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Research paper

Extended Force Density Method for cable nets under self-weight. Part I – Theory and verification

Izabela Wójcik-Grząba[1]

Abstract: This paper presents the Extended Force Density Method which allows for form-finding of cable nets under self-weight. Formulation of the method is based on the curved catenary cable element which assures high accuracy of the results and enables solving wide range of problems. Essential rules of the Force Density Method (FDM) are summarized in the paper. Some well-known formula describing behaviour of a catenary cable element under self-weight are given. Next the improved variant of FDM with all the theoretical and numerical details is introduced. Iterative procedure for solving nonlinear equations is described. Finally a simple verification example proves correctness of methods assumptions. Two further analyses of parameters crucial for correct use of Extended Force Density Method (EFDM) are presented in order to indicate their initial values for other numerical examples. Accuracy of the results are also investigated. A computer program UC-Form was developed in order to perform the calculations and graphically present the results. Some examples of use of EFDM are presented in details in *Part II – Examples of application*.

Keywords: cable nets, form-finding, extended force density method, self-weight

[1]PhD, Eng., Warsaw University of Technology, Faculty of Civil Engineering, al. Armii Ludowej 16, 00-637 Warsaw, Poland, e-mail: iwoj@il.pw.edu.pl, ORCID: 0000-0003-3769-8441

1. Introduction

Cable structures are widely used structural systems providing economical and visually attractive solutions for large span structures like stadiums, sports and concert halls or exhibition pavilions. Due to the facts that cable elements work only in tension and are made of high strength steel wire ropes such structures obtain high cross-section efficiency and low self-weight, especially in the case of large spans [7].

On the other hand there are also some disadvantages, especially associated with design process of cable structures. These appear mainly due to geometrically nonlinear behaviour resulting from large displacements. What is more, cable structures require a preliminary phase of design which is called form-finding in order to establish an initial geometry and tension forces. Finite Element Method, which is the most popular method for static and dynamic analysis, is not efficient for the form-finding process because of the initial geometric instability of the structure and lack of exact cable elements in many commercial programs. Thus, since 1970s there have been made a great effort to develop some new numerical methods for finding initial shape of tensile structures. The most popular are: Transient Stiffness Method [1], Dynamic Relaxation Method [5] and Force Density Method [9]. The main rules of these methods and their comparison was presented by Veenendaal and Block in the paper [11]. The latter method is still being improved and extended, particularly in order to introduce self-weight of cable elements into analysis. There are also some attempts to use FDM in optimisation problems, especially for Michell structures (only in tension or only in compression) [2].

In most cases of correctly designed cable net tension forces in elements should ensure proper spatial stiffness in order to limit displacements. In such situation self-weight of cables is negligibly small compared to the live loads and should have small influence on geometry and forces, see [7]. However, when we deal with long cable elements with large cross-sectional area, self-weight should be included from the beginning of the design process. Similarly, self-weight has great importance in structures with slack cables as main structural elements. Such elements can also appear during erection or after removal of elements as a result of failure or planned action. Correct distribution of forces resulting from self-weight of slack and taut elements is crucial to obtain real geometry and forces in the whole structure. Thera are few papers dealing with this problem. In [4] and [6] authors applied point loads equal to half of elements self-weight which is correct only in the case of taut elements. However in the first paper correction in the force density definition accounts for the exact values of reactions. In the second paper a parabolic (approximate) formulation of cable element is utilized. In both cases the initial FDM system of equations is supplied by additional equations governing behaviour of a cable element. In the paper [3] force density is based on the force value in one end of the element. Further details of method are not clearly presented. Regarding these drawbacks, there is a need for a universal and accurate method of finding shape and forces of cable nets consisting of slack and taut elements under self-weight and external nodal loads.

2. Materials and methods

2.1. The basics of the Force Density Method

Due to the Schek's concept of the Force Density Method [9] a cable net is a system consisting of linear, weightless elements connected by nodes which can be anchored (fixed) or free. Point loads are applied in chosen free nodes. Topology of a cable net is defined by the incidence matrix [g$\boldsymbol{C}\overline{C}$] in which columns of the first submatrix represent free nodes and the second submatrix – fixed nodes. Each row of the matrix is connected with one element and indicates which node is the beginning (number 1) and end (number –1) of the element. For example a simple net and its incidence matrix is presented in Fig. 1. Three rows of matrix correspond to three elements, first column represents a free node 1 next three columns represents subsequent fixed nodes.

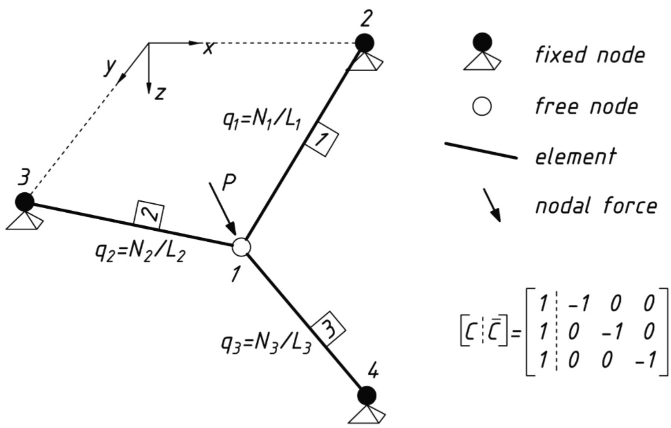

Fig. 1. Example of cable net and its incidence matrix

Using the definition of incidence matrix we can write down the formulas for vectors of element length projections at three axes, where **xyz** are the coordinate vectors of free nodes and $\overline{xyz}$ – of fixed nodes.

$$\boldsymbol{x}_\Delta = \boldsymbol{C}\boldsymbol{x} + \overline{\boldsymbol{C}\boldsymbol{x}}, \qquad \boldsymbol{y}_\Delta = \boldsymbol{C}\boldsymbol{y} + \overline{\boldsymbol{C}\boldsymbol{y}}, \qquad \boldsymbol{z}_\Delta = \boldsymbol{C}\boldsymbol{z} + \overline{\boldsymbol{C}\boldsymbol{z}} \tag{2.1}$$

The main idea of the method is to find coordinates of free nodes satisfying equilibrium equations in each direction presented below:

$$\begin{cases} \boldsymbol{C}^T \boldsymbol{X}_\Delta \boldsymbol{L}^{-1} \boldsymbol{n} = \boldsymbol{p}_x \\ \boldsymbol{C}^T \boldsymbol{Y}_\Delta \boldsymbol{L}^{-1} \boldsymbol{n} = \boldsymbol{p}_y \\ \boldsymbol{C}^T \boldsymbol{Z}_\Delta \boldsymbol{L}^{-1} \boldsymbol{n} = \boldsymbol{p}_z \end{cases} \tag{2.2}$$

where $\boldsymbol{X}_\Delta$, $\boldsymbol{Y}_\Delta$, $\boldsymbol{Z}_\Delta$ are diagonal matrices of element length projections at x-, y-, and z-axes, $\boldsymbol{L}$ is a diagonal matrix of element lengths, $\boldsymbol{n}$ is a column vector of element forces and $\boldsymbol{p}_x, \boldsymbol{p}_y, \boldsymbol{p}_z$

are column vectors of x, y and z components of nodal loads. Taking the definition of element lengths (2.3) into account it can be seen that above equations (2.2) are nonlinear with regard to free nodes coordinates:

$$\boldsymbol{L} = \left(\boldsymbol{X}_\Delta^T \boldsymbol{X}_\Delta + \boldsymbol{Y}_\Delta^T \boldsymbol{Y}_\Delta + \boldsymbol{Z}_\Delta^T \boldsymbol{Z}_\Delta\right)^{\frac{1}{2}} \tag{2.3}$$

In order to rewrite the equilibrium equations in a linear form a force density vector is introduced as shown below:

$$\boldsymbol{q} = \boldsymbol{L}^{-1}\boldsymbol{n} \tag{2.4}$$

With the aid of equations (2.1) and (2.4) the unknown free nodes coordinates can be found from the equilibrium equations (2.2): To simplify the formulas auxiliary matrices are introduced: $\boldsymbol{D} = \boldsymbol{C}^T\boldsymbol{Q}\boldsymbol{C}$, $\overline{D} = \boldsymbol{C}^T\boldsymbol{Q}\overline{C}$, where $\boldsymbol{Q}$ is a diagonal matrix of force densities imposed in the elements. Thus, each set of force density values yields a different configuration of a given cable net.

For further details regarding the Force Density Method [9].

2.2. Catenary cable element

Flexural and shear stiffness in structural cable elements are assumed to be zero which means that we consider them as working only in tension. However such elements can carry transverse loads and in the case of self-weight they take form of a catenary line with variable tensile force.

A cable element along with symbols used subsequently in this paper is presented in Fig. 2. In this case tensile stiffness of the element is assumed to be infinite which is an auxiliary problem for solving a general problem of an elastic cable with stiffness EA. The solution of

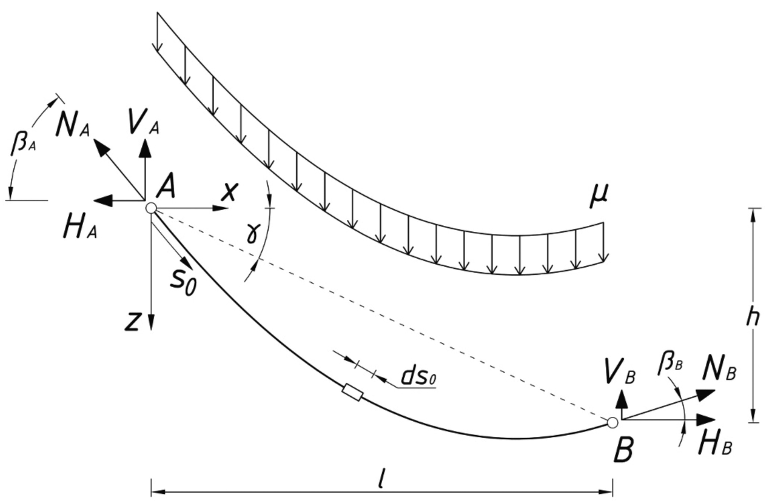

Fig. 2. Cable element under self-weight

differential equations of equilibrium for an elastic cable is given below [10]:

$$
\begin{aligned}
x(s_0) &= \frac{H s_0}{EA} + \frac{H}{\mu}\left[\operatorname{arsinh}\left(\frac{V_A}{H}\right) + \operatorname{arsinh}\left(\frac{\mu s_0 - V_A}{H}\right)\right] \\
z(s_0) &= \frac{V_A s_0}{EA}\left(1 - \frac{\mu s_0}{2V_A}\right) + \frac{H}{\mu}\left[\sqrt{1+\left(\frac{V_A}{H}\right)^2} - \sqrt{1+\left(\frac{V_A - \mu s_0}{H}\right)^2}\right]
\end{aligned}
\tag{2.5}
$$

where: $s_0(x) = \dfrac{H}{\mu}\left[\sinh\left(\dfrac{\mu}{H}x - \zeta\right) + \sinh\zeta\right]$, $\zeta = \operatorname{arsinh}\left(\dfrac{h\eta}{l\sinh\eta}\right) + \eta$, $\eta = \dfrac{\mu l}{2H}$.

As it can be seen in eq. (2.5), the coordinates of points lying on the catenary line are functions of s_0 which is a natural coordinate measuring the length of unstretched cable under self-weight (see Fig. 2). Unknown components of tensile force in a beginning point A of a cable (V_A and H) can be found with the aid of boundary conditions for an end point B: $x = l$ for $s_0 = L_0$, $z = h$ for $s_0 = L_0$.

Eventually, two equations for V_A and H in the elastic catenary with initial (unstretched) length L_0 are presented below:

$$
\begin{aligned}
l &= \frac{H L_0}{EA} + \frac{H}{\mu}\left[\operatorname{arsinh}\left(\frac{V_A}{H}\right) + \operatorname{arsinh}\left(\frac{\mu L_0 - V_A}{H}\right)\right] \\
h &= \frac{V_A L_0}{EA}\left(1 - \frac{\mu L_0}{2V_A}\right) + \frac{H}{\mu}\left[\sqrt{1+\left(\frac{V_A}{H}\right)^2} - \sqrt{1+\left(\frac{V_A - \mu L_0}{H}\right)^2}\right]
\end{aligned}
\tag{2.6}
$$

A function of tensile force in a catenary cable element is given by formula:

$$
N(x) = H\cosh\left(\frac{\mu}{H}x - \frac{\mu l}{2H}\right) \tag{2.7}
$$

Maximum force value occurs always on the upper support and minimum force value occurs in the lowest point of a cable. Horizontal force component is constant along cable because of lack of horizontal load.

3. Theory and calculations

3.1. Extended Force Density Method – stage 1

As the original Force Density Method presented in Section 2.1 has some limitations, the new extended version will be proposed. In the first stage the Force Density Method equations are rephrased in order to simplify the structure definition and to add sliding supports.

As it was shown in Section 2.1 the structure of incidence matrix depends on the boundary conditions because free nodes are always numbered starting from 1. This is not very convenient, especially when we use geometrical data from other programs or we want to shift the fixed nodes. Therefore, in the Extended Force Density Method order of columns in incidence matrix is arbitrary and only after defining numbers of fixed nodes the submatrices $\boldsymbol{C}$, $\overline{\boldsymbol{C}}$ can be built by picking proper columns.

A second disadvantage of the original Force Density Method is only one type of boundary condition which is a hinged support. Adding sliding supports enables defining the symmetry condition in order to reduce the size of a structure model when it is possible. Seven types of sliding supports corresponding to different displacement directions blocked can be distinguished, i.e.: xyz, xy, yz, xz, x, y, z. In such situation different nodes can be free or fixed in different directions. In order to include all the possible types of supports in the Extended Force Density Method submatrices of incidence matrix are defined separately for each direction, i.e.: $\boldsymbol{C}_x, \boldsymbol{C}_y, \boldsymbol{C}_z, \overline{\boldsymbol{C}}_x, \overline{\boldsymbol{C}}_y, \overline{\boldsymbol{C}}_z$. A new system of equilibrium equations is given below:

$$\boldsymbol{x} = \boldsymbol{D}_x^{-1}\left(\boldsymbol{p}_x - \overline{\boldsymbol{D}}_x\overline{\boldsymbol{x}}\right), \qquad \boldsymbol{y} = \boldsymbol{D}_y^{-1}\left(\boldsymbol{p}_y - \overline{\boldsymbol{D}}_y\overline{\boldsymbol{y}}\right), \qquad \boldsymbol{z} = \boldsymbol{D}_z^{-1}\left(\boldsymbol{p}_z - \overline{\boldsymbol{D}}_z\overline{\boldsymbol{z}}\right) \tag{3.1}$$

where: $\boldsymbol{D}_x = \boldsymbol{C}_x^T\boldsymbol{Q}\boldsymbol{C}_x$, $\boldsymbol{D}_y = \boldsymbol{C}_y^T\boldsymbol{Q}\boldsymbol{C}_y$, $\boldsymbol{D}_z = \boldsymbol{C}_z^T\boldsymbol{Q}\boldsymbol{C}_z$, $\overline{\boldsymbol{D}}_x = \boldsymbol{C}_x^T\boldsymbol{Q}\overline{\boldsymbol{C}}_x$, $\overline{\boldsymbol{D}}_y = \boldsymbol{C}_y^T\boldsymbol{Q}\overline{\boldsymbol{C}}_y$, $\overline{\boldsymbol{D}}_z = \boldsymbol{C}_z^T\boldsymbol{Q}\overline{\boldsymbol{C}}_z$.

In the general case number of equations for each direction can be different.

3.2. Extended Force Density Method – stage 2

3.2.1. Introduction

The second and more important stage of enhancement of the Force Density Method is its generalization for nets under self-weight consisting of slack and taut cables. A system of equilibrium equations (3.1) and formulas for elastic catenary cable (2.6) are the basis for proposed Extended Force Density Method.

The main idea of the original version of the method is a concept of force density which is a ratio of tensile force to length of a straight element. As it was shown in the Section 2.2, applying self-weight to a cable element causes a change in the geometry from straight to catenary line and in the tensile force function from constant to 3. Therefore, a substitutive

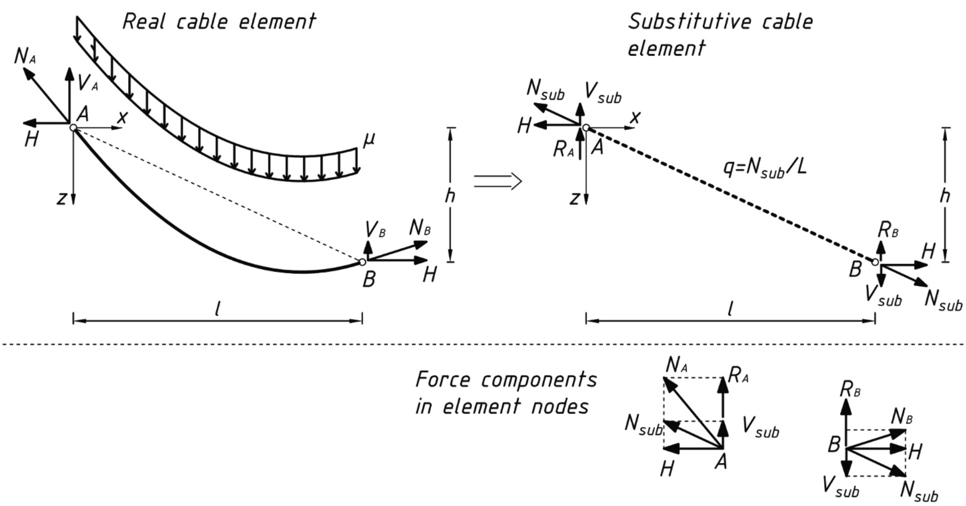

Fig. 3. Real and substitutive cable element under self-weight and force components in A and B nodes

element is introduced as shown in Fig. 3. It is statically equivalent to the real one because of the force components applied to the nodes. Forces N_A and N_B in the beginning and ending of cable are decomposed to a constant part N_{sub} and variable parts (R_A, R_B) which are the reaction forces from self-weight. New definition of the force density is based upon a constant component and due to proportion between forces and dimensions we can introduce the formula for q:

$$q = \frac{N_{\text{sub}}}{L} = \frac{V_{\text{sub}}}{h} = \frac{H}{l} \tag{3.2}$$

According to assumed signs of vertical force components we can write down the relations:

$$V_A = V_{\text{sub}} + R_A, \qquad V_B = V_{\text{sub}} - R_B \tag{3.3}$$

In order to implement the presented idea of a substitutive element to the Force Density Method two steps are necessary:

1) modify Extended Force Density Method equation for z direction by adding a vector of reactions from self-weight $\boldsymbol{p}_r$ to external load vector $\boldsymbol{p}_z$: $\boldsymbol{D}_z\boldsymbol{z} = \boldsymbol{p}_z + \boldsymbol{p}_r - \overline{\boldsymbol{D}}_z\bar{\boldsymbol{z}}$;
2) find force density vector $\boldsymbol{q}$ for a cable net under self-weight.

These steps will be presented in the following subsections.

3.2.2. Finding reactions from cable self-weight

According to Pałkowski [8] reaction forces from self-weight in a cable element can be calculated with the aid of an auxiliary simply-supported beam of a span equal to the horizontal span of a cable (see Fig. 4).

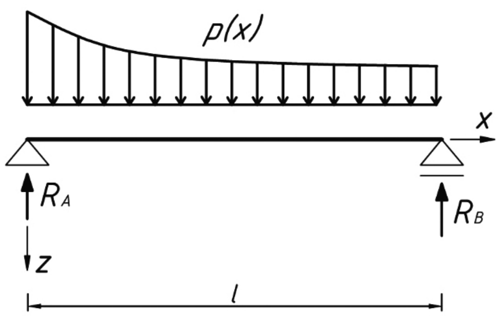

Fig. 4. Auxiliary beam for finding reaction forces

Beam load is a horizontal projection of cable self-weight:

$$p(x) = \mu_s \frac{\mathrm{d}s}{\mathrm{d}x} = \mu \frac{\mathrm{d}s_0}{\mathrm{d}x} \tag{3.4}$$

where s denotes a natural coordinate measuring length of stretched cable, μ_s and μ denote self-weight of stretched and unstretched cable in kN/m. The second equality in (3.4) means that self-weight of an infinitesimal segment is conserved after stretching.

Reaction forces in the auxiliary beam under given load are as shown below:

$$R_A = \mu L_0 - R_B, \qquad R_B = \frac{1}{l}\int_0^l p(x)x\,\mathrm{d}x = \frac{\mu}{l}\int_0^l x(s_0)\frac{\mathrm{d}s_0}{\mathrm{d}x}\,\mathrm{d}x \tag{3.5}$$

In order to simplify calculation of R_A and R_B it is assumed that $x(s_0) = x$, so here we consider unstretched cable and the reaction force can be given as:

$$R_B = \frac{q^2 l}{\mu}\left[2\eta \sinh(2\eta - \zeta) - \cosh(2\eta - \zeta) + \cosh\zeta\right] \tag{3.6}$$

with: $\eta = \dfrac{\mu l}{2H} = \dfrac{\mu}{2q}$, $\zeta = \operatorname{arsinh}\left(\dfrac{h\eta}{l\sinh\eta}\right) + \eta$, $h = z_B - z_A$, $l = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}$.

To define a vector of reaction forces in free nodes of a cable net we must sum up the reactions from each element joined to the node. It will be convenient to define new matrices consisting of zeros and ones on the basis of the matrix $\boldsymbol{C}_z = \boldsymbol{C}_{z,A} - \boldsymbol{C}_{z,B}$. Matrix $\boldsymbol{C}_{z,A}$ has ones in columns representing beginning nodes and $\boldsymbol{C}_{z,B}$ in columns representing ending nodes of elements. Then, according to equation (3.5), vector $\boldsymbol{p}_r$ can be defined as shown below:

$$\boldsymbol{p}_r = \boldsymbol{C}_{z,A}^T\boldsymbol{r}_A + \boldsymbol{C}_{z,B}^T\boldsymbol{r}_B = \boldsymbol{C}_{z,A}^T(\boldsymbol{w} - \boldsymbol{r}_B) + \boldsymbol{C}_{z,B}^T\boldsymbol{r}_B = \boldsymbol{C}_{z,A}^T\boldsymbol{w} - \boldsymbol{C}_z^T\boldsymbol{r}_B$$

where $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ are vectors of reaction forces in beginning and ending nodes of elements, and vector $\boldsymbol{w}$ consists of values of total weight of each element.

Because of the simplifying assumption stated above reaction force calculated according to formula (3.6) is not exact. It can be demonstrated that for slack elements relative error is smaller than 0.05% (compared to an exact solution). For taut elements comparably high accuracy is reached with a reaction force equal to half of a total element weight. Therefore, in EFDM vectors $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ contain values of reaction forces calculated either by formula (3.6) for slack or as $\mu L_0/2$ for taut elements.

System of equilibrium equations in EFDM is given below:

$$\begin{cases} \boldsymbol{D}_x\boldsymbol{x} = \boldsymbol{p}_x - \overline{\boldsymbol{D}}_x\overline{\boldsymbol{x}} \\ \boldsymbol{D}_y\boldsymbol{y} = \boldsymbol{p}_y - \overline{\boldsymbol{D}}_y\overline{\boldsymbol{y}} \\ \boldsymbol{D}_z\boldsymbol{z} = \boldsymbol{p}_z + \boldsymbol{p}_r - \overline{\boldsymbol{D}}_z\overline{\boldsymbol{z}} \end{cases} \tag{3.7}$$

3.2.3. Finding force density vector under self-weight

Purely geometric FDM problem can be summarized as:

find a cable net configuration under given point loads and assumed force densities satisfying equilibrium equations in each free node.

Searching for a configuration of the same net under self-weight demands calculating the force densities which corresponds to this unique geometry. It can be achieved by solving additional system of equations characterizing elastic cables element under self-weight. Substituting (3.2), (3.3), $(3.5)_1$ and (3.6) to equations (2.6) turns $(2.6)_2$ to identity and leaves one constraining equation for each cable element $(2.6)_1$:

$$g_w\left(x_{A,B}(q), y_{A,B}(q), z_{A,B}(q), q\right) = \frac{HL_0}{EA} + \frac{H}{\mu}\left[\operatorname{arsinh}\left(\frac{V_A}{H}\right) + \operatorname{arsinh}\left(\frac{\mu L_0 - V_A}{H}\right)\right] - l \tag{3.8}$$

where: $H = H(q)$, $V_A = V_A\left(x_{(A,B)}(q), y_{(A,B)}(q), z_{(A,B)}(q), q\right)$, $l = l\left(x_{(A,B)}(q), y_{(A,B)}(q)\right)$

For the whole cable net we get a system of nonlinear equations defined by (3.8):

$$\boldsymbol{g}(\boldsymbol{q}, \boldsymbol{x}(\boldsymbol{q}), \boldsymbol{y}(\boldsymbol{q}), \boldsymbol{z}(\boldsymbol{q})) = \boldsymbol{0} \tag{3.9}$$

Let's assume that initial configuration of the net is represented by the force density vector $\boldsymbol{q}_0$ which is not equal to unknown vector $\boldsymbol{q}_w$ for structure under self-weight. Starting from initial vector $\boldsymbol{q}_0$ we can implement the iterative Newton procedure which assumes linearization of equations in every step according to the formula: $\boldsymbol{g}(\boldsymbol{q}_0) + \left.\frac{\mathrm{d}\boldsymbol{g}}{\mathrm{d}\boldsymbol{q}}\right|_{\boldsymbol{q}=\boldsymbol{q}_0} \Delta\boldsymbol{q} = 0$.

The solution of this linear problem may be given as:

$$\Delta\boldsymbol{q} = \boldsymbol{G}^{-1}\boldsymbol{b} \tag{3.10}$$

where: $\boldsymbol{G} = \left.\frac{\mathrm{d}\boldsymbol{g}}{\mathrm{d}\boldsymbol{q}}\right|_{\boldsymbol{q}=\boldsymbol{q}_0}$, $\boldsymbol{b} = -\boldsymbol{g}(\boldsymbol{q}_0)$.

Because constraint function $\boldsymbol{g}$ depends on force density vector $\boldsymbol{q}$ directly and indirectly through coordinates $\boldsymbol{x}(\boldsymbol{q}), \boldsymbol{y}(\boldsymbol{q}), \boldsymbol{z}(\boldsymbol{q})$ matrix $\boldsymbol{G}$ will be calculated with the aid of chain rule:

$$\boldsymbol{G} = \frac{\mathrm{d}\boldsymbol{g}}{\mathrm{d}\boldsymbol{q}} = \frac{\partial\boldsymbol{g}}{\partial\boldsymbol{x}}\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{q}} + \frac{\partial\boldsymbol{g}}{\partial\boldsymbol{y}}\frac{\partial\boldsymbol{y}}{\partial\boldsymbol{q}} + \frac{\partial\boldsymbol{g}}{\partial\boldsymbol{z}}\frac{\partial\boldsymbol{z}}{\partial\boldsymbol{q}} + \frac{\partial\boldsymbol{g}}{\partial\boldsymbol{q}} \tag{3.11}$$

where, after Schek [9], and taking into account adjustments introduced in this section we have:

$$\frac{\partial\boldsymbol{x}}{\partial\boldsymbol{q}} = -\boldsymbol{D}_x^{-1}\boldsymbol{C}_x^T\boldsymbol{X}_\Delta, \qquad \frac{\partial\boldsymbol{y}}{\partial\boldsymbol{q}} = -\boldsymbol{D}_y^{-1}\boldsymbol{C}_y^T\boldsymbol{Y}_\Delta, \qquad \frac{\partial\boldsymbol{z}}{\partial\boldsymbol{q}} = -\left(\boldsymbol{D}_z - \frac{\partial\boldsymbol{p}_r}{\partial\boldsymbol{z}}\right)^{-1}\left(\boldsymbol{C}_z^T\boldsymbol{Z}_\Delta - \frac{\partial\boldsymbol{p}_r}{\partial\boldsymbol{q}}\right)$$

The rest of required in (3.11) derivatives were derived by the author and are included in her PhD thesis [12] but due to their complex form will not be quoted here.

As it was mentioned in subsection 3.2.2, vector of reaction forces from self-weight $\boldsymbol{p}_r$ is calculated approximately. In the PhD thesis [12] it was demonstrated that relative error of vector $\boldsymbol{q}$ calculated according to presented above method is lower than 0.02% for majority of a cable element configurations. Only a cable with particular and rare geometry ($L = L_0$ and large values of l and h) can give a relative error of almost 2% compared to the exact solution.

3.2.4. EFDM – iterative procedure

Finding cable net configuration under self-weight with the use of system of equations (3.7) requires prior determining the force density vector satisfying the (3.9) restrictions. That can be achieved with the iterative procedure described below:

1) assume initial vector of force densities $\boldsymbol{q}_0$ and find initial geometry of cable net with (3.7);
2) determine a vector of force density increments $\Delta\boldsymbol{q}$ from (3.10);

3) calculate a new force density vector: $\boldsymbol{q}_1 = \boldsymbol{q}_0 + \Delta\boldsymbol{q}$ and find new geometry with (3.7);
4) if stop condition is satisfied finish calculations, if not – go to step 2.

A stop condition is used to check if achieved accuracy of solution is sufficient. If $\boldsymbol{q}_w$ is the exact solution, and $\boldsymbol{q}_i$ is a current solution we need to check if:

$$|\boldsymbol{g}\,(\boldsymbol{q}_w)\, g - g\boldsymbol{g}\,(\boldsymbol{q}_i)|\, g \leq \boldsymbol{\varepsilon} \tag{3.12}$$

where $\boldsymbol{\varepsilon}$ is a vector of allowable error defined by the user.

4. Results and discussion of verification and tests

4.1. Computer program

In order to perform calculations by proposed here Extended Force Density Method a computer program UC-Form was developed by the author. Program consists of 25 Scilab files which execute calculations and present results in graphical form and one auxiliary MS Excel file which helps to define input data in most convenient way. There are three main paths of executing calculations to choose by the user:

1) form-finding without self-weight according to subsection 3.1;
2) form-finding without self-weight according to subsection 3.1 and with additional constraints (for details see [9] and [12]);
3) form-finding with self-weight according to subsection 3.2.

4.2. Verification of EFDM

In order to verify the accuracy of proposed EFDM a simple example of a cable element under self-weight is considered. For EFDM procedure a cable is divided into two parts in order to get a value of maximum slack z_2 (Fig. 5) and compare it with the exact value.

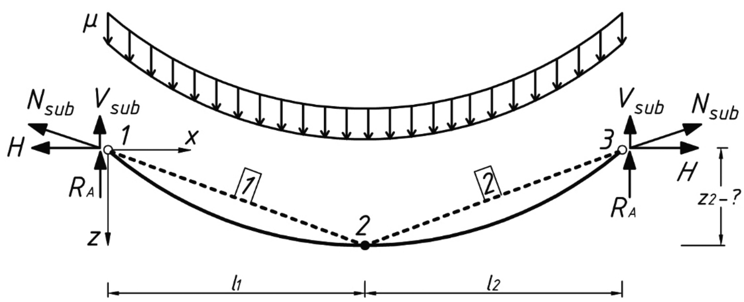

Fig. 5. Verification example

Both elements are made of the same type of steel wire rope with given properties: $\mu = 0.008$ kN/m, $E = 160 \cdot 10^6$ kN/m^2, $A = 80 \cdot 10^{-6}$ m^2, $L_0 = 5.5$ m and span of supports is $l_1 + l_2 = 5 + 5 = 10.0$ m.

Exact catenary shape has maximum value of slack: $z_{exact} = 2003.061$ mm Horizontal cable force component is: $H_{exact} = 0.052$ kN and reaction force from self-weight is: $R_{A,exact} = 0.023$ kN. According to these values force density is: $q_{exact} = 0.0105$ kN/m.

The results of calculation with EFDM for initial value $q_0 = 1.0$ kN/m are presented in Table 1 (iterations from 6 to 9) and in Fig. 6.

Table 1. Results of EFDM verification

Size	Iteration no. / calculation error (m)			
	$6 / 2.5 \cdot 10^{-2}$	$7 / 9.8 \cdot 10^{-4}$	$8 / 1.3 \cdot 10^{-6}$	$9 / 2.6 \cdot 10^{-12}$
z_2 (mm)	1932.498	2005.808	2003.074	2003.071
R_A (kN)	0.023	0.023	0.023	0.023
q (kN/m)	$1.083 \cdot 10^{-2}$	$1.047 \cdot 10^{-2}$	$1.048 \cdot 10^{-2}$	$1.048 \cdot 10^{-2}$

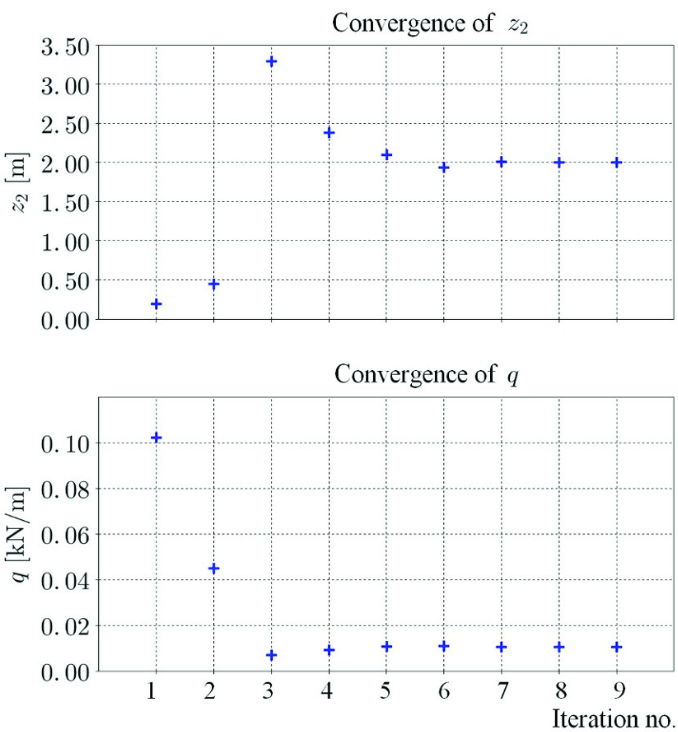

Fig. 6. Convergence of results in verification example

As it is shown on the graphs below (Fig. 6), both maximum slack and force density converge to exact values which proves that theory and numerical implementation of EFDM provide good results for catenary cables under self-weight.

4.3. Influence of initial force density values on convergence of EFDM

Because EFDM utilizes iterative procedure to find force density values in cable elements under self-weight it is crucial to investigate the influence of initial force density values on convergence and accuracy. A simple cable net shown in Fig. 7 is employed for this purpose.

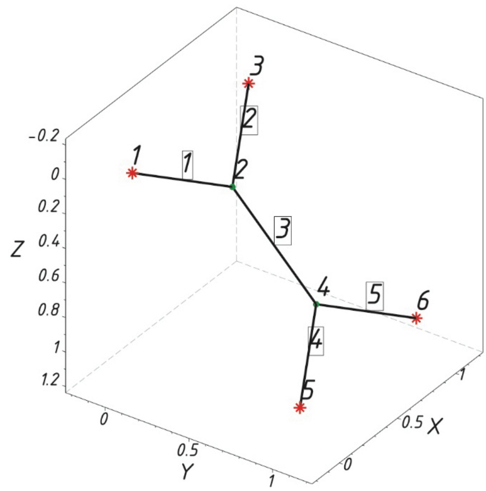

Fig. 7. Cable net for initial force densities analysis

Coordinates of fixed nodes are given in Table 2. Self-weight and tensile stiffness of a rope are assumed as: $\mu = 0.1$ kN/m, $EA = 12800$ kN. Initial lengths of elements are assumed as: $L_{01} = 0.67$ m, $L_{02} = 0.67$ m, $L_{03} = 0.55$ m, $L_{04} = 0.60$ m, $L_{05} = 1.29/0.95/0.60$ m. In the fifth cable three different values of initial length are applied to analyse different possible configurations.

Table 2. Coordinates of fixed nodes

No. of fixed node	Coordinates (m)		
	x	y	z
1	0.0	0.0	0.0
3	0.5	0.5	0.0
5	0.0	1.0	1.0
6	1.0	1.0	1.0

In each case initial force density value q_0 is the same in every element. Allowable error of EFDM calculations is assumed $\varepsilon = 1 \cdot 10^{-3}$ m. Table 3 summarizes number of iterations and calculation errors obtained for different initial force density values.

Calculation error is lower than allowable value in each case of performed calculations except from the one with $L_{05} = 1.29$ m and $q_0 = 0.1$ m. In this case a singular matrix appeared in the ninth iteration and there was no solution achieved. The lowest numbers of iterations in the first

Table 3. Number of iterations and calculation errors

q_0 (kN/m)	L_{05} = 1.29 m		L_{05} = 0.95 m		L_{05} = 0.60 m	
	Number of iterations	Calculation error (m)	Number of iterations	Calculation error (m)	Number of iterations	Calculation error (m)
20	52	$8.77 \cdot 10^{-4}$	11	$8.62 \cdot 10^{-4}$	3	$4.21 \cdot 10^{-4}$
15	30	$4.68 \cdot 10^{-4}$	16	$8.30 \cdot 10^{-4}$	3	$4.17 \cdot 10^{-4}$
10	46	$8.81 \cdot 10^{-4}$	8	$3.50 \cdot 10^{-4}$	3	$8.02 \cdot 10^{-4}$
5	36	$5.08 \cdot 10^{-4}$	13	$3.13 \cdot 10^{-4}$	4	$3.35 \cdot 10^{-5}$
2	37	$9.71 \cdot 10^{-4}$	34	$8.33 \cdot 10^{-4}$	4	$2.62 \cdot 10^{-4}$
1	206	$9.13 \cdot 10^{-4}$	15	$4.31 \cdot 10^{-4}$	4	$3.01 \cdot 10^{-4}$
0.5	5	$9.62 \cdot 10^{-4}$	5	$8.32 \cdot 10^{-4}$	4	$2.36 \cdot 10^{-4}$
0.2	15	$4.20 \cdot 10^{-4}$	6	$9.86 \cdot 10^{-4}$	5	$3.53 \cdot 10^{-4}$
0.1	–	–	5	$9.50 \cdot 10^{-4}$	7	$3.28 \cdot 10^{-4}$
0.05	7	$8.63 \cdot 10^{-4}$	6	$6.07 \cdot 10^{-4}$	8	$6.87 \cdot 10^{-4}$
0.01	41	$4.33 \cdot 10^{-4}$	9	$5.36 \cdot 10^{-4}$	6	$6.70 \cdot 10^{-4}$
0.005	84	$6.27 \cdot 10^{-4}$	21	$5.46 \cdot 10^{-4}$	4	$4.99 \cdot 10^{-4}$

case are associated with initial force density values close to the final ones but the variability is not regular. It can be attributed to the fact that final values are different for each element. In the second case tendency is similar to the first case but the iteration numbers are lower and more regularly distributed. In the third case, where completely prestressed cable net is achieved, the iteration numbers are low in each analysed case of initial force density.

A graph in Fig. 8 summarizes obtained above results. It can be observed that initial force density values close to the final ones contribute to lower number of iterations, although with some exceptions. These irregularities result from different values of final force densities in

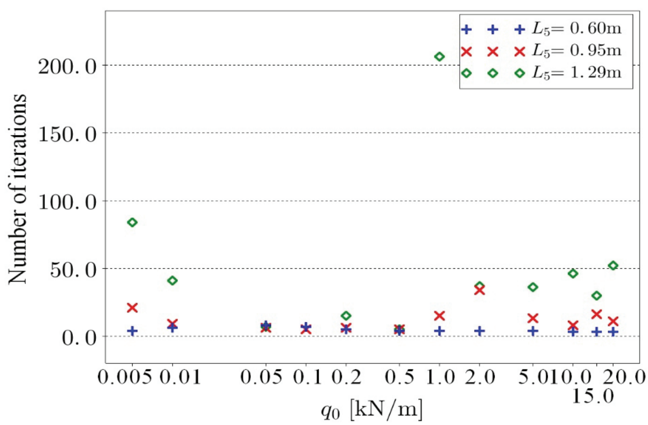

Fig. 8. Dependence of number of iterations on initial force density value

different elements and from iterative procedure in which elements may change from slack to taut (or vice versa) in subsequent steps. It can be noticed that more prestressed cable net corresponds to more flat q_0 distribution. All three final configurations with force density values are shown in Fig. 9.

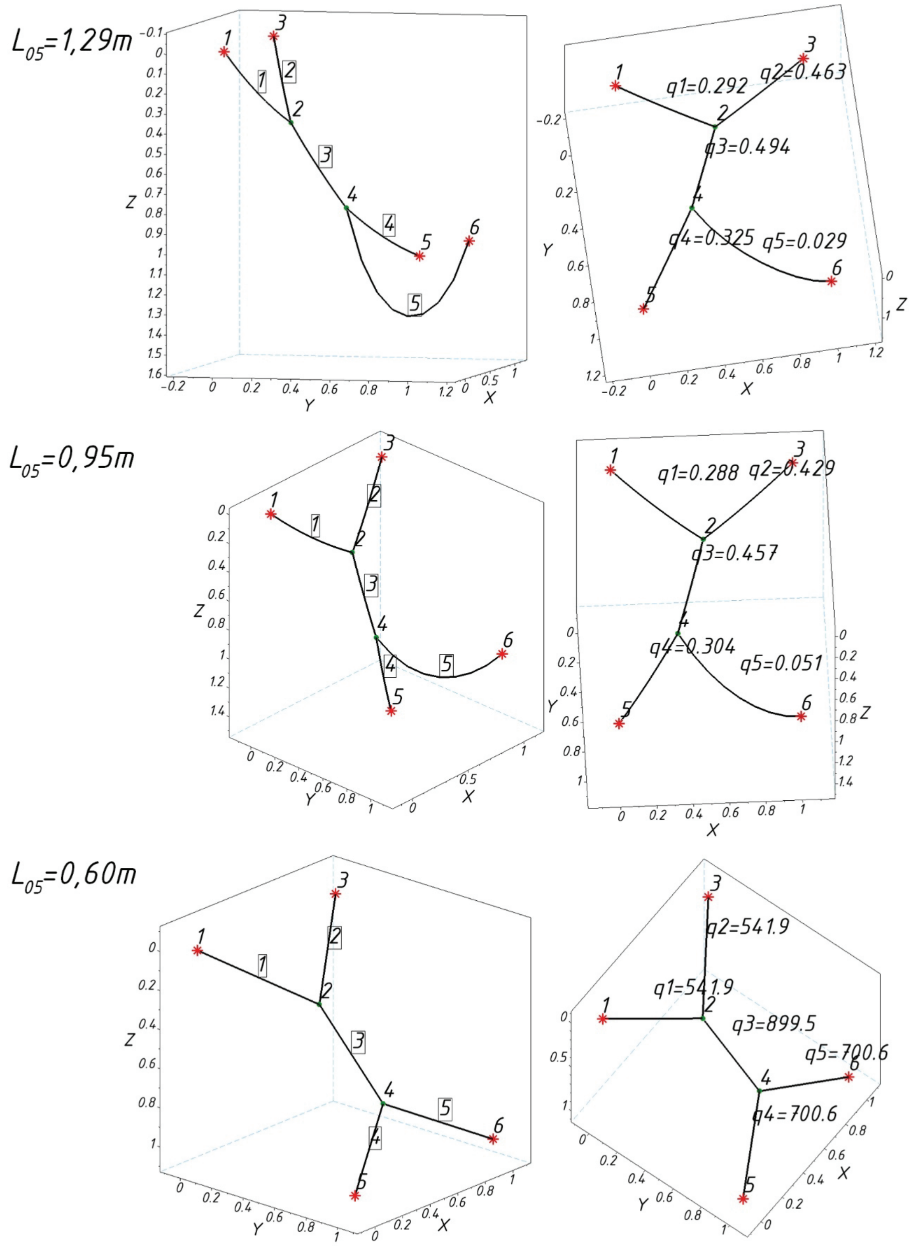

Fig. 9. Final configuration and force densities for $L_{05} = 1.29$ m

Taking above results and comments into account it is recommended to assume lower values of initial force-densities which means approximately from 0.1 to 1.0 kN/m. In the case of slow or lack of convergence it is good to try different values because even slight change can be effective as it can be seen in the first case. In some situations it may be helpful to obtain a configuration of a cable net without self-weight but with other restrictions (forces in elements, distances between given nodes) if we can predict that it will be close to the final, searched geometry. Obtained solution should be a starting point for analysis with self-weight.

4.4. Influence of allowable error of EFDM calculations on results

Second important in EFDM analysis parameter is allowable error of calculations ε which defines a stop condition 4). In this subsection influence of its value on the accuracy of resulting coordinates and forces is analysed. A simple cable net consisting of five elements is used. Fixed nodes coordinates are given in Table 4.

Table 4. Coordinates of fixed nodes

No. of fixed node	Coordinates (m)		
	x	y	z
1	–4.0	–3.0	0.0
3	–3.0	4.0	0.0
4	3.0	2.0	0.0
5	5.0	–2.0	0.0
6	2.0	–5.0	0.0

Weight and tensile stiffness of assumed rope are: μ = 0.1 kN/m, EA = 12800 kN. Initial lengths of elements are summarised in Fig. 10.

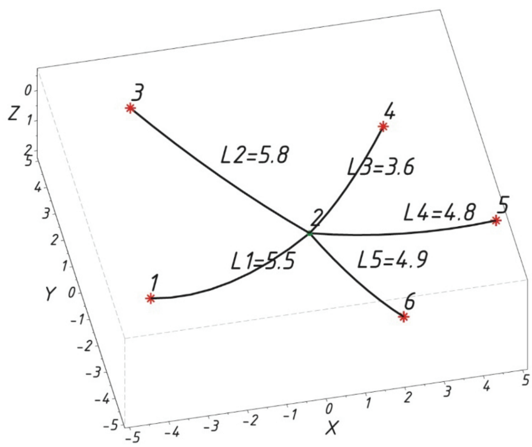

Fig. 10. Configuration of cable net under self-weight

In the first analysis the allowable error is assumed as: $\varepsilon = 1 \cdot 10^{-4}$ m and obtained error value is: $\varepsilon_{\mathrm{real}} = 3.54 \cdot 10^{-6}$ m so this result is further considered as the exact solution shown in Fig. 10.

Coordinates of free node and maximum values of forces in elements are given in Table 5.

Table 5. "Exact" coordinates and forces

Free node coordinates (m)			Maximum element forces (kN)				
x_2	y_2	z_2	N_1	N_2	N_3	N_4	N_5
0.598	–0.427	0.966	0.764	2.349	0.653	1.334	1.570

In order to assess the influence of allowable error on results three different values are considered: $\varepsilon_1 = 0.1$ m, $\varepsilon_2 = 0.01$ m, $\varepsilon_3 = 0.001$ m in subsequent analyses. Relative (to given in Table 5 values) errors of free node coordinates and element forces are shown in Fig. 11 and Fig. 12. It should be noticed that real calculation error values are approximately an order of magnitude smaller than allowable ones.

Both diagrams show that assuming higher precision of calculations (lower allowable error) is associated with lower relative errors of coordinates and forces. Precision of $\varepsilon = 0.01$ m is

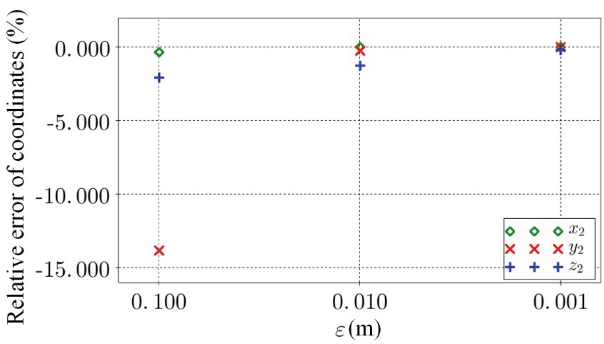

Fig. 11. Relative error of free node coordinates

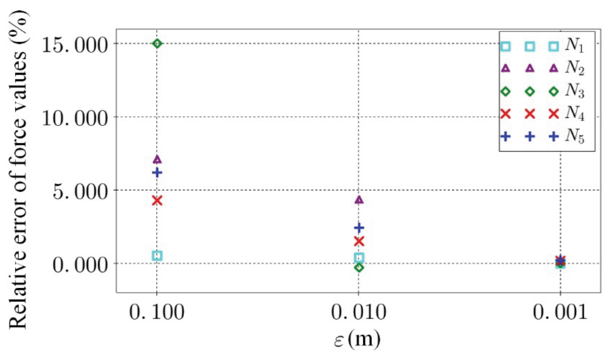

Fig. 12. Relative error of element forces

not sufficient yielding relative errors of coordinates of about 1% and of forces of about 5%. It is hence recommended to apply allowable error value not greater than 0.001 m. Similar tests as shown here are also advisable to analyse convergence of results.

5. Conclusions

Presented here Extended Force Density Method combines the Schek's idea of force density and well-known solution of a catenary cable. Such an approach allows for solving wider range of cable nets problems. Because of the exact, catenary formulation of a cable element almost no restrictions for geometry are present, as opposed to approximate, parabolic formulation. Thanks to adding self-weight of structure into a form-finding process obtained geometry and forces are approximately equal to exact solution.

Verification example shown in subsection 4.2 proves that methodology and theory applied in EFDM provide good compliance with the exact solution. Two subsequent analyses show that obtaining high accuracy results in possibly short time depend on the appropriate parameters. Applying proper values of initial force densities is necessary to get the solution with the lowest possible number of iterations. Allowable error of calculations is crucial for obtaining accurate results. Levels of accuracy for coordinates and force values are different so it is important to analyse errors for both.

Possible applications of EFDM along with some numerical examples obtained with use of UC-Form program will be presented in a second part of the article [13].

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Rozszerzona metoda gęstości sił do analizy siatek cięgnowych pod ciężarem własnym. Część I – Teoria i weryfikacja

Słowa kluczowe: siatki cięgnowe, kształtowanie, rozszerzona metoda gęstości sił, ciężar własny

Streszczenie:

Jest to pierwsza część artykułu dotyczącego Rozszerzonej Metody Gęstości Sił(RMGS). Zaprezentowano w niej założenia i zasady RMGS, a także proste przykłady weryfikacyjne. Metoda ta służy do kształtowania konstrukcji cięgnowych pod wpływem ciężaru własnego, a także dowolnych obciążeń węzłowych.

Cięgno jako element konstrukcyjny zachowuje się odmiennie od powszechnie stosowanych elementów nośnych. Zazwyczaj zakłada się jego zerową sztywność na zginanie. Z tego powodu wymaga również stosowania innych metod projektowania, analizy statycznej, dynamicznej, montażu czy nawet eksploatacji. Element cięgnowy o ustalonym przekroju oraz długości może pod wpływem ciężaru własnego przyjąć nieograniczoną liczbę kształtów zależnie od rozstawu podpór i dodatkowych obciążeń. W przypadku siatki cięgnowej te możliwości gwałtownie rosną. Z tego powodu proces projektowania konstrukcji cięgnowych wymaga etapu wstępnego zwanego kształtowaniem (ang. form-finding). Jego efektem jest uzyskanie stabilnej geometrycznie konfiguracji początkowej potrzebnej do dalszych analiz. Stosowane powszechnie metody kształtowania zakładają nieważkość konstrukcji lub w przybliżony sposób uwzględniają ciężar własny. Co za tym idzie służą one głównie do uzyskania pożądanej konfiguracji, ale nie rozkładu sił w cięgnach. Włączenie rzeczywistego ciężaru własnego konstrukcji stwarza znacznie szersze możliwości wykorzystania takiej metody, a także zapewnia dokładniejsze wyniki.

W artykule zaprezentowano podstawowe założenia Metody Gęstości Sił wprowadzonej przez Scheka [9]. Polega ona na poszukiwaniu współrzędnych węzłów niezamocowanych siatek cięgnowych na podstawie równań równowagi tych węzłów. W celu uzyskania liniowej formy równań względem poszukiwanych współrzędnych wprowadza się pojęcie gęstości siły zdefiniowanej jako stosunek siły do długości danego elementu. W oryginalnej wersji metody każdy element cięgnowy jest prostoliniowy i nieważki, a obciążenia i podpory przegubowe nieprzesuwne zakłada się w dowolnych węzłach. Dla łatwiejszego opisu geometrii siatki wprowadza się macierz połączeń, która wskazuje numery węzłów początkowych i końcowych poszczególnych elementów. Każdemu przyjętemu zestawowi gęstości siłw elementach odpowiada inna konfiguracja siatki cięgnowej i na tej podstawie poszukuje się geometrii siatki spełniającej wymagania wytrzymałościowe, użytkowe i architektoniczne.

W pracy przedstawiono podstawowe wzory opisujące zachowanie cięgna sprężystego pod działaniem ciężaru własnego. Oryginalna wersja Metody Gęstości Siłoraz te wzory są podstawą zaproponowanej Rozszerzonej Metody Gęstości Sił, która w sposób dokładny uwzględnia ciężar własny elementów cięgnowych. Może ona zatem służyć zarówno do analizy siatek cięgnowych wstępnie sprężonych (o elementach prostoliniowych), jak również siatek całkowicie lub częściowo luźnych. W RMGS wprowadzono możliwość łatwiejszego opisu geometrii siatki, w którym nie ma potrzeby wcześniejszego uwzględniania

rozmieszczenia węzłów wolnych i zamocowanych. Budowa macierzy połączeń jest niezależna od definicji warunków brzegowych. Umożliwiono również wprowadzenie podpór przegubowych przesuwnych.

Dodanie ciężaru własnego elementów cięgnowych polega na uwzględnieniu równań opisujących rzeczywiste zachowanie cięgna sprężystego, czyli linię zwisu w kształcie krzywej łańcuchowej. W celu zachowania idei gęstości siły wprowadza się prostoliniowy element zastępczy, statycznie równoważny z elementem zakrzywionym. Na końcach tego elementu przykłada się dodatkowo siły skupione równe wartościom reakcji od ciężaru własnego w rzeczywistym cięgnie. Siły te dodaje się od obciążeń zewnętrznych w kierunku pionowym, przez co modyfikacji ulegają równania równowagi węzłów dla tego kierunku.

W RMGS poszukuje się konfiguracji siatki cięgnowej pod ciężarem własnym. W takim przypadku gęstości sił nie są znane i nie można ich z góry narzucić. Są one poszukiwane na podstawie dodatkowego, nieliniowego układu równań składającego się z warunków opisujących zachowanie cięgna pod ciężarem własnym. Znajdowanie ostatecznych wartości gęstości siłoraz odpowiadającej im konfiguracji jest procesem iteracyjnym zrealizowanym przy użyciu metody Newtona.

W niniejszej pracy przedstawiono przykład weryfikacji zaproponowanej RMGS, w którym pokazano, że uzyskano zbieżność metody do rozwiązania prawidłowego. Przeanalizowano również proste przykłady mające na celu ustalenie optymalnych wartości danych początkowych do obliczeń. Przebadano wpływ wartości początkowych gęstości sił na dokładność i szybkość obliczeń, a także wpływ dopuszczalnego błędu obliczeń na uzyskiwane wyniki.

Uzupełnieniem części I artykułu jest *część II – Przykłady zastosowania*, w której przedstawiono praktyczne przykłady wykorzystania możliwości zaproponowanej metody obliczeniowej.

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