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Projection of the area of Poland in wide Gauss-Krüger zone

The article presents the elementary coordinate calculating formulas (simple and reverse problems) of the Gauss-Krüger projection in wide meridional zone. The formulas that make it possible to calculate the values of local area distortions and reductions of directions, angles and lengths, have been presented as well.

1. *The Gauss-Krüger projection as a triple one*

1.1. The simple aspect. Conversion of geodetic ellipsoidal coordinates B,L to flat rectangular coordinates *x,y*

The length of an arc of the meridian of a flattened spheroid described within ellipsoidal coordinates B, L and referred to quatorial semi-axis a, eccentric e, or the third oblateness n

$$s=s(B)=\int_{0}^{B} M(B)dt$$

$$M=M(B)=\frac{a(1-e^2)}{(\sqrt{1-e^2\sin^2 B})^3}=\frac{a(1-n)(1-n^2)}{(\sqrt{1+2n\cos 2B+n^2})^3} \tag{1}$$

might be expressed by means of the following formula

$$s=a(1-n)(1-n^2)\left[B\sum_{i=1}^{\infty}\binom{-\frac{3}{2}}{i}\binom{-\frac{3}{2}}{i}n^{2i}+\sum_{r=1}^{\infty}\sum_{l=0}^{\infty}\binom{-\frac{3}{2}}{l}\binom{-\frac{3}{2}}{r+l}\frac{n^r}{r}\sin 2rB\right] \tag{2}$$

or in the form of a few first terms of expansion as

$$s=s(B)=a(1-n)(1-n^2)\left[BC_0+nC_1\sin2B+\frac{n^2}{2}C_2\sin4B+\frac{n^3}{3}C_3\sin6B+\frac{n^4}{4}C_4\sin8B+\ldots\right] \quad (3)$$

Coefficients C_i in (3), according to n, will be expressed in the following way:

$$C_0=1+\frac{9}{4}n^2+\frac{225}{64}n^4+\frac{11025}{2304}n^6+\ldots$$

$$C_1=-\frac{3}{2}-\frac{45}{16}n^2-\frac{1575}{384}n^4-\ldots$$

$$C_2=\frac{15}{8}+\frac{105}{32}n^2+\frac{14175}{3072}n^4+\ldots \quad (4)$$

$$C_3=-\frac{35}{16}-\frac{2835}{768}n^2-\ldots$$

$$C_4=\frac{315}{128}+\frac{31185}{7680}n^2+\ldots$$

The radius of a sphere whose length of the meridional arc equals to the length of the meridional arc of the area of the spheroid might be presented as follows:

$$\left[R\frac{\pi}{2}=s(B)_{\mid B=\frac{\pi}{2}}\right]\equiv\left[R=\frac{2}{\pi}s(B)_{\mid B=\frac{\pi}{2}}=a(1-n)(1-n^2)C_0\right] \quad (5)$$

After appropriate reductions it might be presented as a fast-convergent power series

$$R=\frac{a}{1+n}\left(1+\frac{n^2}{4}+\frac{n^4}{64}+\frac{n^6}{256}+\frac{25n^8}{16384}+\ldots\right) \quad (6)$$

The Gauss-Krüger projection as a triple one consists in successive execution (Fig. 1) of the following transformations:

1. Conversion from geodetic ellipsoidal coordinates (B, L), $l=L-L_0$, L_0 — axial meridian of an area, to the geographical coordinates (φ, λ) of a sphere of radius R, by means of the (conformal) Lagrange projection

$$\tan\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)=\left[\tan\left(\frac{\pi}{4}+\frac{B}{2}\right)\right]\left(\frac{1-e\sin B}{1+e\sin B}\right)^{\frac{e}{2}}\Rightarrow$$

$$\varphi=2\left\{\arctan\left\{\left[\tan\left(\frac{\pi}{4}+\frac{B}{2}\right)\right]\left(\frac{1-e\sin B}{1+e\sin B}\right)^{\frac{e}{2}}\right\}-\frac{\pi}{4}\right\}, \quad (7)$$

$$\lambda=l$$

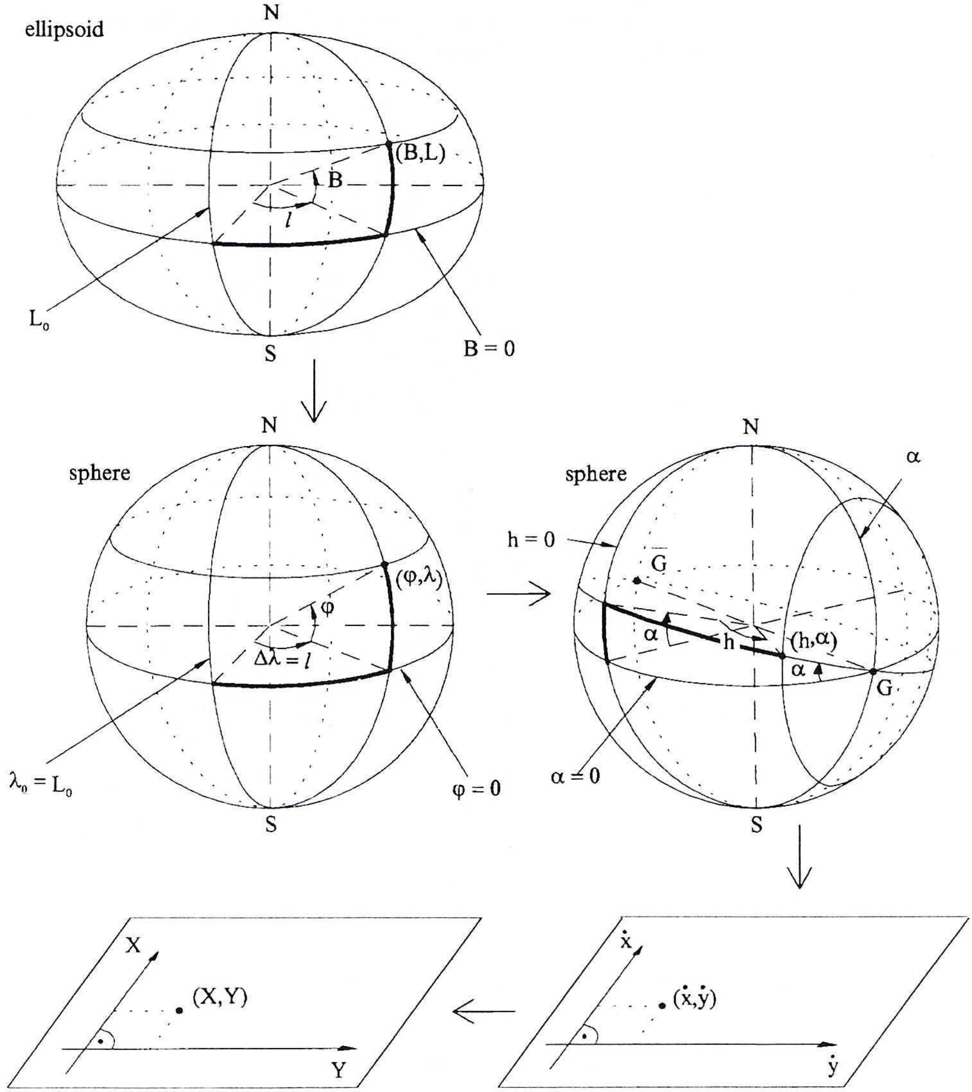

Fig. 1

Value R having been assumed as the radius of the sphere implies equality of the lengths of the arc of the spheroid's meridian and that of the sphere, scale on the meridian unidentically equalling to one.

2. Conversion from geographical coordinates φ and λ of the sphere to coordinates h and α of a spherical system of the polar type by means of dependencies solving spherical triangle GNP (Fig. 2) with directed angles.

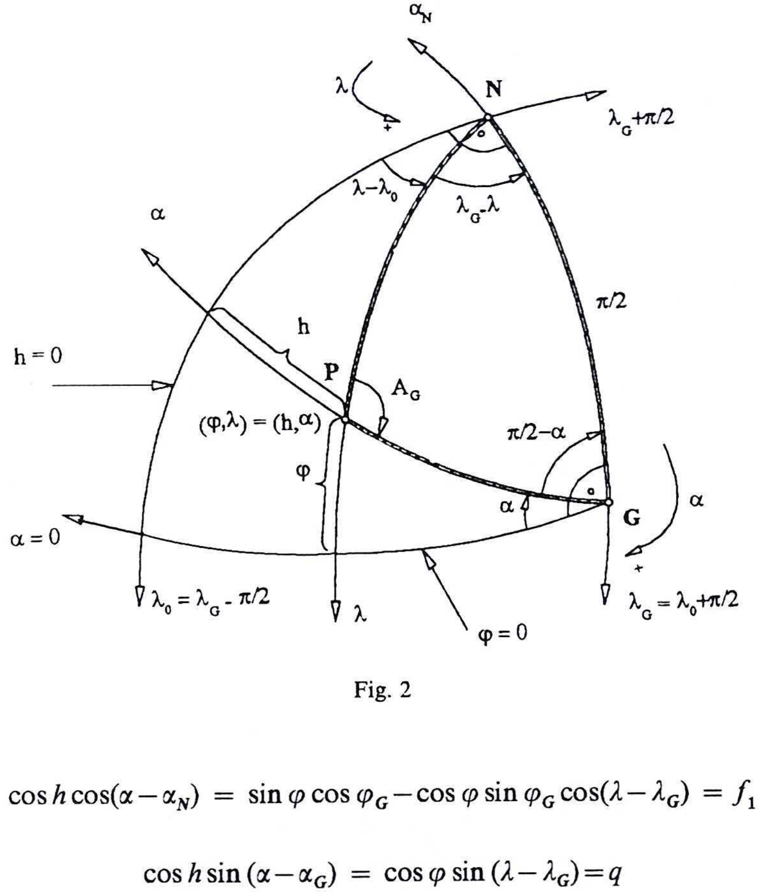

Fig. 2

$$\cos h \cos(\alpha-\alpha_N) = \sin\varphi\cos\varphi_G-\cos\varphi\sin\varphi_G\cos(\lambda-\lambda_G) = f_1$$

$$\cos h \sin(\alpha-\alpha_G) = \cos\varphi\sin(\lambda-\lambda_G)=q \tag{8}$$

$$\sin h = \sin\varphi\sin\varphi_G+\cos\varphi\cos\varphi_G\cos(\lambda-\lambda_G)=f_2$$

where:

$$\varphi_G=0, \qquad \lambda_G=L_0+\frac{\pi}{2}$$

$$\lambda-\lambda_G=L-\left(L_0+\frac{\pi}{2}\right) = l-\frac{\pi}{2}, \qquad \alpha_N = \frac{\pi}{2} \tag{9}$$

Out of (8) and (9) the following dependencies will be obtained:

$$\cos h \sin \alpha = \sin \varphi = f_1$$
$$\cos h \cos \alpha = \cos \varphi \cos l = q \qquad (10)$$
$$\sin h = \cos \varphi \sin l = f_2$$

or

$$\sin h = f_2 = \cos \varphi \sin l$$
$$\tan \alpha = \frac{q}{f_1} = \frac{\sin \varphi}{\cos \varphi \cos l} \qquad (11)$$

where f_1, f_2 and q stand for the right sides of dependencies (8). Pole G of system h, α lies on equator $\varphi_G = 0$ and meridian $\lambda_G = L_0 + \frac{\pi}{2}$, variable α will be calculated positively clockwise.

3. Conversion from coordinates (h, α) of the sphere to coordinates ($\dot{x}$, $\dot{y}$) in the so called Mercator transverse projection:

$$\dot{x} = R\alpha$$
$$\dot{y} = R \ln \tan\left(\frac{\pi}{4} + \frac{h}{2}\right) \qquad (12)$$

whence

$$\xi = \frac{\dot{x}}{R} = \alpha = \arctan \frac{\sin \varphi}{\cos \varphi \cos l}$$
$$\eta = \frac{\dot{y}}{R} = \frac{1}{2} \ln \frac{1 + \cos \varphi \sin l}{1 - \cos \varphi \sin l} \qquad (13)$$

Horizon $h = 0$ overlaps with axial meridian $L = L_0$, while vertical circle $\alpha = 0$ overlaps with equator $\varphi = 0$. Local length scale upon the meridian within plane $\dot{x}O\dot{y}$ varies.

4. Construction of a conformal projection of plane $\dot{x}O\dot{y}$ onto plane xOy the way axial meridian $L = L_0$ projects onto a segment of the x axis, local length scale identically equalling to one. Thus, it is necessary to find a form of the function that will describe the length of the arc of axial meridian $L = L_0$ of the surface of flattened spheroid upon spherical horizon $h = 0$. Since spherical horizon $h = 0$ on the surface of the sphere of a unit radius identifies with axial spherical meridian $\lambda = \lambda_0$, parameter α identify with parameter φ, i.e. the spherical Lagrange conformal latitude. Thus the

function that is being looked for is a composition of function (2) describing the length of the meridian of the surface of the flattened spheroid as a function of parameter B and the reverse Lagrange function (7), describing the spherical conformal latitude marked by parameter φ. Spherical latitude φ will be expressed [2] by means of a trigonometric series of the following form:

$$\varphi = B + \sum_{\nu=1}^{\infty} k_{2\nu} \sin 2\nu B \tag{14}$$

where:

$$\begin{aligned} k_2 &= -2n + \frac{2}{3}n^2 + \frac{4}{3}n^3 - \frac{82}{45}n^4 + \ldots \\ k_4 &= \frac{5}{3}n^2 - \frac{16}{15}n^3 - \frac{13}{9}n^4 + \ldots \\ k_6 &= -\frac{26}{15}n^3 + \frac{34}{21}n^4 + \ldots \\ k_8 &= \frac{1237}{630}n^4 + \ldots \end{aligned} \tag{15}$$

After trigonometric series (14) has been reversed [2], the following will be obtained:

$$B = \varphi + \sum_{\lambda=1}^{\infty} \dot{k}_{2\lambda} \sin 2\lambda\varphi \tag{16}$$

Necessary coefficients $\dot{k}$ in (16) will be calculated by means of the following dependence:

$$\begin{aligned} \dot{k}_2 &= 2n - \frac{2}{3}n^2 - 2n^3 + \frac{116}{45}n^4 + \ldots \\ \dot{k}_4 &= \frac{7}{3}n^2 - \frac{8}{5}n^3 - \frac{227}{45}n^4 + \ldots \\ \dot{k}_6 &= \frac{56}{15}n^3 - \frac{136}{35}n^4 + \ldots \\ \dot{k}_8 &= \frac{4279}{630}n^4 + \ldots \end{aligned} \tag{17}$$

Composition of series (2) with trigonometric series (16) leads to final series

$$\frac{s}{R}=\varphi+\sum_{\lambda=1}^{\infty} i_{2\lambda}\sin 2\lambda\varphi \tag{18}$$

where

$$\begin{aligned}
\dot{l}_2&=\frac{1}{2}n-\frac{2}{3}n^2+\frac{5}{16}n^3+\frac{41}{180}n^4+\ldots\\
\dot{l}_4&=\frac{13}{48}n^2-\frac{3}{5}n^3+\frac{557}{1440}n^4+\ldots\\
\dot{l}_6&=\frac{61}{240}n^3-\frac{103}{140}n^4+\ldots\\
\dot{l}_8&=\frac{49561}{161280}n^4+\ldots
\end{aligned} \tag{19}$$

On the basis of trigonometric series (18), the Gauss-Krüger projection represented by variable $x+iy$, expressed by complex variable $\dot{x}+i\dot{y}$ determining the position of a point within the Mercator plane, might be presented in the following form:

$$x+iy=R\left[\left(\frac{\dot{x}}{R}+i\frac{\dot{y}}{R}\right)+\sum_{\lambda=1}^{\infty}\dot{l}_{2\lambda}\sin 2\lambda\left(\frac{\dot{x}}{R}+i\frac{\dot{y}}{R}\right)\right]. \tag{20}$$

Expansion in series (20) to real and imaginary part leads to two trigonometric series of the following form:

$$\begin{aligned}
x&=\dot{x}+R\left[\sum_{\lambda=1}^{\infty}\dot{l}_{2\lambda}\sin\left(2\lambda\frac{\dot{x}}{R}\right)\cosh\left(2\lambda\frac{\dot{y}}{R}\right)\right]\\
y&=\dot{y}+R\left[\sum_{\lambda=1}^{\infty}\dot{l}_{2\lambda}\cos\left(2\lambda\frac{\dot{x}}{R}\right)\sinh\left(2\lambda\frac{\dot{y}}{R}\right)\right]
\end{aligned} \tag{21}$$

or

$$\begin{aligned}
x&=\dot{x}+R\left[\sum_{\lambda=1}^{\infty}\dot{l}_{2\lambda}\sin(2\lambda\xi)\cosh(2\lambda\eta)\right]\\
y&=\dot{y}+R\left[\sum_{\lambda=1}^{\infty}\dot{l}_{2\lambda}\cos(2\lambda\xi)\sinh(2\lambda\eta)\right]
\end{aligned} \tag{22}$$

5. The value of local length scale in the Gauss-Krüger projection — for shrinkage coefficient $m_0=1$ of axial meridian L_0 — which is a product of three successive local scales of partial projections, might be presented by means of the following formula:

$$m=\frac{R\cos\varphi}{N\cos B}\sqrt{\frac{\left(\frac{x}{R}\right)_{\xi}^{2}+\left(\frac{y}{R}\right)_{\xi}^{2}}{1-\cos^{2}\varphi\sin^{2}l}} \tag{23}$$

where

$$\begin{aligned}\frac{\partial\left(\frac{x}{R}\right)}{\partial\xi}&=\left(\frac{x}{R}\right)_{\xi}=1+\sum_{\lambda=1}^{\infty}2\lambda\dot{l}_{2\lambda}\cos(2\lambda\xi)\cosh(2\lambda\eta)\\ \frac{\partial\left(\frac{y}{R}\right)}{\partial\xi}&=\left(\frac{y}{R}\right)_{\xi}=-\sum_{\lambda=1}^{\infty}2\lambda\dot{l}_{2\lambda}\sin(2\lambda\xi)\sinh(2\lambda\eta)\end{aligned} \tag{24}$$

stand for the values of partial derivatives.

6. Convergence of meridians γ in the Gauss-Krüger projection at point of coordinates (x, y) will be calculated by means of the following formula:

$$\gamma=\gamma_1+\gamma_2=\arctan(\sin\varphi\tan l)-\arctan\left(\frac{\left(\frac{y}{R}\right)_{\xi}}{\left(\frac{x}{R}\right)_{\xi}}\right) \tag{25}$$

The following dependence comes about between geodetic azimuth A_g and topographic azimuth A_t within the projection plane: $A_g=A_t+\gamma$.

1.2. The reverse aspect — conversion of flat rectangular coordinates x,y to geodetic ellipsoidal coordinates B,L

1. Trignometric series (18) of the standard length of the arc of the meridian of the spheroid will be presented in a reverse form:

$$\varphi=\frac{s}{R}+\sum_{\lambda=1}^{\infty}l_{2\lambda}\sin\left(2\lambda\frac{s}{R}\right) \tag{26}$$

where

$$\begin{aligned}l_2&=-\frac{1}{2}n+\frac{2}{3}n^2-\frac{37}{96}n^3+\frac{1}{360}n^4+\dots\\ l_4&=\qquad-\frac{1}{48}n^2-\frac{1}{15}n^3+\frac{437}{1440}n^4+\dots\end{aligned} \tag{27}$$

$$l_6 = \qquad -\frac{17}{480}n^3+\frac{37}{840}n^4+\ldots \tag{27}$$

$$l_8 = \qquad -\frac{4397}{161280}n^4+\ldots$$

Coming from real variables φ and s to complex variables, expressing standard Gauss-Krüger coordinates (20) and standard Mercator coordinates (13), the following trigonometric series will be obtained:

$$\frac{\dot{x}}{R}+i\frac{\dot{y}}{R}=\xi+i\eta=\left(\frac{x}{R}+i\frac{y}{R}\right)+\sum_{\lambda=1}^{\infty}l_{2\lambda}\sin\left[2\lambda\left(\frac{x}{R}+i\frac{y}{R}\right)\right] \tag{28}$$

Distribution of series (28) into its real and imaginary part results in standard coordinates in the Mercator plane in a transverse position

$$\xi=\frac{x}{R}+\sum_{\lambda=1}^{\infty}l_{2\lambda}\sin\left(2\lambda\frac{x}{R}\right)\cosh\left(2\lambda\frac{y}{R}\right)$$
$$\eta=\frac{y}{R}+\sum_{\lambda=1}^{\infty}l_{2\lambda}\cos\left(2\lambda\frac{x}{R}\right)\sinh\left(2\lambda\frac{y}{R}\right) \tag{29}$$

2. Dependences meant for calculation of azimuth coordinates h and α on the basis of coordinates ξ and η will be obtained after relationships (12) have been reversed

$$\alpha=\xi$$
$$h=2\arctan(\exp(\eta))-\frac{\pi}{2} \tag{30}$$

3. Reversal of dependences (8)–(10)

$$\cos\varphi\cos(\lambda-\lambda_G)=\sin h\cos\varphi_G-\cos h\sin\varphi_G\cos(\alpha-\alpha_N)=\bar{f}_1$$
$$\cos\varphi\sin(\lambda-\lambda_G)=\cos h\sin(\alpha-\alpha_N)=\bar{q} \tag{31}$$
$$\sin\varphi=\sin h\sin\varphi_G+\cos h\cos\varphi_G\cos(\alpha-\alpha_N)=\bar{f}_2$$

$$\varphi_G=0,\quad \lambda_G=L_0+\frac{\pi}{2},\quad \lambda-\lambda_G=l-\frac{\pi}{2},\quad \alpha_N=\frac{\pi}{2} \tag{32}$$

leads to dependence

$$\sin\varphi=\bar{f}_2=\cos h\sin\alpha,\quad \tan(\lambda-\lambda_G)=\frac{\bar{q}}{\bar{f}_1}\Rightarrow\tan l=\frac{\sin h}{\cos h\cos\alpha} \tag{33}$$

This makes it possible to calculate the value of spherical latitude φ and longitude l.

4. Calculation of geodetic ellipsoidal latitude B by means of proceeding from spherical latitude φ to the value of parameter B will be accomplished on the basis of series (16).

The geodetic coordinates system "1992" is based on the points of networks EUREFPOL and POLREF. Flat rectangulated coordinates X, Y for the area of Poland will be calculated within a 10 degree meridional zone of the surface of the ellipsoid of reference (GRS 80)[1] in the Gauss-Krüger projection.

The metric parameters of ellipsoid GRS 80 (Geodetic Reference System 1980) as follows:

$a = 6\,378\,137$ m, equatorial semi-axis

$b = 6\,356\,752.3141$ m, polar semi-axis

$e^2 = 0.006\,694\,380\,022\,90$, square of eccentric $\left(e^2 = \frac{a^2 - b^2}{a^2}\right)$

$e'^2 = 0.006\,739\,496\,77548$, square of the second eccentric $\left(e'^2 = \frac{e^2}{1 - e^2}\right)$

$n = 0.001\,679\,220\,39463$, thrid oblateness $\left(n = \frac{a-b}{a+b} = \frac{1-\sqrt{1-e^2}}{1-\sqrt{1-e^2}}\right)$

$Q = 10\,001\,965.7293$ m, length of arc of meridian from equator to pole.

Axial meridian for the area of Poland in the „1992" system has been accepted as amounting to $L_0 = 19°$.

Plane rectangular coordinates for the area of Poland will be calculated from the following formulas:

$$X = (x^* m_0) - X_0, \quad Y = (y^* m_0) + Y_0$$

where $m_0 = 0.9993$ has been accepted as the shrinkage coefficient of the axial meridian in the Gauss-Krüger projection.

Gauss-Krüger shift parameters of the beginning of the system xoy amount to:

$$X_0 = 5\,3000\,000.0 \text{ m}, \qquad Y_0 = 500\,000.0 \text{ m}.$$

2. *Calculation of the values of projecting reductions*

2.1. Transfer of flat rectangular coordinates X, Y in the Gauss-Krüger projection along the image of the geodesic line

1. The simple problem

The following have been given: coordinates (X_1, Y_1) of point $P_1'(X_1, Y_1)$, azimuth A_{g12} of the geodesic line at point P_1 and ellipsoidal length S_{12} of an arc of the geodesic line connecting points P_1 and P_2 upon the ellipsoid (Fig. 3).

[1] The metric parameters of the ellipsoid of reference GRS 80 are identical with those of the ellipsoid of system WGS 84

On the basis of the coordinates of point P_1 the following will be calculated in succession:

1) parameters B_1, L_1 of point P_1 and convergence of meridians γ_1,
2) azimuth $A_{t12}=A_{g12}-\gamma_1$ of the image of the geodesic line at point P'_1,
3) flat rectangular coordinates X_2, Y_2 of point P'_2.

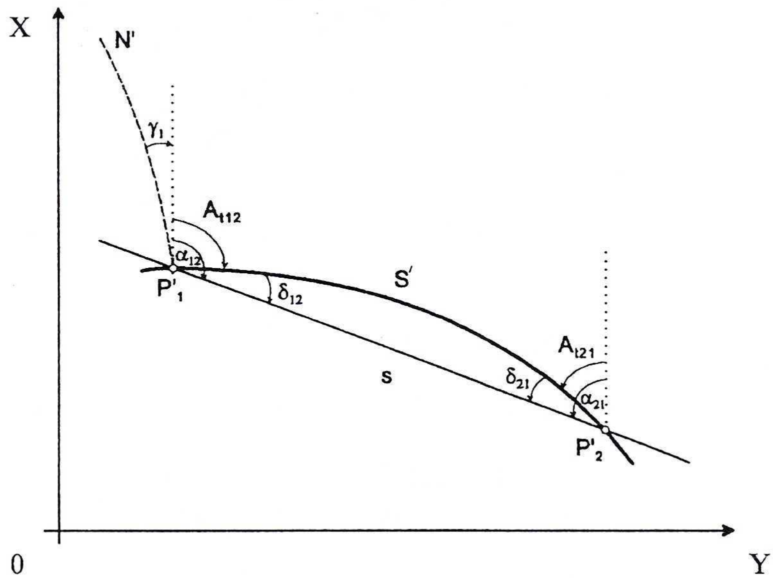

Fig. 3

Coordinates X_2, Y_2 of point P'_2 will be calculated from the following formulas:

$$X_2=X_1+m_0\Bigg[\left(1+\frac{(Y_1-Y_0)^2}{2R_1^2}\frac{1}{m_0^2}+\frac{(Y_1-Y_0)^4}{24R_1^4}\frac{1}{m_0^4}+\frac{(Y_1-Y_0)^6}{720R_1^6}\frac{1}{m_0^6}\right)u+$$

$$-\frac{(Y_1-Y_0)^2}{R_1^3}\frac{1}{m_0^2}t_1\eta_1^2u^2+\left(\frac{(Y_1-Y_0)}{R_1^2}\frac{1}{m_0}+\frac{2}{3}\frac{(Y_1-Y_0)}{R_1^4}\frac{1}{m_0^3}\right)uv+ \tag{34}$$

$$+\frac{(Y_1-Y_0)^2}{R_1^3}\frac{1}{m_0^2}t_1\eta_1^2v^2-\frac{1}{3}\frac{(Y_1-Y_0)^2}{R_1^4}\frac{1}{m_0^2}u^3-\frac{2(Y_1-Y_0)}{R_1^3}\frac{1}{m^0}t_1\eta_1^2u^2v+$$

$$+\left(\frac{1}{3R_1^2}+\frac{7}{6}\frac{(Y_1-Y_0)^2}{R_1^4}\frac{1}{m_0^2}\right)uv^2+\frac{2}{3}\frac{(Y_1-Y_0)}{R_1^3}\frac{1}{m_0}t_1\eta_1^2v^3+\ldots\Bigg]$$

$$
\begin{aligned}
Y_2 = Y_1 + m_0 \Bigg[& \left(1 + \frac{(Y_1 - Y_0)^2}{2R_1^2}\frac{1}{m_0^2} + \frac{(Y_1 - Y_0)^4}{24R_1^4}\frac{1}{m_0^4} + \frac{(Y_1 - Y_0)^6}{720R_1^6}\frac{1}{m_0^6}\right)v + \\
& -\left(\frac{(Y_1 - Y_0)}{2R_1^2}\frac{1}{m_0} + \frac{(Y_1 - Y_0)^3}{3R_1^4}\frac{1}{m_0^3}\right)u^2 - \frac{2(Y_1 - Y_0)^2}{R_1^3}\frac{1}{m_0^2}t_1\eta_1^2 uv + \\
& +\left(\frac{(Y_1 - Y_0)}{2R_1^2}\frac{1}{m_0} + \frac{(Y_1 - Y_0)^3}{3R_1^4}\frac{1}{m_0^3}\right)v^2 + \frac{2}{3}\frac{(Y_1 - Y_0)}{R_1^3}\frac{1}{m_0}t_1\eta_1^2 u^3 + \\
& -\left(\frac{1}{61R_1^2} + \frac{13}{12}\frac{(Y_1 - Y_0)^2}{R_1^4}\frac{1}{m_0^2}\right)u^2 v - \frac{2(Y_1 - Y_0)}{R_1^3}\frac{1}{m_0}t_1\eta_1^2 uv^2 + \\
& +\left(\frac{1}{6R_1^2} + \frac{5}{12}\frac{(Y_1 - Y_0)^2}{R_1^4}\frac{1}{m_0^2}\right)v^3 + \ldots \Bigg]
\end{aligned}
\tag{35}
$$

The value of root $\sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$ gives length s_{12} reduced within the projecting plane.

Reflexive azimuth A_{t21} at point P_2' will be calculated from the following formula:

$$
\begin{aligned}
A_{t21} = A_{t12} - & \left(\frac{(Y_1 - Y_0)}{R_1^2}\frac{1}{m_0} + \frac{(Y_1 - Y_0)^3}{6R_1^4}\frac{1}{m_0^3}\right)u + \\
& - \frac{2(Y_1 - Y_0)^2}{R_1^3}\frac{1}{m_0^2}t_1\eta_1^2 v + \frac{2(Y_1 - Y_0)^2}{R_1^3}\frac{1}{m_0}t_1\eta_1^2 u^2 + \\
& -\left(\frac{1}{2R_1^2} + \frac{(Y_1 - Y_0)^2}{R_1^4}\frac{1}{m_0^2}\right)uv - \frac{2(Y_1 - Y_0)}{R_1^3}\frac{1}{m_0}t_1\eta_1^2 v^2 + \ldots
\end{aligned}
\tag{36}
$$

where

$$u = S_{12}\cos A_{t12}, \qquad v = S_{12}\sin A_{t12}$$

$$t_1 = \tan B_1, \qquad \eta_1^2 = e'^2\cos^2 B_1, \qquad R_1 = \sqrt{M_1 N_1} = \frac{a\sqrt{1-e^2}}{1 - e^2\sin^2 B_1}$$

Topographic azimuths α_{12}, α_{21}, will be calculated from the flat rectangular coordinates X_1, Y_1 of point P_1' and the flat rectangular coordinates X_2, Y_2 of point P_2'.

Reductions δ_{12}, δ_{21} of direction angles A_{t12} and A_{t21} will be calculated from the following formulas:

$$
\begin{aligned}
\delta_{12} &= \alpha_{12} - A_{t12} \\
\delta_{21} &= \alpha_{21} - A_{t21}
\end{aligned}
\tag{37}
$$

2. The reverse problem

Two points $P'_1(X_1, Y_1)$, $P'_2(X_2, Y_2)$ have been given in the Gauss-Krüger projection. We are looking for distance $S=S_{12}=S_{21}$ between points P_1 and P_2 upon the ellipsoid and topographic azimuths A_{t12} and A_{t21} of the geodesic line at points P'_1 and P'_2, respectively.

In the *first* version the following will be calculated in succession:

1) parameters B_1, L_1 point P_1 from formulas (26—33) and (16), auxiliary parameters $t_1=\tan B_1$, and $\eta_1^2=e'^2\cos^2 B_1$;
2) values of reductions of direction angles δ_{12} and δ_{21};
3) value of projecting reduction coefficient of length $\frac{S}{s}$, i.e. proportion of the length of a segment of the geodesic line on the surface of the spheroid and the length of the chord corresponding to it in the projection plane.

$$\delta_{12}=-\left(\frac{(Y_1-Y_0)}{2R_1^2}\frac{1}{m_0}-\frac{(Y_1-Y_0)^3}{6R_1^4}\frac{1}{m_0^3}\right)\frac{1}{m_0}\Delta X_{12}-\frac{(Y_1-Y_0)^2}{R_1^3}\frac{1}{m_0^3}t_1\eta_1^2\Delta Y_{12}+$$

$$+\frac{2(Y_1-Y_0)}{3R_1^3}\frac{1}{m_0^3}t_1\eta_1^2\Delta X_{12}^2-\left(\frac{1}{6R_1^2}-\frac{(Y_1-Y_0)^2}{4R_1^4}\frac{1}{m_0^2}\right)\frac{1}{m_0^2}\Delta X_{12}\Delta Y_{12}+ \tag{38}$$

$$-\frac{2(Y_1-Y_0)}{3R_1^3}\frac{1}{m_0^3}t_1\eta_1^2\Delta Y_{12}^2+\ldots,$$

$$\alpha_{12}-A_{t12}=\delta_{12}, \qquad A_{t12}=\alpha_{12}-\delta_{12}$$

$$\delta_{21}=\left(\frac{(Y_1-Y_0)}{2R_1^2}\frac{1}{m_0}-\frac{(Y_1-Y_0)^3}{6R_1^4}\frac{1}{m_0^3}\right)\frac{1}{m_0}\Delta X_{12}+\frac{(Y_1-Y_0)^2}{R_1^3}\frac{1}{m_0^3}t_1\eta_1^2\Delta Y_{12}+$$

$$-\frac{4(Y_1-Y_0)}{3R_1^3}\frac{1}{m_0^3}t_1\eta_1^2\Delta X_{12}^2+\left(\frac{1}{3R_1^2}-\frac{(Y_1-Y_0)^2}{4R_1^4}\frac{1}{m_0^2}\right)\frac{1}{m_0^2}\Delta X_{12}\Delta Y_{12}+ \tag{39}$$

$$+\frac{4(Y_1-Y_0)}{3R_1^3}\frac{1}{m_0^3}t_1\eta_1^2\Delta Y_{12}^2+\ldots$$

$$\alpha_{21}-A_{t21}=\delta_{21}, \qquad A_{t21}=\alpha_{21}-\delta_{21}$$

$$\frac{S}{s}=\frac{1}{m_0}\left[1-\frac{(Y_1-Y_0)^2}{2R_1^2}\frac{1}{m_0^2}+\frac{5(Y_1-Y_0)^4}{24R_1^4}\frac{1}{m_0^4}-\frac{61(Y_1-Y_0)^6}{720R_1^6}\frac{1}{m_0^6}+\right.$$

$$+\frac{(Y_1-Y_0)^2}{R_1^3}\frac{1}{m_0^3}t_1\eta_1^2\Delta X_{12}-\frac{(Y_1-Y_0)}{2R_1^2}\frac{1}{m_0^2}\Delta Y_{12}+\frac{5(Y_1-Y_0)^3}{12R_1^4}\frac{1}{m_0^4}\Delta Y_{12}+ \tag{40}$$

$$-\frac{(Y_1-Y_0)^2}{24R_1^4}\frac{1}{m_0^4}\Delta X_{12}^2+\frac{4(Y_1-Y_0)}{3R_1^3}\frac{1}{m_0^3}t_1\eta_1^2\Delta X_{12}\Delta Y_{12}+$$

$$-\frac{1}{6R_1^2m_0^2}\Delta Y_{12}^2+\frac{5(Y_1-Y_0)^2}{12R_1^4}\frac{1}{m_0^4}\Delta Y_{12}^2+\ldots\Bigg] \tag{40}$$

In the alternative version the following will be calculated in succession:
1) parameters B_m, L_m of the midpoint of the chord,
2) values of reductions of direction angles δ_{12} and δ_{21},
3) value of projecting reduction coefficient of length $\frac{S}{s}$.

$$\delta_{12}=-\frac{(Y_m-Y_0)}{2R_m^2}\frac{1}{m_0^2}\Delta X_{12}+\frac{1}{12R_m^2}\frac{1}{m_0^2}\Delta X_{12}\Delta Y_{12}+\frac{(Y_m-Y_0)^3}{6R_m^4}\frac{1}{m_0^4}\Delta X_{12}+$$

$$-\frac{(Y_m-Y_0)^2}{R_m^3}\frac{1}{m_0^3}t_m\eta_m^2\Delta Y_{12}-\frac{(Y_m-Y_0)}{3R_m^3}\frac{1}{m_0^3}t_m\eta_m^2\Delta X_{12}^2+ \tag{41}$$

$$+\frac{(Y_m-Y_0)^2}{3R_m^3}\frac{1}{m_0^3}t_m\eta_m^2\Delta Y_{12}^2+\ldots$$

$$\delta_{21}=-\frac{(Y_m-Y_0)}{2R_m^2}\frac{1}{m_0^2}\Delta X_{12}+\frac{1}{12R_m^2}\frac{1}{m_0^2}\Delta X_{12}\Delta Y_{12}-\frac{(Y_m-Y_0)^3}{6R_m^4}\frac{1}{m_0^4}\Delta X_{12}+$$

$$+\frac{(Y_m-Y_0)^2}{R_m^3}\frac{1}{m_0^3}t_m\eta_m^2\Delta Y_{12}-\frac{(Y_m-Y_0)}{3R_m^3}\frac{1}{m_0^3}t_m\eta_m^2\Delta X_{12}^2+ \tag{42}$$

$$+\frac{(Y_m-Y_0)}{3R_m^3}\frac{1}{m_0^3}t_m\eta_m^2\Delta Y_{12}^2+\ldots,$$

$$\frac{S}{s}=\frac{1}{m_0}\left[1-\frac{(Y_m-Y_0)^2}{2R_m^2}\frac{1}{m_0^2}-\frac{1}{24R_m^2}\frac{1}{m_0^2}\Delta Y_{12}^2+\frac{5(Y_m-Y_0)^4}{24R_m^4}\frac{1}{m_0^4}+\right.$$

$$-\frac{61(Y_m-Y_0)^6}{720R_m^6}\frac{1}{m_0^6}+\frac{(Y_m-Y_0)}{3R_m^3}\frac{1}{m_0^3}t_m\eta_m^2\Delta X_{12}\Delta Y_{12}+ \tag{43}$$

$$\left.-\frac{(Y_m-Y_0)^2}{24R_m^4}\frac{1}{m_0^4}\Delta X_{12}^2+\frac{5(Y_m-Y_0)^2}{48R_m^4}\frac{1}{m_0^4}\Delta Y_{12}^2+\ldots\right]$$

where:

$$t_m=\tan B_m, \quad \eta_m^2=\frac{e^2\cos^2 B_m}{(1-e^2)}, \quad R_m=\sqrt{M_m N_m}=\frac{a\sqrt{1-e^2}}{1-e^2\sin^2 B_m},$$

$$X_m=\frac{(X_1+X_2)}{2}, \quad Y_m=\frac{(Y_1+Y_2)}{2}$$

The sign of the value of angular reduction δ_{ik} depends on the position of point P_i according to the central meridian, and the value of azimuth (direction angle) α_{ik}, which is illustrated by Fig. 4 and Fig. 5, respectively.

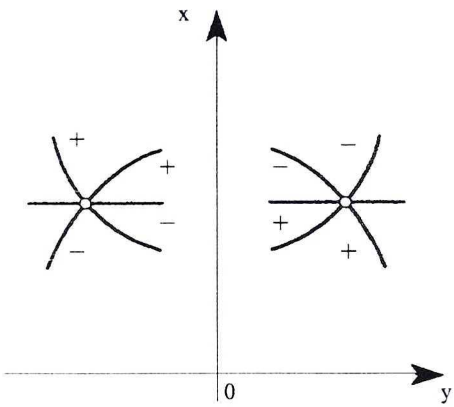

Fig. 4

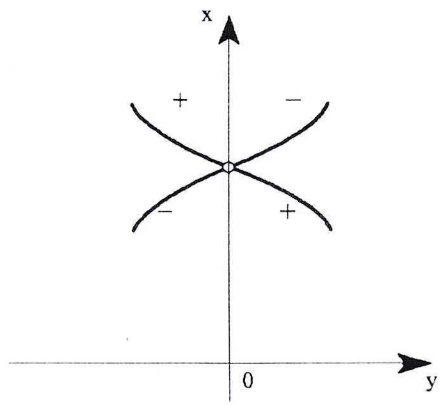

Fig. 5

2.2. Reductions of the area of an ellipsoid in the Gauss-Krüger projection

Integral growth of area dP of a given area on an ellipsoid is connected with its topological equivalent $d'P$ within a Gauss-Krüger projection plane by means of the following dependence:

$$d'P=m_0^2\, m_{GK}^2\, dP=m^2 dP$$

Area P' of the topological equivalent to ellipsoidal area P will by expressed by the following formula:

$$P'=\iint_P m^2(B,l)dP=\lim_{\Delta P_i\to 0}\sum_{i=1}^{\infty} m_i^2\Delta P_i$$

Area betwen image S' of a segment of the geodesic line and its chord s within the projection plane will be approximately expressed by the following formula:

$$\Delta' F=\frac{s^2}{4}\left(\frac{\delta-0.5\sin 2\delta}{\sin^2\delta}\right)$$

where s — length of chord P_1', P_2', $\delta=0.5(\delta_{12}+\delta_{21})$.

3. *Final remarks*

The foregoing calculating formulas of coordinates in the Gauss-Krüger projection make it possible to determine with geodetic accuracy the value of coordinates within the projecting zone of a width of up to 70°. Therefore, the width of the zone covering the area of the entire country might be considerably extended, accuracy not being lost. Such a situation may occur in case of special studies, reaching beyond Poland's boundaries.

Reduction formulas meant for calculation of corrections to reduction of direction and length secure geodetic accuracy within a range of up to 50 km. Works of a local range and distances up to 300 m may make use of simplified formulas, restricted down to the first few assends of partial sums, or even down to terms depending exclusively on the first powers of a local length scale.

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Odwzorowanie obszaru Polski w szerokiej strefie Gaussa-Krügera

S t r e s z c z e n i e

W pracy przedstawiono podstawowe formuły obliczania współrzędnych prostokątnych płaskich x, y w zadaniu prostym i współrzędnych geodezyjnych elipsoidalnych B, L w zadaniu odwrotnym, odwzorowania Gaussa-Krügera w szerokiej strefy południkowej. Odwzorowanie Gaussa-Krügera rozumiane jest tu jako odwzorowanie potrójne, polegające na przejściu z powierzchni elipsoidy obrotowej spłaszczonej jako powierzchni oryginału na powierzchnię kuli, zmianie układu współrzędnych powierzchni kuli i odworowaniu powierzchni kuli w płaszczyznę. Przedstawiono także wzory pozwalające na obliczanie wartości zbieżności południków, lokalnych skal długości, pól i kątów oraz redukcji odwzorowawczych figur geodezyjnych.

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Отображение территории Польши в широкой зоне Гаусса-Крюгера

Резюме

В работе представлены основные формулы вычисления координат, в прямой и обратной задаче, отображения Гаусса-Крюгера в широкой меридианной зоне. Представлены тоже формулы, дающие возможность вычисления величин местных деформаций полей, а также редукций направлений, углов и длин.