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Limits and Bounds – Boundaries in Mathematics

On the one hand, mathematics draws inspiration from everyday life and has the ambition to model it through a precise logical system, while on the other, it contributes new, sometimes very abstract ideas which expand our imagination and broaden our understanding of the surrounding reality. This is clearly illustrated when we scrutinize various mathematical concepts centered around the notion of "boundary."

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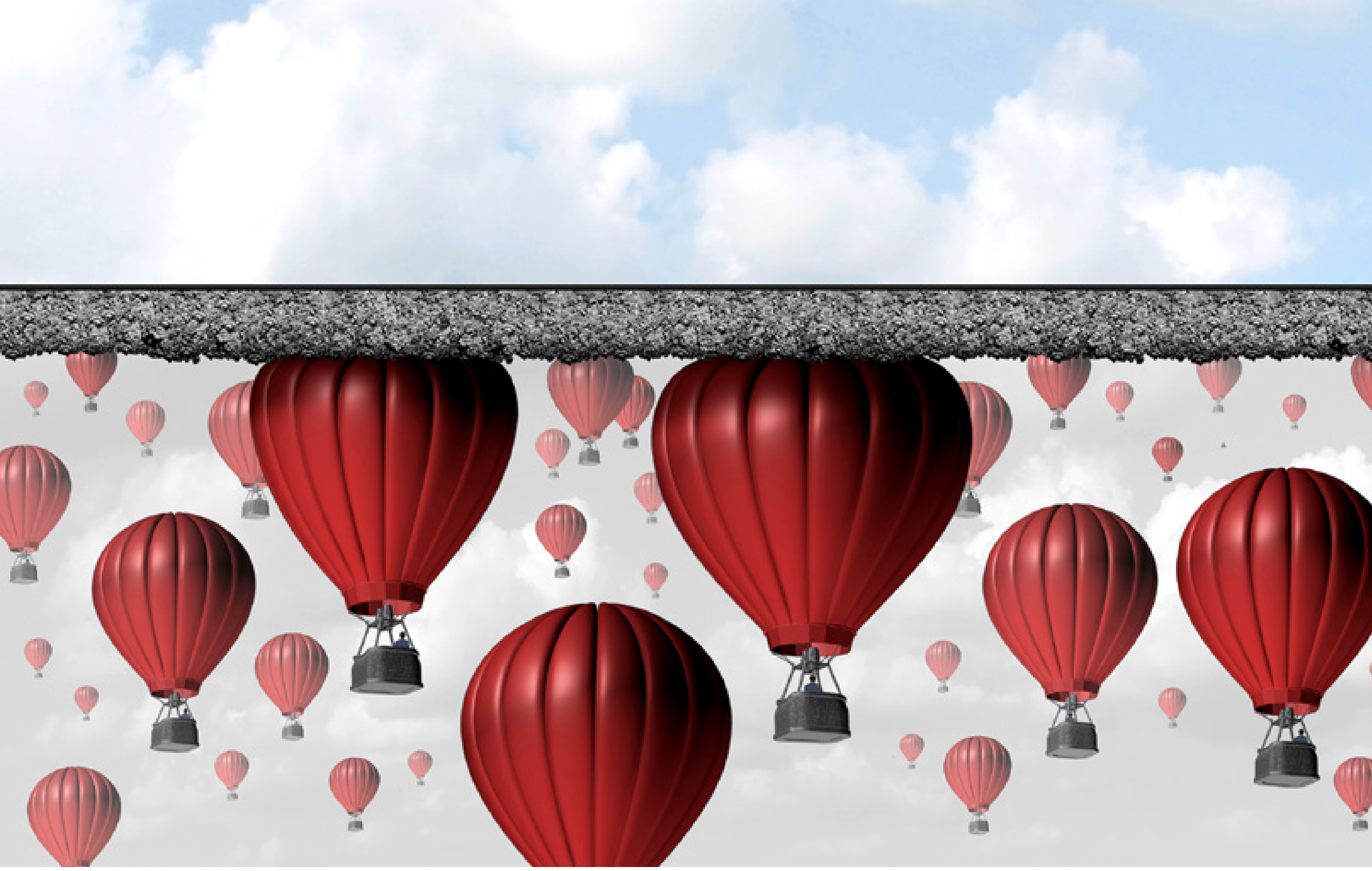
Let us first examine various ways the notion of "boundary" turns up in a few everyday situations, and how the same notions are reflected in mathematics. For instance, when we say that Poland's border with Germany partly runs along the Odra River, we mean a line which delimits a specific area, in this particular case – a country. We find a very similar concept in mathematics: the boundary of a set in a metric space. Such a space is one where we are able to measure the distance between any two of its elements and therefore can easily define open balls – sets of elements whose distance from the center of the ball is less than a certain positive constant, known as the radius of the ball. The balls whose center falls at a specific point define the neighborhood of that point, and the boundary of set A is a set of points whose neighborhood always includes at least one element in A and one element not in A. The "affiliation" of the boundary point itself is of no consequence: such a point may or may not belong to set A.

Next, in the sentence: "The nurse's tolerance for her patients knows no bounds," we have yet another notion of boundary in mind. Here, we want to underline the unlimited nature of a given feature of behavior, which also finds its counterpart in mathematics. We will say that a given set (in a metric space) is bounded if it can be contained within a ball. If not, such a set – just like the nurse's tolerance – is unbounded.

Yet another context in which the notion of a boundary crops us is that of record achievements, as in sports, the Guinness Book of Records or our ordinary, private lives. For instance, the sentence "I could spend 200 euro on a purse, tops" refers to a certain maximum amount, or upper bound. In mathematics, this same notion is reflected in the concept of extremum (maximum or minimum) of a function over a given set and, in a subtler version, of the greatest lower and least upper bound (also known as the infimum and supremum) of the function over that set,



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which can be of particular use when the function does not actually attain its extrema.

The mathematical concept of "limit"

Now let us explore some of the ways mathematics has taken the intuitive, everyday notion of boundary and developed it into more rigorous, abstract concepts – many of them are known by the term "limit" (from Latin *limes*), and can be applied in various ways.

• Limits of sequences

In its basic definition, the limit of a numerical sequence is the number which that sequence "tends to" or converges on. More specifically, we say that a sequence of numbers that can be labeled with the successive natural numbers (and which we could visualize as the numbers of subsequent moments on the axis of time) has *g* as its limit when, if choosing any arbitrarily close neighborhood for *g*, we will find all the future terms of the sequence starting from a given moment in time inside that neighborhood. (We might add here, that, in the world of mathematical sequences, we might wait for such a given moment arbitrarily long.) Intuitively, the sequence for which a limit thus defined exists has a specific "target," so to speak, and tends towards it in time. Such as, for example, the sequence of inverse consecutive natural numbers: 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, …, which converges to the limit 0.

Conversely, in a situation when the "appetite" of a sequence grows larger and larger, i.e. regardless what level we set, all the successive terms in the sequence after a certain moment will exceed that level. One example here is a geometric sequence in which successive items are multiplied by a factor of 2 (1, 2, 2^2, 2^3 …). We say that such a sequence diverges to infinity (and we can define divergence to negative infinity in a similar way).

Finally, a sequence which has no limit (neither convergent nor divergent) can be compared to an individual who is extremely irresolute and who, throughout his or her life – infinite in this case – keeps wavering over the choice of objective. This can be simply illustrated by the sequence 1, −1, 1, −1, 1, −1, …, which ever keeps changing its direction, flip-flopping back and forth.

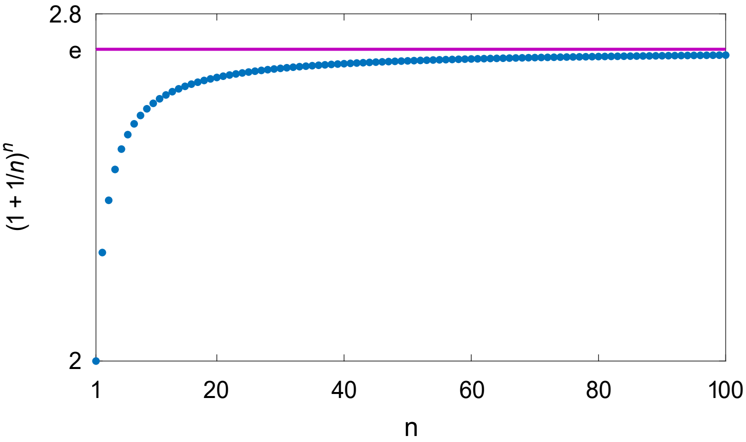

Fig. 1
Convergence of the sequence $(1 + 1/n)^n$ to Euler's number, e

The world of sequence limits is governed by its own distinctive laws, such as for example the theorems for calculating the sum or product of convergent sequences or the "sandwich" rule, which allow for a sequence to be identified as convergent and its limit found more easily than when directly referring to its definition. This world also has its mysteries, such as so-called indeterminate forms, when all intuitive bets seem to be off (for instance, even though two sequences each tend to the same limit, equal to 0, their quotient does not necessarily tend to 1 – indeed, for any given number, we can actually provide examples of such sequences where the limit of the quotient is equal to that number).

For some sequences, we know a limit exists even though we lack detailed information about it. Here, we can refer to the monotonic and bounded sequence theorem, which states that if the subsequent term in a sequence is always larger than the previous one, and if all the terms lie below a certain "ceiling," then the sequence must be convergent, and its limit lies somewhere below or at that ceiling. Although we do not know the exact location of the limit, it is approximated by the values of the sequence terms with high numbers. Also, if we know that the limit exists, we can examine many of its properties.

To illustrate this, let us take the sequence $(1 + 1/n)^n$, (cf. Fig. 1). It is increasing and is bounded from above, and therefore is convergent, and its limit is known as Euler's number, e. In other words, the constant $e \approx 2.7182818284590452...$ (strongly linked with logarithms and the exponential function, and therefore with many applications of mathematics) is defined as the limit of the above sequence. We have known since back in the 18th century that it is an irrational number, much like the number π, so it cannot have a full decimal expansion, but only rational approximations which can be obtained, for instance, by calculating the value of $(1 + 1/n)^n$ for some specific, large n. The correctness of such a method for making approximated calculations stems directly from the relationship between a sequence and its limit assumed in the definition according to which the difference between e and $(1 + 1/n)^n$ does not exceed desired accuracy ε for a sufficiently large n (and what the "sufficiently large" actually stands for depends on the selected value of ε). Incidentally, the above example also serves as an excellent illustration of the mystery of indeterminate forms: although the base of $1 + 1/n$ tends to 1, the sequence $(1 + 1/n)^n$ does not converge to 1, but instead to $e \neq 1$. This is because the exponent n tends to infinity, a situation when the usual principles of arithmetic simply do not apply…

• Theory of series

Defining the limit of a sequence and being able to examine and apply such a limit even in situations when its value is unknown or even in a sense unknowable (as in the example of Euler's number) allows us to enter the theory of series, i.e. infinite sums – first numerical series, and then functional series. In the latter, Taylor series and Fourier series play a major role as they allow many functions to be represented as infinite sums of polynomials or simple trigonometric functions (cf. Fig. 2). In turn, such representation significantly informs the development of a host of numerical methods for calculating approximations. And who of us, ordinary users of a calculator, might think that, somewhere deep in its entrails, it actually works based on the notion of the limit of a sequence?

• Limits of functions, derivatives and integrals

The concept of the limit of a numerical sequence can be expanded in another direction. Instead of restricting ourselves to sequences alone, we can analyze a function as such, and analyze its values when the arguments tend towards a specific target. Naturally, to do this, we need to define not only what we mean by saying that the arguments tend to a target, but also need to describe the manner in which we measure/assess so-called asymptotic notation of the function $f(x)$. The general concept can be formulated relatively simply, when both the arguments and the function values come from metric spaces. In such a situation, we can say that the function $f(x)$ has the limit g (or that the values of the function $f(x)$ tend to g) as x converges to a, when, if we select, from the set of function values, a ball K_g with center g and an arbitrarily small radius, we will find such a ball K_a with center a in the domain of the function, that all the function values for the arguments from K_a will fall within ball K_g. In other words, g is the limit of function $f(x)$ as x converges to a if the function's values are in a close neighborhood of g (i.e. with a small radius) for arguments with in a sufficiently close neighborhood of a. (For the sake of accuracy, all the details concerning the nature of element a, which obviously needs to be related to the domain of the function, though not necessarily its part, are omitted here.)

Such an approach to the idea of a function limit is, naturally, presented in very general terms and could be elaborated or expanded upon in a variety of ways. For instance, when we look at the basic case of functions whose arguments and values are real numbers, we can quite easily add a way of defining the limit of that function (defined on a certain half-line) as the arguments tend to infinity (by analogy to the definition of the limit of a sequence) or to minus infinity, as well as one-sided limits, in the definition of which we take into account the neighborhood of a given point on one side only (which is clear and obvious for arguments on the real line). We can also allow divergent rather than convergent limits, just as in the case of sequences.

But that is not all. Armed with such a tool, we can use it to introduce new concepts which are of cardinal importance for both the theory and applications of mathematics, such as function continuity and differentiability – notions without which we could hardly imagine classical physics or the basic model of the world that surrounds us. We can also define various types of integrals as limits of certain more complicated objects, such as the Riemann integral, which are as important for the applications of mathematics in basic and engineering sciences.

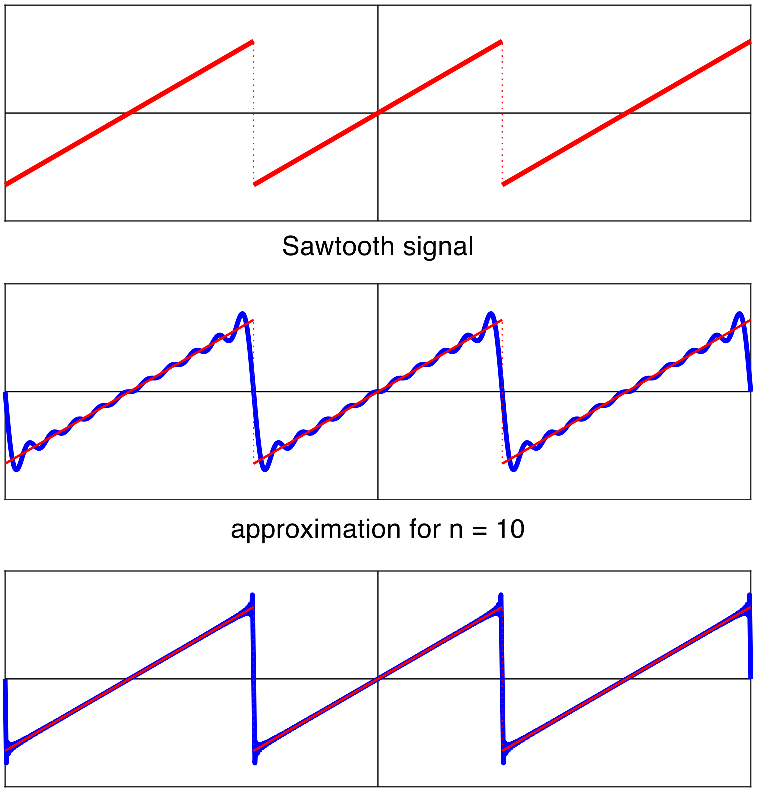

Fig. 2
Convergence of the Fourier series for the sawtooth signal to that signal (n is the number of applied series elements to obtain a signal approximation)

• Gaussian functions and Monte Carlo methods

However, the mathematical notion of limit is the foundation for even more than just the mathematical fields of differential and integral calculus. In fact, the concept can be encountered, more or less openly, in nearly every sub-field of the "queen of the sciences." Personally, I am fascinated by problems related to the laws of large numbers and limit theorems in probability theory, which studies limits with the probability of 1 and limits in distribution of certain sequences of random variables which, in the basic version, are partial sums of sequences of independent and identically distributed random variables. It has been found that, with very general assumptions on the distribution of the elements (or, in other words, on the nature of their randomness), such sums, having been sufficiently normalized, behave in a sense universally at the limit, when the number of the elements increases to infinity.

For instance, it is sufficient if there exists an expected value of the distribution of an element, for the sums divided by the number of elements (i.e. arithmetical means) to be no longer random and converge with probability 1 to a constant equal to that expected value (as stated in Kolmogorov's strong law of large numbers). This theorem provides the basis for an important group of numerical methods in which approximated deterministic values are obtained through random sampling. This concept was first introduced by Polish mathematician Stanisław Ulam during the work on the atom bomb in Los Alamos and was subsequently dubbed the Monte Carlo method, although it had in some specific cases been applied before. One well-known example is the eitheenth-century "Buffon's needle problem." In this particular case, a needle is repeatedly dropped randomly on a board marked with parallel lines of equal distance from one another (the needle being shorter than the distance between the lines), and we count how often the needle lands in a position crossing any of the lines. It can be demonstrated, e.g. on the basis of the law of large numbers, that the obtained frequency plus knowledge of the length of the needle and the distance between the lines is sufficient to accurately estimate the value of the number π.

If the distribution of all elements of the sum has a finite non-zero variance, then the sums centered to have zero mean and scaled to have unit standard deviation follow, asymptotically, the same standard normal (or Gaussian) distribution regardless of the actual distribution of the elements. This fact is known as the Lindeberg-Levy central limit theorem and explains the prevalence of the Gaussian function (bell curve) in statistical data analysis, commonly used in such fields as finance, medicine, psychology, and social sciences.

In closing, I hope that this handful of examples showing how the notion of "boundary" manifests itself in mathematics – especially in various uses of the term "limit" – well illustrates the key role it plays and the extent to which we all benefit from it, even if we rarely realize it. ■