hat{\mathbf{X}}) = -\mathbf{A}^T \mathbf{D}(\hat{\mathbf{X}}) \mathbf{A}.$$

$\mathbf{D}(\hat{\mathbf{X}})$ is a diagonal matrix of elements:

$$[\mathbf{D}(\hat{\mathbf{X}})]_{ii} = 1 - \frac{\sum_{i=1}^6 h_i'' \left(1 + \sum_{i=1}^6 h_i\right) - \left(\sum_{i=1}^6 h_i'\right)^2}{\left(1 + \sum_{i=1}^6 h_i\right)^2}$$

where

$$h_i' = \frac{dh_i}{d\varepsilon}, \quad h_i'' = \frac{d^2 h_i}{d\varepsilon^2}.$$

If the number of additional observations approaches infinity then $\mathbf{C}_{\hat{x}}^{\infty} \rightarrow \mathbf{C}_{\hat{x}}$. From the practical point of view it is just impossible thus Wiśniewski [3] proposed that the covariance matrix of measurement results \mathbf{C}_x could be the basis for computation of $\mathbf{C}_{\hat{x}}$ (with application of the covariance propagation law). The following property is used here:

If

$$l^E(\hat{\mathbf{X}}) = \max_{\mathbf{X} \in X_0} l^E(\mathbf{X})$$

then

$$\mathbf{g}(\hat{X}) = \mathbf{A}^T \mathbf{b}(\hat{X}) = \mathbf{0} \quad (2)$$

where

$$[\mathbf{b}(\hat{X})]_i = \varepsilon_i - \frac{\sum_{i=1}^6 h'_i}{\sum_{i=1}^6 h_i} \quad (3)$$

Let $\delta_{\hat{X}}$ be the error of the estimator \hat{X} , i. e.

$$\hat{X} = X + \delta_{\hat{X}} \quad (4)$$

Thus if $E\{\delta_{\hat{X}}\} = \mathbf{0}$ then $E\{\hat{X}\} = X$.

Consequently (on the basis of a covariance matrix definition)

$$\mathbf{C}_{\hat{X}} = E\{(\hat{X} - E\{\hat{X}\})(\hat{X} - E\{\hat{X}\})^T\} = E\{(\hat{X} - X)(\hat{X} - X)^T\} = E\{\delta_{\hat{X}} \delta_{\hat{X}}^T\} = \mathbf{C}_{\delta_{\hat{X}}} \quad (5)$$

The matrix $\mathbf{C}_{\delta_{\hat{X}}}$ is obtained with use of the law of covariance matrix propagation, but firstly a linear relationship between the vectors $\delta_{\hat{X}}$ and ε (with a covariance matrix \mathbf{C}_{ε}) should be drawn up. Since

$$\hat{\varepsilon} = \mathbf{x} - \mathbf{A}\hat{X} = \mathbf{x} - \mathbf{A}(X + \delta_{\hat{X}}) = \mathbf{x} - \mathbf{A}X - \mathbf{A}\delta_{\hat{X}} = -\mathbf{A}\delta_{\hat{X}} \quad (6)$$

then $\mathbf{b}(\delta_{\hat{X}}, \varepsilon)$ is a vector function of the following elements

$$[\mathbf{b}(\hat{X})]_i = \varepsilon_i - A_{(i,\cdot)} \delta_{\hat{X}} - \frac{\sum_{i=1}^6 h'_i(\delta_{\hat{X}}, \varepsilon)}{1 + \sum_{i=1}^6 h_i(\delta_{\hat{X}}, \varepsilon)} = [\mathbf{b}(\delta_{\hat{X}}, \varepsilon)]_i \quad (7)$$

Expanding $\mathbf{b}(\delta_{\hat{X}}, \varepsilon)$ into Taylor series in neighbourhood of $\delta_{\hat{X}}, \varepsilon$, ($E\{\delta_{\hat{X}}\} = \mathbf{0}$, $E\{\varepsilon\} = \mathbf{0}$) one obtains

$$\mathbf{b}(\delta_{\hat{X}}, \varepsilon) = \mathbf{D}(X)\varepsilon - \mathbf{D}(X)\mathbf{A}\delta_{\hat{X}} + \mathbf{b} \quad (8)$$

where $\mathbf{b} = E\{\mathbf{b}(\delta_{\hat{X}}, \varepsilon)\}$ — non-random residue of the expansion.

Placing the expansion (8) into the equation (2) (the necessary condition of an extremum) one can write

$$\mathbf{g}(\hat{\mathbf{X}}) = \mathbf{A}^T[\mathbf{D}(\mathbf{X})\boldsymbol{\varepsilon} - \mathbf{D}(\mathbf{X})\mathbf{A}\delta_{\hat{\mathbf{X}}} + \mathbf{b}] = -\mathbf{A}^T\mathbf{D}(\mathbf{X})\mathbf{A}\delta_{\hat{\mathbf{X}}} + \mathbf{A}^T\mathbf{D}(\mathbf{X})\boldsymbol{\varepsilon} + \mathbf{A}^T\mathbf{b} = \mathbf{0} \quad (9)$$

Solving the equation (9) with respect to $\delta_{\hat{\mathbf{X}}}$ it is obtained that

$$\delta_{\hat{\mathbf{X}}} = (\mathbf{A}^T\mathbf{D}(\mathbf{X})\mathbf{A})^{-1}\mathbf{A}^T\mathbf{D}(\mathbf{X})\boldsymbol{\varepsilon} + (\mathbf{A}^T\mathbf{D}(\mathbf{X})\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}.$$

or

$$\delta_{\hat{\mathbf{X}}} = \mathbf{S}\boldsymbol{\varepsilon} + \mathbf{s}_0 \quad (10)$$

where

$$\mathbf{S} = (\mathbf{A}^T\mathbf{D}(\mathbf{X})\mathbf{A})^{-1}\mathbf{A}^T\mathbf{D}(\mathbf{X})$$

and $\mathbf{s}_0 = (\mathbf{A}^T\mathbf{D}(\mathbf{X})\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$ is a non-random vector.

Applying the law of covariance matrix propagation to (10) one can obtain

$$\mathbf{C}_{\delta_{\hat{\mathbf{X}}}} = \mathbf{S}\mathbf{C}_{\boldsymbol{\varepsilon}}\mathbf{S}^T = \mathbf{C}_{\hat{\mathbf{X}}} \quad (11)$$

The matrix $\mathbf{C}_{\hat{\mathbf{X}}}$ (11) has a property that can be written down in the following theorem:

THEOREM 1

If $NW - E \xrightarrow{\gamma_1=0, \gamma_2=0} NK$, then $\mathbf{C}_{\hat{\mathbf{X}}} \rightarrow \mathbf{C}_{\hat{\mathbf{X}}}^{NK} = (\mathbf{A}^T\mathbf{C}_{\boldsymbol{\varepsilon}}^{-1}\mathbf{A})^{-1}$

Proof

The theorem is true if one can prove that:

$$\lim_{\gamma_1 \rightarrow 0, \gamma_2 \rightarrow 0} \mathbf{C}_{\hat{\mathbf{X}}} = \mathbf{C}_{\hat{\mathbf{X}}}^{NK}.$$

Since

$$\lim_{\gamma_1 \rightarrow 0, \gamma_2 \rightarrow 0} [\mathbf{D}(\hat{\mathbf{X}})]_{ii} = \lim_{\gamma_1 \rightarrow 0, \gamma_2 \rightarrow 0} \left(\frac{1}{\sigma_i^2} - \frac{\sum_{i=1}^6 h_i'' \left(1 + \sum_{i=1}^6 h_i \right) - \left(\sum_{i=1}^6 h_i' \right)^2}{\left(1 + \sum_{i=1}^6 h_i \right)^2} \right)$$

(if $\gamma_1 \rightarrow 0$ i $\gamma_2 \rightarrow 0$ to $h_i, h'_i, h''_i \rightarrow 0$)
so

$$\lim_{\gamma_1 \rightarrow 0, \gamma_2 \rightarrow 0} \mathbf{D}(X) = \text{diag} \left(\frac{1}{\sigma_1^2}, \dots, \frac{1}{\sigma_n^2} \right) = \mathbf{C}_\varepsilon^{-1}.$$

On this basis it can be written that

$$\lim_{\gamma_1 \rightarrow 0, \gamma_2 \rightarrow 0} \mathbf{S} = (\mathbf{A}^T \mathbf{C}_\varepsilon^{-1} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{C}_\varepsilon^{-1}.$$

and finally that

$$\lim_{\gamma_1 \rightarrow 0, \gamma_2 \rightarrow 0} \mathbf{C}_{\hat{X}} = \lim_{\gamma_1 \rightarrow 0, \gamma_2 \rightarrow 0} \mathbf{S} \mathbf{C}_\varepsilon \mathbf{S}^T = (\mathbf{A}^T \mathbf{C}_\varepsilon^{-1} \mathbf{A})^{-1} \mathbf{A}^T \mathbf{C}_\varepsilon^{-1} \mathbf{C}_\varepsilon \mathbf{C}_\varepsilon^{-1} \mathbf{A} (\mathbf{A}^T \mathbf{C}_\varepsilon^{-1} \mathbf{A})^{-1} = (\mathbf{A}^T \mathbf{C}_\varepsilon^{-1} \mathbf{A})^{-1}$$

Let us assume that \mathbf{C}_ε matrix can be written down using the following model

$$\mathbf{C}_\varepsilon = \sigma_0^2 \tilde{\mathbf{C}}_\varepsilon$$

where $\tilde{\mathbf{C}}_\varepsilon$ is a known cofactor matrix, and σ_0^2 is an unknown variance coefficient.
Thus

$$\mathbf{C}_{\hat{X}} = \mathbf{S} \mathbf{C}_\varepsilon \mathbf{S}^T = \sigma_0^2 \mathbf{S} \tilde{\mathbf{C}}_\varepsilon \mathbf{S}^T = \sigma_0^2 \tilde{\mathbf{C}}_{\hat{X}}$$

where $\tilde{\mathbf{C}}_{\hat{X}} = \mathbf{S} \tilde{\mathbf{C}}_\varepsilon \mathbf{S}^T$ ($\tilde{\mathbf{C}}_\varepsilon$ – cofactor matrix of the vector \hat{X}).

The following matrix $\hat{\mathbf{C}}_{\hat{X}}$ is the estimate of $\mathbf{C}_{\hat{X}}$

$$\hat{\mathbf{C}}_{\hat{X}} = \hat{\sigma}_0^2 \tilde{\mathbf{C}}_{\hat{X}} = \hat{\mathbf{S}} \tilde{\mathbf{C}}_\varepsilon \hat{\mathbf{S}}^T$$

at

$$\hat{\mathbf{C}}_\varepsilon = \sigma_0^2 \tilde{\mathbf{C}}_\varepsilon$$

THEOREM 2

If $NW - E \xrightarrow{\gamma_1=0, \gamma_2=0} NK$, then $\tilde{\mathbf{C}}_{\hat{X}} \rightarrow \tilde{\mathbf{C}}_{\hat{X}}^{NK} = (\mathbf{A}^T \mathbf{P} \mathbf{A})^{-1}$

where $\mathbf{P} = \text{diag} \left(\frac{1}{\tilde{\sigma}_1^2}, \dots, \frac{1}{\tilde{\sigma}_n^2} \right)$ – weight matrix of NK method.

Proof

Since (Theorem 1) $\lim_{\gamma_1 \rightarrow 0, \gamma_2 \rightarrow 0} \mathbf{C}_{\hat{x}} = (\mathbf{A}^T \mathbf{C}_\varepsilon^{-1} \mathbf{A})^{-1}$ thus

$$\lim_{\gamma_1 \rightarrow 0, \gamma_2 \rightarrow 0} \tilde{\mathbf{C}}_{\hat{x}} = \frac{1}{\sigma_1^2} \lim_{\gamma_1 \rightarrow 0, \gamma_2 \rightarrow 0} \mathbf{C}_{\hat{x}} = \frac{1}{\sigma_0^2} (\mathbf{A}^T \mathbf{C}_\varepsilon^{-1} \mathbf{A})^{-1} = (\mathbf{A}^T \mathbf{P} \mathbf{A})^{-1}$$

(if only $\mathbf{C}_\varepsilon = \sigma_0^2 \tilde{\mathbf{C}}_\varepsilon = \sigma_0^2 \mathbf{P}^{-1}$).

3. Numerical example

Numerical tests were carried out using a simulated level network shown in the following Fig. 1.

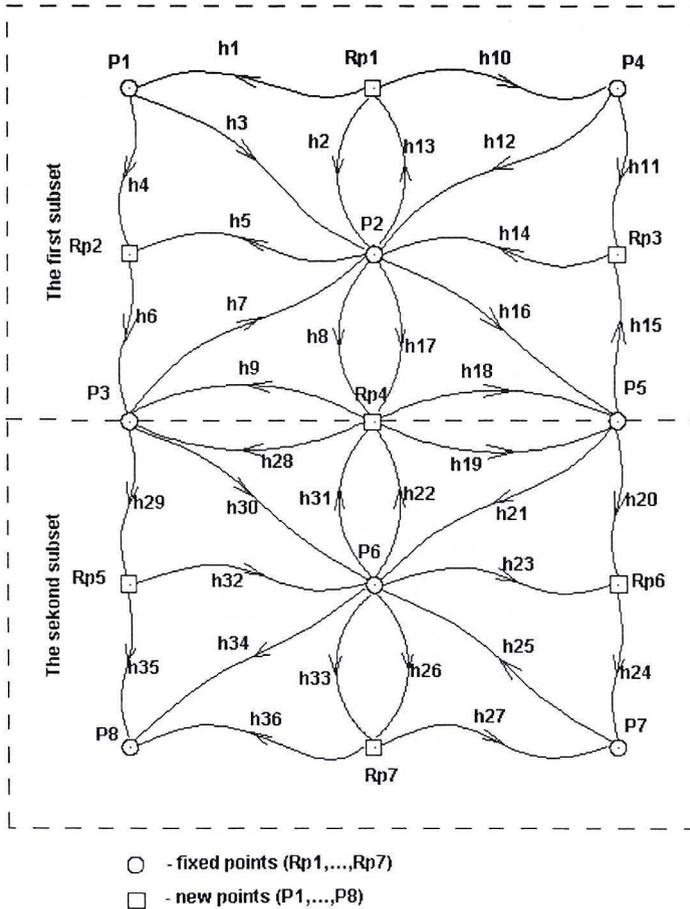


Fig. 1

The network was divided into two sets of observations of the same standard deviation σ but different values of the excess coefficient γ_2 . Each subset contains 4 fixed points and 5 new ones, however one fixed and two new points belong to both subsets. For the first subset the excess coefficient was assumed as zero, and "measurement results" belonging to the second subset were simulated on the assumption of non-zero excess ($\gamma_2 \neq 0$). The variance analyses were carried out for three variants differing from one another in the value of the excess coefficient in the second subset. The following covariance matrices were obtained after the adjustment of the tested network:

Variant I ($\gamma_2 = 0.5$)

$$C_{\hat{X}}^{NK} = (A^T P A)^{-1} = \begin{bmatrix} 0.069 & 0.007 & 0.001 & 0.002 & 0.001 & 0 & 0 & 0 \\ & 0.022 & 0.004 & 0.007 & 0.004 & 0.001 & 0 & 0 \\ & & 0.035 & 0.001 & 0.001 & 0.004 & 0.001 & 0.001 \\ & & & 0.069 & 0.001 & 0 & 0 & 0 \\ & & & & 0.035 & 0.004 & 0.001 & 0.001 \\ & & & & & 0.022 & 0.007 & 0.007 \\ & & & & & & 0.069 & 0.002 \\ & & & & & & & 0.069 \end{bmatrix}$$

$$C_{\hat{X}} = S C_{\epsilon} S^T = \begin{bmatrix} 0.077 & 0.009 & 0.001 & 0.003 & 0.002 & 0 & 0 & 0 \\ & 0.028 & 0.004 & 0.009 & 0.005 & 0.001 & 0 & 0 \\ & & 0.044 & 0.001 & 0.001 & 0.004 & 0.001 & 0.001 \\ & & & 0.076 & 0.002 & 0 & 0 & 0 \\ & & & & 0.038 & 0.004 & 0.001 & 0.001 \\ & & & & & 0.027 & 0.009 & 0.009 \\ & & & & & & 0.092 & 0.003 \\ & & & & & & & 0.085 \end{bmatrix}$$

$$C_{\hat{X}}^{\infty} = (A^T D(\hat{X}) A)^{-1} = \begin{bmatrix} 0.075 & 0.010 & 0.001 & 0.003 & 0.002 & 0 & 0 & 0 \\ & 0.028 & 0.004 & 0.009 & 0.005 & 0.001 & 0 & 0 \\ & & 0.044 & 0.001 & 0.001 & 0.004 & 0.001 & 0.001 \\ & & & 0.076 & 0.002 & 0 & 0 & 0 \\ & & & & 0.038 & 0.004 & 0.001 & 0.001 \\ & & & & & 0.027 & 0.009 & 0.009 \\ & & & & & & 0.092 & 0.003 \\ & & & & & & & 0.085 \end{bmatrix}$$

Variant II ($\gamma_2 = 1.01$)

$$\mathbf{C}_{\dot{\mathbf{x}}}^{NK} = \begin{bmatrix} 0.069 & 0.007 & 0.001 & 0.002 & 0.001 & 0 & 0 & 0 \\ & 0.022 & 0.004 & 0.007 & 0.004 & 0.001 & 0 & 0 \\ & & 0.035 & 0.001 & 0.001 & 0.004 & 0.001 & 0.001 \\ & & & 0.069 & 0.001 & 0 & 0 & 0 \\ & & & & 0.035 & 0.004 & 0.001 & 0.001 \\ & & & & & 0.022 & 0.007 & 0.007 \\ & & & & & & 0.069 & 0.002 \\ & & & & & & & 0.069 \end{bmatrix}$$

$$\mathbf{C}_{\ddot{\mathbf{x}}} = \begin{bmatrix} 0.240 & 0.016 & 0.002 & 0.006 & 0.003 & 0.001 & 0 & 0 \\ & 0.037 & 0.004 & 0.013 & 0.008 & 0.002 & 0 & 0 \\ & & 0.057 & 0.001 & 0.004 & 0.011 & 0 & 0 \\ & & & 0.096 & 0.003 & 0.001 & 0 & 0 \\ & & & & 0.080 & 0.017 & 0.001 & 0 \\ & & & & & 0.048 & 0.001 & 0 \\ & & & & & & 0.159 & -0.001 \\ & & & & & & & 0.122 \end{bmatrix}$$

$$\mathbf{C}_{\ddot{\mathbf{x}}}^{\infty} = \begin{bmatrix} 0.149 & 0.0127 & 0.001 & 0.004 & 0.002 & 0.001 & 0 & 0 \\ & 0.027 & 0.003 & 0.009 & 0.005 & 0.001 & 0 & 0 \\ & & 0.040 & 0.001 & 0.002 & 0.006 & 0.001 & 0.001 \\ & & & 0.076 & 0.002 & 0 & 0 & 0 \\ & & & & 0.049 & 0.009 & 0.001 & 0.001 \\ & & & & & 0.031 & 0.005 & 0.004 \\ & & & & & & 0.102 & 0.001 \\ & & & & & & & 0.100 \end{bmatrix}$$

Variant III ($\gamma_2 = 1.51$)

$$\mathbf{C}_{\dot{\mathbf{x}}}^{NK} = \begin{bmatrix} 0.069 & 0.007 & 0.001 & 0.002 & 0.001 & 0 & 0 & 0 \\ & 0.022 & 0.004 & 0.007 & 0.004 & 0.001 & 0 & 0 \\ & & 0.035 & 0.001 & 0.001 & 0.004 & 0.001 & 0.001 \\ & & & 0.069 & 0.001 & 0 & 0 & 0 \\ & & & & 0.035 & 0.004 & 0.001 & 0.001 \\ & & & & & 0.022 & 0.007 & 0.007 \\ & & & & & & 0.069 & 0.002 \\ & & & & & & & 0.069 \end{bmatrix}$$

$$C_{\hat{X}} = \begin{bmatrix} 0.113 & 0.045 & 0.011 & 0.004 & 0.013 & 0.002 & 0.001 & 0 \\ & 0.107 & 0.027 & 0.009 & 0.030 & 0.004 & 0 & 0.001 \\ & & 0.089 & 0.002 & 0.008 & 0.001 & 0 & -0.001 \\ & & & 0.154 & 0.003 & 0 & 0 & 0 \\ & & & & 0.070 & 0.011 & 0.004 & 0.001 \\ & & & & & 0.045 & 0.019 & 0.002 \\ & & & & & & 0.096 & 0.001 \\ & & & & & & & 0.200 \end{bmatrix}$$

$$C_{\hat{X}^\infty} = \begin{bmatrix} 0.069 & 0.014 & 0.003 & 0.002 & 0.004 & 0.001 & 0 & 0 \\ & 0.034 & 0.008 & 0.005 & 0.009 & 0.001 & 0.001 & 0 \\ & & 0.047 & 0.001 & 0.002 & 0.003 & 0.001 & 0.001 \\ & & & 0.077 & 0.001 & 0 & 0 & 0 \\ & & & & 0.039 & 0.004 & 0.002 & 0.001 \\ & & & & & 0.026 & 0.010 & 0.005 \\ & & & & & & 0.064 & 0.002 \\ & & & & & & & 0.091 \end{bmatrix}$$

The obtained, for the various values of γ_2 , covariance matrices of the vector \hat{X} confirm the earlier claimed theoretical statements especially this one saying that if $\gamma_1 \rightarrow 0$ and $\gamma_2 \rightarrow 0$ then $C_{\hat{X}}$ approaches $C_{\hat{X}}^{NK}$. One should also notice that if $\gamma_1 = 0$ and $\gamma_2 = 0$ then $C_{\hat{X}} = \tilde{C}_{\hat{X}}^{NK} = C_{\hat{X}}^\infty$.

However, from the equality $\tilde{C}_{\hat{X}}^{NK} = C_{\hat{X}}^\infty$ one cannot deduce that $\tilde{C}_{\hat{X}}^{NK} = C_{\hat{X}}^\infty$, because usually $\hat{\sigma}^{NK} \neq 1$. The values of the variance coefficient estimator $\hat{\sigma}^{NK}$, for each of the presented variants, are shown in Table 1.

Table 1

| Variant | $\hat{\sigma}^{NK}$ |
|---------|---------------------|
| I | 0.92 |
| II | 0.46 |
| III | 0.93 |

From the practical point of view it is important to analyse how a misjudged value of the excess coefficient can influence adjustment outcomes. To carry it out one can compare values of the norms: $\|\hat{X} - X^r\|_2$ (at $X^r = \mathbf{0}$) including all the network points, $\|\hat{X}_I\|_2$ including the points belonging to the first subset, $\|\hat{X}_{II}\|_2$ to the second one and $\|\hat{X}_p\|_2$ including the points belonging to the both subsets. The set of the measurement results was simulated with

the following values: $\gamma_1 = 0, \gamma_2 = 0.5, \sigma = 3$. The outcomes of adjustment (values of the above presented norms) with application of *NW-E* method and with various disturbances of the excess coefficient γ_2 are presented in Table 2.

Table 2

| γ | $\ \hat{\mathbf{X}}\ _2$ | $\ \hat{\mathbf{X}}_1\ _2$ | $\ \hat{\mathbf{X}}_2\ _2$ | $\ \hat{\mathbf{X}}_p\ _2$ |
|------------|--------------------------|----------------------------|----------------------------|----------------------------|
| 0 | 2.68 | 2.28 | 1.37 | 0.33 |
| 0.1 | 2.61 | 2.26 | 1.27 | 0.27 |
| 0.2 | 2.56 | 2.26 | 1.18 | 0.22 |
| 0.3 | 2.53 | 2.27 | 1.09 | 0.18 |
| 0.4 | 2.51 | 2.29 | 1.02 | 0.17 |
| 0.5 | 2.51 | 2.32 | 0.94 | 0.19 |
| 0.6 | 2.51 | 2.34 | 0.88 | 0.23 |
| 0.7 | 2.51 | 2.36 | 0.82 | 0.28 |
| 0.8 | 2.51 | 2.37 | 0.76 | 0.34 |
| 0.9 | 2.52 | 2.38 | 0.71 | 0.40 |
| 1.0 | 2.52 | 2.38 | 0.66 | 0.47 |

The norm values for the proper value of γ_2 ($\gamma_2 = 0.5$) are shown in bold type. The analyses show that the best results (the minimal values of the norms) were obtained for the best estimate of the excess coefficient. If it is underestimated one can notice increase in the norm values for the whole set as well as for the second subset and for the joint points. On the other hand if it is overestimated the norm of the second subset is on the decrease however the norms for the first subset and for the joint points are still on the increase. In conclusion, bad estimates of the excess coefficient worsen the final results of *NW-E* method.

Additionally, analyses, how a standard deviation estimate σ could influence adjustment result, were carried out. Comparisons of the earlier applied norms $\|\hat{\mathbf{X}}\|_2, \|\hat{\mathbf{X}}_1\|_2, \|\hat{\mathbf{X}}_2\|_2, \|\hat{\mathbf{X}}_p\|_2$ were still the base for it. The measurement results were simulated assuming that $\gamma_1 = 0, \gamma_2 = 0.5, \sigma = 3.1$. Disturbances led to the outcomes presented in Table 3.

The results for the proper estimate of σ are presented in bold type. The disturbances make the norms be usually on the increase especially for the norms of the all network points, the second subset and the joint points. It means that an improper estimate of σ leads to worse adjustment outcomes (it is similar to the earlier shown influence of excess disturbances). It is especially noticeable if the estimate of the standard deviation is underestimated. One can also notice that the norm

$\|\hat{\mathbf{X}}^{NK} - \mathbf{X}^t\|_2$ is independent of σ . It follows from the fact that if a network is homogeneous with regard to the value of σ then one can obtain (\mathbf{E} — unit matrix)

$$\hat{\mathbf{X}}^{NK} = (\mathbf{A}^T \sigma^{-2} \mathbf{E} \mathbf{A})^{-1} \mathbf{A}^T \sigma^{-2} \mathbf{E} \mathbf{x} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{x}$$

Table 3

| σ | $\ \hat{\mathbf{X}}\ _2$ | $\ \hat{\mathbf{X}}_1\ _2$ | $\ \hat{\mathbf{X}}_2\ _2$ | $\ \hat{\mathbf{X}}_p\ _2$ |
|------------|--------------------------|----------------------------|----------------------------|----------------------------|
| 1.0 | 2.66 | 2.21 | 1.44 | 0.35 |
| 2.0 | 2.47 | 2.29 | 0.91 | 0.21 |
| 3.1 | 2.51 | 2.32 | 0.94 | 0.19 |
| 3.2 | 2.53 | 2.33 | 0.98 | 0.18 |
| 3.4 | 2.56 | 2.34 | 1.02 | 0.17 |
| 3.6 | 2.57 | 2.33 | 1.07 | 0.17 |
| 4 | 2.58 | 2.31 | 1.13 | 0.19 |

CONCLUSION

The analyses show the new important properties of NW-E method. The most important is that this method and NK method converge if the probabilistic model of measurement errors is not disturbed. Thus NW-E method can be treated as a universal instrument of adjustment taking into consideration also standard assumptions about probabilistic properties of measurement results. This paper proved that a proper estimating of the control parameters and especially γ_2 (in presented case) is essential for the quality of final results. Effects of disturbances of standard deviation are rather similar to effects of an improper weighing in NK method. Thus an application of NW-E method requires not only a good estimating of σ (like for NK method) but also of the excess coefficient γ_2 .

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Received November 26, 2002

Accepted January 4, 2003

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Analiza dokładności właściwości metody NW-E

Streszczenie

W niniejszej pracy przedstawiono ogólną koncepcję metody wyrównania obserwacji geodezyjnych z zastosowaniem szeregów Edgewortha. Zaproponowano zastąpienie rozkładów prawdopodobieństw aproksymantą reprezentowaną szeregiem Edgewortha, który byłby uniwersalnym, probabilistycznym modelem zadania wyrównawczego. Szczególną uwagę zwrócono na analizę dokładności. Wyznaczono macierz kowariancji $C_{\hat{X}}$ wektora \hat{X} , jej estymatora $\hat{C}_{\hat{X}}$ oraz estymator współczynnika wariancji $\hat{\sigma}_0$. Ustalono niektóre własności metody.

Андрей Думальски

Дисперсионный анализ и особенности метода NW-E

Резюме

В работе представлена общая концепция метода уравнивания геодезических наблюдений с применением Енджилорфа. Предложена замена распределений вероятностей аппроксимирующей функцией представленной при помощи ряда Енджилорфа, который являлся бы универсальной, вероятностной моделью уравнивательной задачи. Особое внимание было обращено на анализ точности. Была определена ковариационная матрица $C_{\hat{X}}$ вектора \hat{X} , её оценки $\hat{C}_{\hat{X}}$, а также оценки вариационного коэффициента $\hat{\sigma}_0$. Определены некоторые особенности метода.