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Research paper

Determination of vertical displacements by using the hydrostatic levelling systems with the variable location of the reference sensor

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Abstract: In this paper, the author proposed a new method for determination of vertical displacements with the use of hydrostatic levelling systems. The traditional method of hydrostatic levelling uses a rule in which a position of reference sensor is stable. This assumption was not adapted in the proposed method. Regarding the issue mentioned above, the reference sensor is treated in the same way as the others sensors that measure the liquid level. As a consequence of this approach there is a possibility of vertical displacement determination of both the reference sensor as well as the remaining controlled sensors. A theoretical considerations were supplemented with the practical examples. The possibility of calculating the vertical displacement of reference sensor is an undoubted advantage of the submitted proposal. This information enables more detailed interpretation of the vertical displacements results obtained from hydrostatic levelling systems. Thus, wider knowledge about maintenance of the entire examined object treated as the rigid body is obtained. The tests that were carried out confirm the theoretical assumptions and encourage to perform further, more precise empirical analyses.

Keywords: vertical displacement, hydrostatic levelling systems, reference sensor, controlled sensors

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1. Introduction

The determination of building constructions displacements and strains is a current problem of geodetic engineering. One of the most important stage of these works is a problem of stable reference points (RPs) positioning. In relation to these points the displacements of remaining controlled points (CPs) will be determined. The RPs must keep the stability of their position in order to provide the reliable measurements. The rule is that the RPs are located outside the zones of influence of the moving and deforming object. In order to determine the displacements the following geodetic technologies can be used e.g.: GNSS systems, electronic tachometers, precise levelling and hydrostatic levelling systems (HLSs). The hydrostatic levelling system (HLS) allows to determine vertical displacements with very high accuracy (of hundredths of millimetre). Such accuracies cause that HLSs are very widely used, especially in the monitoring of engineering structures. The examples of using HLSs are among others monitoring of: river dams [11], road and railway bridges [2, 5, 11], buildings monitoring carried out near deep excavations related to the construction of the tunnel [14], sports halls [15], CERN's synchrotron [1, 10, 11] transmission towers' foot [18], churches [12] and many others application available in the subject of literature.

Simultaneously with the development of highly accurate HLSs, works analysing in detail the impact of HLS components on gained displacements' results, appear. The examples can be the papers of [4, 18] in which the impact of temperature on deformation measurements obtained with the help of HLS was analysed in detail. In the paper [9] the authors proposed the method of the differences of observation for determining the vertical displacements. This approach allows to eliminate the systematic errors burdening the HLSs measurements' results. The authors provided also the solutions enabling to conduct the accuracy analysis of the results of vertical displacements as well as their prediction using the Kalman's filter method.

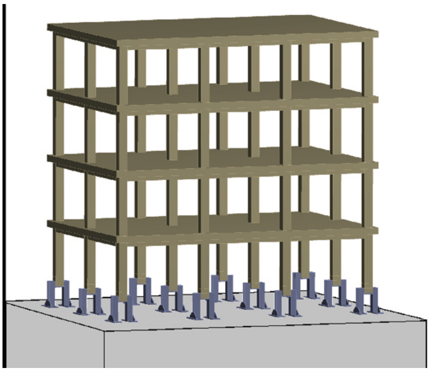
While realising the HLS, generally it is assumed that one of the points with sensor placed on it does not change its positon and that the displacements of remaining sensors placed on the CPs will be determined according to this point. This sensor is treated as the reference sensor (*RS*). However, the determination of the stable position of such sensor is a difficult task.

This work presents the new method for determining the vertical displacements of points measured by HLSs. The theoretical and empirical problem analysed in this work assumes the instability of *RS*. During the estimation both the vertical displacement value of the *RS* as well as of CP are determined. This means that the *RS* is treated in the same way as other sensors placed on CPs. This work is an extension analyses presented in [3, 6–8, 16]. In the paper [3] the author proposed a certain algorithm for determining vertical displacements in situations where the stable RS cannot be determined. In analyses carried out, the existence of redundant observations was assumed, thus it is possible to adjust of the observations' results. In this paper the author proposes a procedure of determining of vertical displacements in situations, when it is not possible to adjust of observations' results. Examples of such geometric construction can be: line (travers) of levelling tied up at one end or HLSs. Section 2 presents the theoretical basis of the novel method. In Section 3, the covariance matrices (that make the accuracy analyses of obtained parameters possible) are derived. The practical examples and discussion about obtained results are the content of Section 4. The conclusions are presented in Section 5 and the summary is the content of Section 6.

In conclusion, it can already be said, that the presented results encourage for further more detailed research on real engineering objects.

2. Theoretical background

In the paper [3] the authors proposed a way of performing calculations in geodetics measurements where there are some redundant observations. The example of these constructions is e.g. a closed geometric or hydrostatic levelling network. Fig. 1b, shows the closed HLS network.

a) 

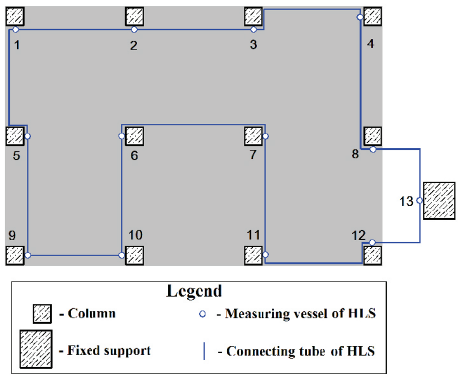
b) 

Fig. 1. a) The tested structure, b) the location of the HLS on columns) [17]

Figure 1 shows the layout of the HLS on the tested engineering structure. The measuring vessels (sensors) of the HLS are located on columns. The connecting pipes were laid between them. The sensor no. 13 is placed outside the tested structure.

The estimation rules given in the paper [3] can be used because there are the redundant observations in the calculations on the construction presented in Figure 1.

In the aforementioned work, the following observational equations were adopted for numerical calculations

$$v_{lk} = s_k^j - s_l^j - \left(Y_k^j - Y_l^j\right) \varepsilon_X^j + \left(X_k^j - X_l^j\right) \varepsilon_Y^j - h_{lk}^{\text{obs}(j)} \tag{2.1}$$

in which $h_{lk}^{\text{obs}(j)}$ – the observed heights difference between the points (l, k), v_{lk} – the correction of the observed magnitude, s_k^j, s_l^j – the spatial distances between the optimal plane (obtained from adjustment with the use of the least squares method (LSM)) and the surface on which the points to be measured are located, X_k^j, X_l^j, Y_k^j, Y_l^j – the CPs' coordinates (points' to be measured), ε_X^j, ε_Y^j – the rotations' angles around the axis respectively X and Y, $j = 0, 1, 2, \ldots$ – the subsequent measuring epochs.

The relationship (2.1) can be presented in the following matrix notation

$$\mathbf{v}^j = \mathbf{A}\mathbf{s}^j + \mathbf{B}\boldsymbol{\varepsilon}^j - \mathbf{L}^j \tag{2.2}$$

where: $\mathbf{A}$, $\mathbf{B}$ – the known matrices of coefficients, $\mathbf{v}$ – the vector of the corrections of the observed magnitude, $\mathbf{s} = [s_1, s_2, s_n]^T$ – the vector of the spatial distances between the optimal plane and the surface on which the measured points are located, $\mathbf{L} = [\ldots, h_{l,k}^{\text{obs}}, \ldots]^T$ – the observations' vector, $\boldsymbol{\varepsilon} = [\varepsilon_X, \varepsilon_Y]^T$ – the vector of rotations angles, $j = 0, 1, \ldots$ – the subsequent measuring epochs. The rotations' angels $\varepsilon_X, \varepsilon_Y$ are the parameters of the object treated as a rigid body. The information about displacement of the examined object can be gained by creating their difference. The optimisation problem of the proposed method is defined in the following form [3]

$$\left.\begin{aligned} &\mathbf{v}^T \mathbf{P_L} \mathbf{v} = \min \\ &\mathbf{s}^T \mathbf{P_s} \mathbf{s} = \min \\ &\mathbf{C_L} = m_0^2 \mathbf{Q_L} = m_0^2 \mathbf{P_L}^{-1} \\ &\mathbf{v} = \mathbf{A}\mathbf{s} + \mathbf{B}\boldsymbol{\varepsilon} - \mathbf{L} \end{aligned}\right\} \tag{2.3}$$

(the matrix of cofactors of the observations' results $\mathbf{Q_L} = \mathbf{P_L^{-1}}$, and $\mathbf{P_L} = \mathbf{Q_L^{-1}}$ – the weights' matrix, $\mathbf{P_s}$ – the weights' matrix determined for the parameters $\mathbf{s}$).

The details of the solution of the optimisation problem (2.3) together with the practical application are the content of the work [3] and will not be further analysed.

This work concerns the situation where there are not any redundant observations, hence the number of observations n is equal to the number of parameters m, ($n = m$, then there is no corrections v_{lk} to the observations). Hence the observational Eq. (2.2) takes the form

$$\mathbf{A}\mathbf{s}^j + \mathbf{B}\boldsymbol{\varepsilon}^j - \mathbf{L}^j = \mathbf{0} \tag{2.4}$$

Because vector $\mathbf{v} = \mathbf{0}$, hence the first relationship ($\mathbf{v}^T \mathbf{P_L} \mathbf{v} = \min$) in the optimisation problem (2.3) disappears. Therefore, the traditional coefficient (used to calculate the appropriate covariance matrix after the adjustment process) $m_0^2 = \dfrac{\mathbf{v}^T \mathbf{P_L} \mathbf{v}}{n - m}$ cannot be determined.

The example of this problem is the open hydrostatic levelling line in which the sensors are connected in series, as in Fig. 2.

Hence, in the place of the adjustment problem (2.3) the new optimisation problem can be formulated in following form

$$\left.\begin{aligned} &\mathbf{s}^T \mathbf{P_s} \mathbf{s} = \min \\ &\mathbf{A}\mathbf{s} + \mathbf{B}\boldsymbol{\varepsilon} - \mathbf{L} = \mathbf{0} \end{aligned}\right\} \tag{2.5}$$

The LSM with use of Lagrange multipliers can be applied while searching for the solution of problem (2.5). Then the following optimisation criterion is obtained

$$\rho = \left(\mathbf{s}^j\right)^T \mathbf{P_s} \mathbf{s}^j - 2\left(\mathbf{k}^j\right)^T \left(\mathbf{A}\mathbf{s}^j + \mathbf{B}\boldsymbol{\varepsilon} - \mathbf{L}^j\right) = \min \tag{2.6}$$

where: $\mathbf{k}$ – the vector of Lagrange multipliers.

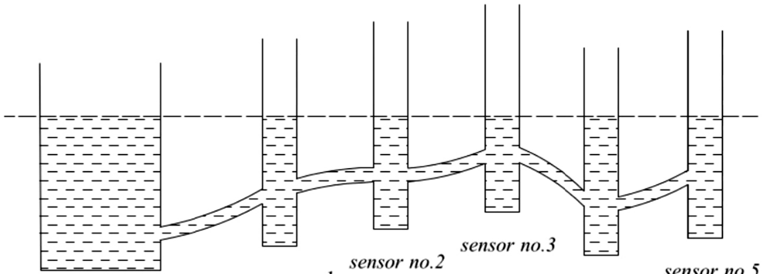


Fig. 2. An example configuration of measuring sensors [9]

Searching for minimum of the function (2.6) the following partial derivatives must be computed

$$\frac{\partial \rho}{\partial \mathbf{s}} = 2\left(\mathbf{s}^j\right)^T \mathbf{P_s} - 2\left(\mathbf{k}^j\right)^T \mathbf{A} = \mathbf{0} \tag{2.7}$$

and

$$\frac{\partial \rho}{\partial \boldsymbol{\varepsilon}} = 2\left(\mathbf{k}^j\right)^T \mathbf{B} = \mathbf{0} \Rightarrow \mathbf{B^T k^j} = \mathbf{0} \tag{2.8}$$

Transforming Eq. (2.7), the following expression is obtained

$$\hat{\mathbf{s}}^j = \mathbf{P_s^{-1} A}^T \mathbf{k}^j \tag{2.9}$$

Inserting Eq. (2.9) into Eq. (2.4) and performing elementary transformations the following formulas are gained (for $\mathbf{B}^T \mathbf{k}^j = \mathbf{0}$)

$$\left.\begin{aligned} \hat{\mathbf{k}}^j &= -\left(\mathbf{AP_s^{-1}A}^T\right)^{-1} (\mathbf{B}\boldsymbol{\varepsilon} - \mathbf{L}) \\ \hat{\boldsymbol{\varepsilon}} &= \left[\mathbf{B}^T \left(\mathbf{AP_s^{-1}A}^T\right)^{-1} \mathbf{B}\right]^{-1} \mathbf{B}^T \left(\mathbf{AP_s^{-1}A}^T\right)^{-1} \mathbf{L} \end{aligned}\right\} \tag{2.10}$$

and then the vector

$$\hat{\mathbf{s}}^j = \mathbf{P_s^{-1}A}^T \hat{\mathbf{k}} = -\mathbf{P_s^{-1}A}^T \left(\mathbf{AP_s^{-1}A}^T\right)^{-1} (\mathbf{B}\hat{\boldsymbol{\varepsilon}} - \mathbf{L}) \tag{2.11}$$

Assuming that the weights' matrix $\mathbf{P_s} = \mathrm{diag}(1, \ldots, 1)$, the solutions Eq. (2.10) and Eq. (2.11) can be described as follows

$$\hat{\boldsymbol{\varepsilon}} = \left[\mathbf{B}^T \left(\mathbf{AA}^T\right)^{-1} \mathbf{B}\right]^{-1} \mathbf{B}^T \left(\mathbf{AA}^T\right)^{-1} \mathbf{L} \tag{2.12}$$

$$\hat{\mathbf{k}} = -\left(\mathbf{AA}^T\right)^{-1} (\mathbf{B}\hat{\boldsymbol{\varepsilon}} - \mathbf{L}) \tag{2.13}$$

$$\hat{\mathbf{s}}^j = \mathbf{A}^T \hat{\mathbf{k}} \tag{2.14}$$

The next stage of calculation is the determination of the differences $\Delta\hat{\boldsymbol{\varepsilon}}$ between the values of the rotations' angles $\hat{\boldsymbol{\varepsilon}}^{j+1}$ and $\hat{\boldsymbol{\varepsilon}}^{j=0}$ obtained from the considered measuring periods. The magnitudes of differences are determined for the entire object treated as the rigid body. The differences of the rotations' angles $\Delta\hat{\boldsymbol{\varepsilon}}$ can suggest the possibility of displacements of the entire object and can be presented in the following formula:

$$\Delta\hat{\boldsymbol{\varepsilon}} = \hat{\boldsymbol{\varepsilon}}^{j+1} - \hat{\boldsymbol{\varepsilon}}^{j=0} \tag{2.15}$$

Formula (2.15) cannot determine the vertical displacements of individual points of the examining object.

The magnitudes $\hat{\lambda}_i = X_i^j \hat{\varepsilon}_Y^j - Y_i^j \hat{\varepsilon}_X^j$ denoting the linear values of vertical displacements caused by the rotations of all CPs to the horizontal plane must be determined when looking for the displacements values of the individual points.

The values of the horizontal coordinates X_i^j, Y_i^j of all sensors are needed to calculate the values $\hat{\lambda}_i$ besides the estimated angles values $\hat{\varepsilon}_Y^j$, $\hat{\varepsilon}_X^j$. The origin of the horizontal coordinate system should be taken at any point of the tested object which do not take part in calculating in order to perform the calculations correctly. Otherwise the errors in the assessment can occur because $X = 0$, $Y = 0$, $\Rightarrow \hat{\lambda} = 0$. While calculating the vertical displacements using the proposed method, it is assumed that the RS is the origin of vertical, local reference system, hence $Z_{RS} = 0.00$ mm. Assuming that $i = RS, 1, \ldots, m$, m – the number of the points at which the measurements were made, $j = 0, 1, 2, \ldots$ – subsequent measuring epochs and $\mathbf{x} = \left[X_i^j,\ Y_i^j\right]$ the following formula is used:

$$\hat{\boldsymbol{\lambda}} = \mathbf{x}\hat{\boldsymbol{\varepsilon}} = \left[X_i^j,\ Y_i^j\right] \begin{bmatrix} \hat{\varepsilon}_Y^j \\ \hat{\varepsilon}_X^j \end{bmatrix} \tag{2.16}$$

The obtained magnitudes of the vector $\hat{\boldsymbol{\lambda}} = \left[\hat{\lambda}_1, \ldots, \hat{\lambda}_n\right]^T$ allow to determine the heights of the points in the local coordinates system both in the initial epoch $\hat{\mathbf{Z}}^{j=0}$ and the current epoch $\hat{\mathbf{Z}}^{j+1}$ using the following expression

$$\left.\begin{aligned} \hat{\mathbf{Z}}^{j=0} &= \hat{\mathbf{s}}^{j=0} + \hat{\boldsymbol{\lambda}}^{j=0} \\ \hat{\mathbf{Z}}^{j+1} &= \hat{\mathbf{s}}^{j+1} + \hat{\boldsymbol{\lambda}}^{j+1} \end{aligned}\right\} \tag{2.17}$$

The graphic presentation of the dependencies (2.17) for any epochs $j = 0, 1, \ldots$, in one-dimensional system is shown at Fig. 3. The straight line "a" is fitted into the set of points $P\{RS, 1, \ldots, i, \ldots, n\}$ according to rule of the LSM, satisfying the condition $\mathbf{s}^T\mathbf{P}_\mathbf{s}\mathbf{s} = \min$ (Eq. (2.5)). The values $\hat{\lambda}_1, \ldots, \hat{\lambda}_n$ are gained as a result of the straight line "a" rotation by the angle ε (Eq. (2.16)) to the horizontal position $a(\varepsilon = 0)$. The next step is the calculation of the heights values $\hat{Z}_i$ in the local coordinates system for given epochs from the sum of values $\hat{s}_i$ and $\hat{\lambda}_i$, $\left(\hat{Z}_i = \hat{s}_i + \hat{\lambda}_i\right)$.

The dependence (2.17) allows to realise the next stage of calculation where the CPs' vertical displacements between of measurements epochs are assessed. While starting the

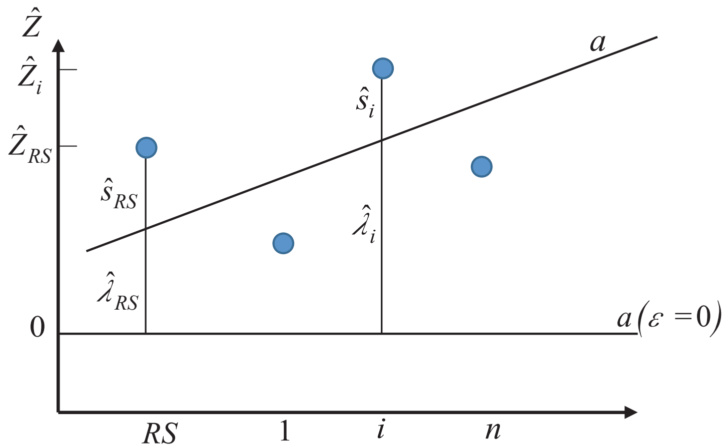


Fig. 3. The demonstrative estimators: $\hat{Z}$, $\hat{\lambda}$, $\hat{s}$

calculation, it is essential to determine the *RS's* vertical displacement $\hat{d}_{RS}$ (if it exists) using the following relationship

(2.18) $$\hat{d}_{RS} = \hat{Z}_{RS}^{j+1} - \hat{Z}_{RS}^{j=0}$$

The vertical displacements of CPs are gained by using the expression (for $i = 1, 2, \ldots, m$)

(2.19) $$\hat{d}_i = \left(\hat{Z}_i^{j+1} - \hat{Z}_{RS}^{j=1}\right) - \left(\hat{Z}_i^{j=0} - \hat{Z}_{RS}^{j=0}\right)$$

or in the matrix notation

(2.20) $$\hat{\mathbf{d}} = \hat{\mathbf{Z}}^{j+1} - \hat{\mathbf{Z}}^{j=0} + \left(\hat{Z}_{RS}^{j=0} - \hat{Z}_{RS}^{j+1}\right)\mathbf{I}_m$$

where $\mathbf{I}_m = [1, \ldots, 1]^T$ – the unit vector with the m dimension corresponding to the number of CPs in the examined network.

3. Accuracy analysis

The covariance matrices are used to assess the accuracy of calculations' results, further denoted as $\mathbf{C}$. The law of propagation of mean errors is used for this purpose. Most often it is presented in the following form $\mathbf{C} = \mathbf{DC_LD}^T = m_0^2\mathbf{DQ_LD}^T$, $(\mathbf{C_L} = m_0^2\mathbf{Q_L})$. Assuming, that there is a lack of redundant observations and $m_0^2 = 1$ the covariance matrix $\mathbf{C}$ is equal to the cofactors matrix $\mathbf{Q}$. Hence

(3.1) $$\mathbf{Q} = \mathbf{DQ_LD}^T$$

where: $\mathbf{D}$ – the transformation matrix.

Searching for the cofactors matrix $\mathbf{Q}_{\hat{\varepsilon}}$ of vector $\hat{\varepsilon} = \left[\mathbf{B}^T \left(\mathbf{A}\mathbf{P}_s\mathbf{A}^T\right)^{-1} \mathbf{B}\right]^{-1} \mathbf{B}^T \left(\mathbf{A}\mathbf{P}_s\mathbf{A}^T\right)^{-1} \mathbf{L}$ and assuming the transformation matrix is in the form

$$\mathbf{D} = \mathbf{M}^{-1}\mathbf{B}^T\mathbf{N}^{-1} \tag{3.2}$$

(where: $\mathbf{N} = \mathbf{A}\mathbf{P}_s\mathbf{A}^T$, $\mathbf{M} = \mathbf{B}^T\mathbf{N}^{-1}\mathbf{B}$) it is obtained

$$\mathbf{Q}_{\hat{\varepsilon}} = \mathbf{M}^{-1}\mathbf{B}^T\mathbf{N}^{-1}\mathbf{Q}_\mathbf{L}\mathbf{N}^{-1}\mathbf{B}\mathbf{M}^{-1} \tag{3.3}$$

While calculating the cofactors matrix $\mathbf{Q}_{\hat{\mathbf{k}}}$ of the Lagrange multipliers' vector $\hat{\mathbf{k}} = -\mathbf{N}^{-1}(\mathbf{B}\hat{\varepsilon} - \mathbf{L})$, (for $\hat{\varepsilon} = \mathbf{M}^{-1}\mathbf{B}^T\mathbf{N}^{-1}\mathbf{L}$) it can be noted that

$$\hat{\mathbf{k}} = -\mathbf{N}^{-1}\left(\mathbf{B}\mathbf{M}^{-1}\mathbf{B}^T\mathbf{N}^{-1}\mathbf{L} - \mathbf{L}\right) \tag{3.4}$$

or

$$\hat{\mathbf{k}} = \left(\mathbf{N}^{-1} - \mathbf{\Omega}\right)\mathbf{L} \tag{3.5}$$

where: $\mathbf{\Omega} = \mathbf{N}^{-1}\mathbf{B}\mathbf{M}^{-1}\mathbf{B}^T\mathbf{N}^{-1}$.

Assuming that the transformation matrix $\mathbf{D} = \mathbf{N}^{-1} - \mathbf{\Omega}$, the cofactors matrix $\mathbf{Q}_{\hat{\mathbf{k}}}$ of the vector $\hat{\mathbf{k}}$ can be written in following form

$$\left.\begin{aligned} \mathbf{Q}_{\hat{\mathbf{k}}} &= (\mathbf{N}^{-1} - \mathbf{\Omega})\,\mathbf{Q}_\mathbf{L}\,(\mathbf{N}^{-1} - \mathbf{\Omega})^T \\ \mathbf{Q}_{\hat{\mathbf{k}}} &= \mathbf{R}\mathbf{Q}_\mathbf{L}\mathbf{R}^T \end{aligned}\right\} \tag{3.6}$$

where: $\mathbf{R} = \mathbf{N}^{-1} - \mathbf{\Omega}$.

To determine the cofactors matrix $\mathbf{Q}_{\hat{\mathbf{s}}}$, the vector $\hat{\mathbf{s}}$ is presented in the following form (for $\hat{\mathbf{k}} = \left(\mathbf{N}^{-1} - \mathbf{\Omega}\right)\mathbf{L}$)

$$\hat{\mathbf{s}}^j = \mathbf{P}_s^{-1}\mathbf{A}^T\hat{\mathbf{k}} = \left(\mathbf{H}\mathbf{N}^{-1} - \mathbf{H}\mathbf{\Omega}\right)\mathbf{L} \tag{3.7}$$

where: $\mathbf{H} = \mathbf{P}_s^{-1}\mathbf{A}^T$.

Assuming, that $\mathbf{D} = \mathbf{H}\mathbf{N}^{-1} - \mathbf{H}\mathbf{\Omega}$, the cofactors matrix is defined by the form

$$\mathbf{Q}_{\hat{\mathbf{s}}} = \left(\mathbf{H}\mathbf{N}^{-1} - \mathbf{H}\mathbf{\Omega}\right)\mathbf{Q}_\mathbf{L}\left(\mathbf{H}\mathbf{N}^{-1} - \mathbf{H}\mathbf{\Omega}\right)^T = \mathbf{\Psi}\mathbf{C}_\mathbf{L}\mathbf{\Psi}^T \tag{3.8}$$

($\mathbf{\Psi} = \mathbf{H}\mathbf{N}^{-1} - \mathbf{H}\mathbf{\Omega}$).

Determining of the cofactors matrix of the vectors of the parameters $\hat{\lambda} = \mathbf{x}\hat{\varepsilon} = \mathbf{x}\mathbf{M}^{-1}\mathbf{B}^T\mathbf{N}^{-1}\mathbf{L}$ for $\mathbf{D} = \mathbf{x}\mathbf{M}^{-1}\mathbf{B}^T\mathbf{N}^{-1}$ the following formula is obtained

$$\mathbf{Q}_{\hat{\lambda}} = \mathbf{x}\mathbf{M}^{-1}\mathbf{B}^T\mathbf{N}^{-1}\mathbf{Q}_\mathbf{L}\mathbf{N}^{-1}\mathbf{B}\mathbf{M}^{-1}\mathbf{x} \tag{3.9}$$

The vector $\hat{\mathbf{Z}}^j = \hat{\mathbf{s}}^j + \hat{\lambda}^j$ for $\hat{\lambda} = \mathbf{x}\mathbf{M}^{-1}\mathbf{B}^T\mathbf{N}^{-1}\mathbf{L}$ and $\hat{\mathbf{s}}^j = \left(\mathbf{H}\mathbf{N}^{-1} - \mathbf{H}\mathbf{\Omega}\right)\mathbf{L}$ can be written in the following form

$$\hat{\mathbf{Z}} = \left(\mathbf{H}\mathbf{N}^{-1} - \mathbf{H}\mathbf{\Omega}\right)\mathbf{L} + \mathbf{x}\mathbf{M}^{-1}\mathbf{B}^T\mathbf{N}^{-1}\mathbf{L} \tag{3.10}$$

or

$$\hat{\mathbf{Z}} = \left(\mathbf{HN}^{-1} - \mathbf{H\Omega} + \mathbf{xM}^{-1}\mathbf{B}^T\mathbf{N}^{-1}\right)\mathbf{L} \tag{3.11}$$

The cofactors matrix $\mathbf{Q}_{\hat{\mathbf{Z}}}$, for $\mathbf{D} = \mathbf{HN}^{-1} - \mathbf{H\Omega} + \mathbf{xM}^{-1}\mathbf{B}^T\mathbf{N}^{-1}$ takes the form

$$\mathbf{Q}_{\hat{\mathbf{Z}}} = \mathbf{DQ_L D}^T \tag{3.12}$$

The values of the mean errors $m_{\hat{d}_i}$ of received displacements $\hat{d}_i$ are calculated using the form of Eq. (2.19)

$$\begin{aligned} \hat{d}_i &= \left(\hat{Z}_i^{j+1} - \hat{Z}_{RS}^{j+1}\right) - \left(\hat{Z}_i^{j=0} - \hat{Z}_{RS}^{j=0}\right) \\ &= X_i\hat{\varepsilon}_Y^{j+1} - Y_i\hat{\varepsilon}_X^{j+1} + \hat{s}_i^{j+1} - \hat{Z}_{RS}^{j+1} - X_i\hat{\varepsilon}_Y^{j=0} + Y_i\hat{\varepsilon}_X^{j=0} - \hat{s}_i^{j=0} + \hat{Z}_{RS}^{j=0} \end{aligned} \tag{3.13}$$

Hence

$$\begin{aligned} m_{\hat{d}_i}^2 = &\left(\frac{\partial\hat{d}_i}{\partial\hat{\varepsilon}_Y^{j+1}}\right)^2 m_{\hat{\varepsilon}_Y^{j+1}}^2 + \left(\frac{\partial\hat{d}_i}{\partial\hat{\varepsilon}_X^{j+1}}\right)^2 m_{\hat{\varepsilon}_X^{j+1}}^2 + \left(\frac{\partial\hat{d}_i}{\partial\hat{s}^{j+1}}\right)^2 m_{\hat{s}^{j+1}}^2 + \left(\frac{\partial\hat{d}_i}{\partial\hat{Z}_{RS}^{j+1}}\right)^2 m_{Z_{RS}^{j+1}}^2 \\ &+ \left(\frac{\partial\hat{d}_i}{\partial\hat{\varepsilon}_Y^{j=0}}\right)^2 m_{\hat{\varepsilon}_Y^{j=0}}^2 + \left(\frac{\partial\hat{d}_i}{\partial\hat{\varepsilon}_X^{j=0}}\right)^2 m_{\hat{\varepsilon}_X^{j=0}}^2 + \left(\frac{\partial\hat{d}_i}{\partial\hat{s}^{j=0}}\right)^2 m_{\hat{s}^{j=0}}^2 + \left(\frac{\partial\hat{d}_i}{\partial\hat{Z}_{RS}^{j=0}}\right)^2 m_{\hat{Z}_{RS}^{j=0}}^2 \end{aligned} \tag{3.14}$$

and the final expression

$$m_{\hat{d}_i}^2 = X_i^2 m_{\hat{\varepsilon}_Y^{j+1}}^2 + Y_i^2 m_{\hat{\varepsilon}_X^{j+1}}^2 + m_{\hat{s}^{j+1}}^2 + m_{Z_{RS}^{j+1}}^2 + X_i^2 m_{\hat{\varepsilon}_Y^{j=0}}^2 + Y_i^2 m_{\hat{\varepsilon}_X^{j=0}}^2 + m_{\hat{s}^{j=0}}^2 + m_{\hat{Z}_{RS}^{j=0}}^2. \tag{3.15}$$

Assuming the same values of the mean errors for all measuring periods $j = 1, 2 \ldots$, it means: $m_{\hat{\varepsilon}_Y^{j+1}}^2 = m_{\hat{\varepsilon}_Y^{j=0}}^2 = m_{\hat{\varepsilon}_Y}^2$, $m_{\hat{\varepsilon}_X^{j+1}}^2 = m_{\hat{\varepsilon}_X^{j=0}}^2 = m_{\hat{\varepsilon}_X}^2$, $m_{\hat{s}_i^{j+1}}^2 = m_{\hat{s}_i^{j=0}}^2 = m_{\hat{s}_i}^2$, $m_{\hat{Z}_{RS}^{j+1}}^2 = m_{\hat{Z}_{RS}^{j=0}}^2 = m_{\hat{Z}_{RS}}^2$ the following simplified relationship is gained

$$m_{\hat{d}_i}^2 = 2\left(X_i^2 m_{\hat{\varepsilon}_Y}^2 + Y_i^2 m_{\hat{\varepsilon}_X}^2 + m_{\hat{s}_i}^2 + m_{\hat{Z}_{RS}}^2\right). \tag{3.16}$$

4. Practical examples

In order to verify the obtained relationships, numerical tests were carried out on one HLS (Fig. 2), already analysed in work [9]. The *RS's* stability was adopted (standard procedure) in above mentioned paper. This work is about the new concept that does not assume the stability of *RS*. Therefore, further calculations and analyses will be related to the results obtained from the standard procedure. In the calculations it will be assumed that the local height of the *RS* is $H_{RS} = 0.00$ mm. Table 1 presents the readings obtained from the measuring sensors adopted for calculations in work [9] supplemented with the rectangular coordinates (X, Y).

Table 1. Readings from the measuring sensors z [mm] and the rectangular coordinates X, Y [mm]

Sensor's number	Epoch $j = 0$	Epoch $j = 1$	Coordinates	
	reading z_i	reading z_i	X_i	Y_i
RS	5.1	45.4	10000	10000
1	66.8	101.4	30100	10220
2	53.3	94.7	50100	10030
3	48.0	90.0	50850	40320
4	38.4	80.7	29160	39860
5	43.4	85.4	29860	20510

The calculation was carried out in two following variants:

– Variant I: calculations were carried out for differences of heights between consecutive sensors.
– Variant II: calculations were carried out for differences of heights between different sensors and the reference sensor always.

4.1. Analysis of the results obtained for Variant I

The values presented in Table 1 were used to calculate the differences $h_{i+1}^{j=0} = z_i^{j=0} - z_{i+1}^{j=0}$ and $h_{i+1}^{j=1} = z_i^{j=1} - z_{i+1}^{j=1}$ (similarly to the principle of calculating of differences of heights used in geometric levelling). Table 2 presents the heights differences obtained in both measuring periods.

Table 2. Heights differences [mm]

Heights difference	
$j = 0$	$j = 1$
$h_{RS,1}^{obs} = -61.7$	$h_{RS,1}^{obs} = -56.0$
$h_{1,2}^{obs} = 13.5$	$h_{1,2}^{obs} = 6.7$
$h_{2,3}^{obs} = 5.3$	$h_{2,3}^{obs} = 4.7$
$h_{3,4}^{obs} = 9.6$	$h_{3,4}^{obs} = 9.3$
$h_{4,5}^{obs} = -5.0$	$h_{4,5}^{obs} = -4.7$

The basic adjustment results of variant 1 are presented below. The values of the rotations' angles in both measuring periods (Eq. (2.12)), their difference $\gamma' = \left|(\hat{\varepsilon})^{j=1} - (\hat{\varepsilon})^{j=0}\right|$ and the values of the mean errors obtained from the estimation (Eq. (3.3)) are shown in Table 3.

Commenting on the results obtained from the estimation, it is easy to notice that differences of the rotations' angles are respectively $\gamma'_X = 7.3151^{cc}$, $\gamma'_Y = 67.1568^{cc}$ and

Table 3. The values of: rotation angles, mean errors and parameter γ'

Angle	Epoch $j = 0$	Epoch j =1	γ'	Mean errors
$\hat{\varepsilon}_X^{cc}$	$-647,9543^{cc}$	$-655,^{cc}\,2694$	7.3151^{cc}	$m_{\varepsilon X} = 1^{cc}, 8369$
$\hat{\varepsilon}_Y^{cc}$	$185,8406^{cc}$	$118,^{cc}\,6838$	67.1568^{cc}	$m_{\varepsilon Y} = 2^{cc}, 6714$

they indicate the existence of displacements of the entire examined object treated as the rigid body. The parameters $\gamma_X = km_{\varepsilon X} = 3 \cdot 1,^{cc}\,8369 = 5,^{cc}\,5107$ and $\gamma_Y = km_{\varepsilon Y} = 3 \cdot 2,^{cc}\,6714 = 8,^{cc}\,0142$ are obtained assuming that for $k = 3.0\gamma = km_{\varepsilon}$ is the limited value for vertical displacements. Both magnitudes $\gamma'_X > \gamma_X$ and $\gamma'_Y > \gamma_Y$. Hence, the conclusion is that the examined object displaced vertically. The formulated conclusion concerns the whole construction treated as the rigid body. In order to determine the vertical displacements of individual points of the object (CPs), further calculation process should be carried out according to the theoretical part of this paper. The estimated parameters $\hat{\mathbf{s}}$ (Eq. (2.14)), $\hat{\boldsymbol{\lambda}}$ (Eq. (2.16)) and $\hat{\mathbf{Z}}$ (Eq. (2.17)) are presented in Table 4. The mean errors were calculated from Eq. (3.8), Eq. (3.9), Eq. (3.12).

Table 4. Basic adjustment parameters [mm]

Parameters	Epoch $j = 0$	Epoch $j = 1$	Mean errors
$\hat{s}_{RS}$	17.1	15.7	$m_{\hat{s}_{RS}} = 0.06$
$\hat{s}_1$	–24.2	–19.6	$m_{\hat{s}_1} = 0.06$
$\hat{s}_2$	9.7	7.7	$m_{\hat{s}_2} = 0.03$
$\hat{s}_3$	6.9	7.5	$m_{\hat{s}_3} = 0.06$
$\hat{s}_4$	–5.4	–5.4	$m_{\hat{s}_4} = 0.04$
$\hat{s}_5$	–4.1	–5.8	$m_{\hat{s}_5} = 0.13$
$\hat{\lambda}_{RS}$	–7.3	–8.4	$m_{\hat{\lambda}_{RS}} = 0.05$
$\hat{\lambda}_1$	–27.7	–29.1	$m_{\hat{\lambda}_1} = 0.09$
$\hat{\lambda}_2$	–48.1	–49.7	$m_{\hat{\lambda}_2} = 0.15$
$\hat{\lambda}_3$	–40.0	–44.8	$m_{\hat{\lambda}_3} = 0.23$
$\hat{\lambda}_4$	–18.0	–22.6	$m_{\hat{\lambda}_4} = 0.19$
$\hat{\lambda}_5$	–24.4	–26.9	$m_{\hat{\lambda}_5} = 0.12$
$\hat{Z}_{RS}$	9.8	7.3	$m_{\hat{Z}_{RS}} = 0.05$
$\hat{Z}_1$	–51.9	–48.7	$m_{\hat{Z}_1} = 0.10$
$\hat{Z}_2$	–38.4	–42.0	$m_{\hat{Z}_2} = 0.16$
$\hat{Z}_3$	–33.1	–37.3	$m_{\hat{Z}_3} = 0.19$
$\hat{Z}_4$	–23.5	–28.0	$m_{\hat{Z}_4} = 0.21$
$\hat{Z}_5$	–28.5	–32.7	$m_{\hat{Z}_5} = 0.22$

Using the data presented in Table 5, the RS's vertical displacement $\hat{d}_{RS}$ can be determined using the relationship (Eq. (2.18))

$$\hat{d}_{RS} = \hat{Z}_{RS}^{j=1} - \hat{Z}_{RS}^{j=0} = 7.3 - 9.8 = -2.5 \text{ mm}$$

The mean error calculated for this displacement from Eq. (3.14) is $m_{\hat{d}_{RS}} = 0.13$ mm. The vertical displacements of the other sensors ($i = 1, 2, 3, 4, 5$) were determined using Eq. (3.13). The assessment of their accuracy was also conducted with the use of Eq. (3.14). The obtained values are presented in Table 5. The vertical displacements determined on the basis of the proposed algorithm are marked as M (K). Whereas, the displacements obtained from the algorithm proposed in the work [9] were marked as M (KM).

Table 5. Values of the vertical displacements and their mean errors [mm]

M (K)		M (KM)	
value of displacements	mean errors	value of displacements	mean errors
$\hat{d}_1 = 5.7$	$m_{\hat{d}_1} = 0.16$	$\hat{d}_1 = 5.7$	$m_{\hat{d}_1} = 0.3$
$\hat{d}_2 = -1.1$	$m_{\hat{d}_2} = 0.22$	$\hat{d}_2 = -1.1$	$m_{\hat{d}_2} = 0.3$
$\hat{d}_3 = -1.7$	$m_{\hat{d}_3} = 0.33$	$\hat{d}_3 = -1.7$	$m_{\hat{d}_3} = 0.4$
$\hat{d}_4 = -2.0$	$m_{\hat{d}_4} = 0.27$	$\hat{d}_4 = -2.0$	$m_{\hat{d}_4} = 0.4$
$\hat{d}_5 = -1.7$	$m_{\hat{d}_5} = 0.25$	$\hat{d}_5 = -1.7$	$m_{\hat{d}_5} = 0.2$

Commenting on the calculation results presented in Table 5, it should be stated that the identical vertical displacements values $\hat{d}_i$ were obtained for all sensors using methods: M (K) and M (KM).

4.2. Analysis of the results obtained for Variant II

In variant II detailed analyses were carried out for the differences of heights determined from the following relationship $h_i = z_{RS} - z_i$, for $j = 0$ and $j = 1$ (Table 6).

Table 6. Heights differences [mm]

Heights difference	
$j = 0$	$j = 1$
$h_{RS,1}^{obs} = -61.7$	$h_{RS,1}^{obs} = -56.0$
$h_{RS,2}^{obs} = -48.2$	$h_{RS,2}^{obs} = -49.3$
$h_{RS,3}^{obs} = -42.9$	$h_{RS,3}^{obs} = -44.6$
$h_{RS,4}^{obs} = -33.3$	$h_{RS,4}^{obs} = -35.3$
$h_{RS,5}^{obs} = -38.3$	$h_{RS,5}^{obs} = -40.0$

Table 7 presents respectively: magnitudes of rotation angles (Eq. (2.12)) and their mean errors (Eq. (3.3)) as well as values $\gamma' = \left|(\hat{\varepsilon})^{j=1} - (\hat{\varepsilon})^{j=0}\right|$.

Table 7. The values of: rotation angles, mean errors and parameter γ'

Angle	Epoch $j = 0$	Epoch $j = 1$	γ'	Mean errors
$\hat{\varepsilon}_X^{cc}$	$-647,\ 9543^{cc}$	$-655^{cc},\ 2694$	7.3151^{cc}	$m_{\varepsilon X} = 1^{cc},\ 5786$
$\hat{\varepsilon}_Y^{cc}$	$185,\ 8406^{cc}$	$118^{cc},\ 6838$	66.1458^{cc}	$m_{\varepsilon Y} = 2^{cc},\ 0263$

Commenting on the calculation results presented in Table 7, it can be noted that: $\gamma'_X = 7.3151^{cc}$ assuming that $\gamma_X = km_{\varepsilon X} = 3 \cdot 1,5786^{cc} = 4.7358^{cc}$ and $\gamma'_Y = 66.1458^{cc}$ assuming that $\gamma_Y = km_{\varepsilon Y} = 3 \cdot 2.0263^{cc} = 6.0789^{cc}$.

Hence, the results are the magnitudes of $\gamma'_X > \gamma_X$ and $\gamma'_Y > \gamma_Y$. It is equivalent to the fact that the tested structure (treated as the rigid body) was vertically displaced. The basic parameters values ($\hat{\mathbf{s}}$ (Eq. (2.14)), $\hat{\boldsymbol{\lambda}}$ (Eq. (2.16)) and $\hat{\mathbf{Z}}$ (Eq. (2.17)) obtained from the estimation are presented in Table 8. The mean errors were calculated from: Eq. (3.8), Eq. (3.9), Eq. (3.12).

Table 8. Basic adjustment parameters [mm]

Parameters	Epoch $j = 0$	Epoch $j = 1$	Mean errors
$\hat{s}_{RS}$	17.1	15.7	$m_{\hat{s}_{RS}} = 0.05$
$\hat{s}_1$	–24.2	–19.6	$m_{\hat{s}_1} = 0.08$
$\hat{s}_2$	9.7	7.7	$m_{\hat{s}_2} = 0.05$
$\hat{s}_3$	6.9	7.5	$m_{\hat{s}_3} = 0.06$
$\hat{s}_4$	–5.4	–5.4	$m_{\hat{s}_4} = 0.06$
$\hat{s}_5$	–4.1	–5.8	$m_{\hat{s}_5} = 0.09$
$\hat{\lambda}_{RS}$	–7.3	–8.4	$m_{\hat{\lambda}_{RS}} = 0.03$
$\hat{\lambda}_1$	–27.7	–29.1	$m_{\hat{\lambda}_1} = 0.06$
$\hat{\lambda}_2$	–48.1	–49.7	$m_{\hat{\lambda}_2} = 0.11$
$\hat{\lambda}_3$	–40.0	–44.8	$m_{\hat{\lambda}_3} = 0.12$
$\hat{\lambda}_4$	–18.0	–22.6	$m_{\hat{\lambda}_4} = 0.11$
$\hat{\lambda}_5$	–24.4	–26.9	$m_{\hat{\lambda}_5} = 0.07$
$\hat{Z}_{RS}$	9.8	7.3	$m_{\hat{Z}_{RS}} = 0.07$
$\hat{Z}_1$	–51.9	–48.7	$m_{\hat{Z}_1} = 0.08$
$\hat{Z}_2$	–38.4	–42.0	$m_{\hat{Z}_2} = 0.13$
$\hat{Z}_3$	–33.1	–37.3	$m_{\hat{Z}_3} = 0.16$
$\hat{Z}_4$	–23.5	–28.0	$m_{\hat{Z}_4} = 0.12$
$\hat{Z}_5$	–28.5	–32.7	$m_{\hat{Z}_5} = 0.09$

The *RS's* vertical displacement should be determined in the same way as in variant I. The following results are gained: $\hat{d}_{RS} = \hat{Z}_{RS}^{j=1} - \hat{Z}_{RS}^{j=0} = 7.3 - 9.8 = -2.5$ mm while carrying out the calculations with mean error $m_{\hat{d}_{RS}} = 0.13$ (Eq. (3.14)). The values of the estimated parameters presented in Table 8 allowed to determine the vertical displacements of the individual CPs ($i = 1, 2, 3, 4, 5$). Table 9 presents the displacements values (Eq. (3.13)) obtained with the use of M (K) method, the comparison of their results with the results obtained using the M (KM) method as well as the mean errors values (Eq. (3.14)) of the vertical displacements.

Table 9. Values of the vertical displacements and their mean errors [mm]

M (K)		M (KM)	
value of displacements	mean errors	value of displacements	mean errors
$\hat{d}_1 = 5.7$	$m_{\hat{d}_1} = 0.16$	$\hat{d}_1 = 5.7$	$m_{\hat{d}_1} = 0.3$
$\hat{d}_2 = -1.1$	$m_{\hat{d}_2} = 0.20$	$\hat{d}_2 = -1.1$	$m_{\hat{d}_2} = 0.3$
$\hat{d}_3 = -1.7$	$m_{\hat{d}_3} = 0.27$	$\hat{d}_3 = -1.7$	$m_{\hat{d}_3} = 0.4$
$\hat{d}_4 = -2.0$	$m_{\hat{d}_4} = 0.22$	$\hat{d}_4 = -2.0$	$m_{\hat{d}_4} = 0.4$
$\hat{d}_5 = -1.7$	$m_{\hat{d}_5} = 0.19$	$\hat{d}_5 = -1.7$	$m_{\hat{d}_5} = 0.2$

Commenting on the results of calculations summarised in Table 9, it can be noted that displacements values obtained with the use of both methods are identical. The magnitudes of mean errors obtained from calculations have comparable values between them.

5. Conclusion

Table 10 presents the results of calculations obtained using the M (K) method and the M (KM) method proposed in this work. To compare the obtained results, the displacements values were determined from raw data M (RD).

The values of vertical displacements with the use of the M (RD) method were obtained as follows:

1. Using the relationship $H_{i+1} = H_i + z_i - z_{i+1}$, the heights of the CPs on which the measuring sensors were placed, were calculated in epochs $j = 0$ and $j = 1$, assuming for both epochs $H_{RS}^{j=0,1} = 0.00$ mm.
2. The values of vertical displacements of the CPs were calculated using the dependence $d_i = H_i^{j=1} - H_i^{j=0}$.

The displacements values obtained using the M (RD) method are presented in the last column of Table 10.

Analysing the magnitudes of displacements $\hat{d}$ presented in Table 10, it can be seen, that the M (K) method (variant I and variant II) provides the same results as results received using the methods: M (KM) and M (RD). The values of *RS's* displacements obtained

Table 10. Summary of obtained displacement values [mm]

Variant I M (K)		Variant II M (K)		M (KM)		M (RD)
$\hat{d}$	$m_{\hat{d}}$	$\hat{d}$	$m_{\hat{d}}$	$\hat{d}$	$m_{\hat{d}}$	$\hat{d}$
$\hat{d}_1 = 5.7$	$m_{\hat{d}_1} = 0.16$	$\hat{d}_1 = 5.7$	$m_{\hat{d}_1} = 0.16$	$\hat{d}_1 = 5.7$	$m_{\hat{d}_1} = 0.3$	$\hat{d}_1 = 5.7$
$\hat{d}_2 = -1.1$	$m_{\hat{d}_2} = 0.22$	$\hat{d}_2 = -1.1$	$m_{\hat{d}_2} = 0.20$	$\hat{d}_2 = -1.1$	$m_{\hat{d}_2} = 0.3$	$\hat{d}_2 = -1.1$
$\hat{d}_3 = -1.7$	$m_{\hat{d}_3} = 0.33$	$\hat{d}_3 = -1.7$	$m_{\hat{d}_3} = 0.27$	$\hat{d}_3 = -1.7$	$m_{\hat{d}_3} = 0.4$	$\hat{d}_3 = -1.7$
$\hat{d}_4 = -2.0$	$m_{\hat{d}_4} = 0.27$	$\hat{d}_4 = -2.0$	$m_{\hat{d}_4} = 0.22$	$\hat{d}_4 = -2.0$	$m_{\hat{d}_4} = 0.4$	$\hat{d}_4 = -2.0$
$\hat{d}_5 = -1.7$	$m_{\hat{d}_5} = 0.25$	$\hat{d}_5 = -1.7$	$m_{\hat{d}_5} = 0.19$	$\hat{d}_5 = -1.7$	$m_{\hat{d}_5} = 0.2$	$\hat{d}_5 = -1.7$
$\hat{d}_{RS} = -2.5$	$m_{\hat{d}_{RS}} = 0.13$	$\hat{d}_{RS} = -2.5$	$m_{\hat{d}_{RS}} = 0.13$	–	–	–

from the estimation in both variants are $\hat{d}_{RS} = -2.5$ mm. The possibility of calculating the vertical displacement of *RS* is an undoubted advantage of the submitted proposal. This information enables for more detailed interpretation of the vertical displacements results obtained from HLS. Thus, wider knowledge about maintenance of the entire examined object treated as the rigid body is obtained. The advantage of the submitted proposal is also the possibility of assessing the accuracy of the obtained values of displacements $\hat{d}$ represented by mean errors $m_{\hat{d}}$. It is worth adding here that the covariance matrices presented in this work allow to assess the accuracy of the obtained parameters at all stages of the estimation of the implemented algorithm. It should also be stated that the proposed method of data evaluation of observations' results obtained from HLS found its practical confirmation.

6. Summary

To sum up, this paper proposes the new way of estimation the results obtained from HLS. The calculations assume the lack of stability of the point where the *RS* is placed. The author derived the dependencies solving the research problem and explicit forms of the covariance matrices enabling the accuracy assessment of obtained determinations. The tests using practical examples were also carried out. The results obtained from practical research confirmed the theoretical considerations.

It is essential to add that number of tests (especially empirical) has to be conducted in order to examine the remaining practical properties of submitted proposal. These tests should be realised on different geometrical constructions (e.g. long bridges, tunnels, etc.), larger number of CPs set, etc. The identification of probable limits in application of proposed estimation method will be possible from these tests. The author does not have the access to such various set of HLSs data so, the author invites to scientific cooperation all who are interested in the problems presented in this paper.

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Wyznaczanie przemieszczeń pionowych metodą niwelacji hydrostatycznej przy zmiennym położeniu sensora referencyjnego

Słowa kluczowe: przemieszczenia pionowe, systemy niwelacji hydrostatycznej, przemieszczenia pionowe sensora referencyjnego

Streszczenie:

Wyznaczanie przemieszczeń i odkształceń obiektów jest aktualnym problemem współczesnej geodezji inżynieryjnej. Jednym z najważniejszych etapów takich prac, jest problem lokalizacji stabilnych punktów referencyjnych w stosunku, do których wyznaczanie będą przemieszczenia pozostałych punktów kontrolowanych. Punkty referencyjne muszą zachować stabilność swojego położenia, aby uzyskiwane wyniki pomiarów były wiarygodne. Zasadą jest, że punkty referencyjne lokalizowane są poza strefami wpływu przemieszczającego i deformującego się obiektu. Celem wyznaczania przemieszczeń można wykorzystać między innymi: systemy satelitarne GNSS, tachimetry, niwelację precyzyjną oraz hydrostatyczną, naziemny skaning laserowy, itp. System niwelacji hydrostatycznej (HLS) pozwala na wyznaczanie przemieszczeń pionowych z wysoką dokładnością (rzędu setnych części milimetra). Takie dokładności powodują, że HLSs są bardzo szeroko wykorzystywane zwłaszcza w monitoringu struktur inżynierskich. Przykładem wykorzystania HLS jest między innymi monitoring: tam [11], mostów i wiaduktów [2,5,11], budowli usytuowanych w pobliżu głębokich wykopów występujących podczas budowy tunelu [14], hal sportowych [15], jak również monitoringu synchrotronu CERN [1,10,11], stopy fundamentowej wież i masztów [18], kościołów [12,13], itp. Realizując HLS najczęściej zakłada się, że jeden z punktów wraz z umieszczonym na nim sensorem nie będzie zmieniałswojego położenia i względem tego punktu będą wyznaczane przemieszczenia pozostałych sensorów umieszczonych na punktach kontrolowanych. Taki sensor jest traktowany jako sensor referencyjny (*RS*). Trudnym zadaniem jest jednak określenie stałości położenia takiego sensora.

Równocześnie z rozwojem wysoko dokładnych HLSs, pojawiają się prace, w których szczegółowo analizowane są wpływy elementów składowych HLS na uzyskiwane rezultaty przemieszczeń. Przykładem może być między innymi prace: [4, 18], w których szczegółowo analizowano wpływ temperatury na wyniki uzyskane przy pomocy HLS. W pracy [9] autorzy proponują metodę różnic obserwacji wyznaczania przemieszczeń pionowych. Takie podejście umożliwia eliminowanie błędów systematycznych obciążających rezultaty pomiarów HLSs. Autorzy podali rozwiązania umożliwiające analizę dokładności uzyskiwanych rezultatów przemieszczeń pionowych, jak również ich predykcję z wykorzystaniem filtru metodą Kalmana. W niniejszej pracy przedstawiono nową metodę wyznaczania przemieszczeń pionowych punktów kontrolowanych mierzonych HLSs. Problemem teoretyczno – empirycznym analizowanym w tej pracy jest złożenie o braku stabilności sensora *RS*. W trakcie estymacji wyznaczane są zatem zarówno wartości przemieszczeń sensora *RS*, jak również przemieszczenia pionowe pozostałych sensorów. Oznacza to, że sensor referencyjny traktowany jest tak samo, jak pozostałe sensory umieszczone na punktach kontrolowanych. Praca jest rozwinięciem problematyki analizowanej w pozycjach: [3,6–9,16]. W pracy [3], zaproponowano pewien algorytm postępowania podczas wyznaczania przemieszczeń pionowych w sytuacjach, gdy nie można określić stabilnych punktów referencyjnych. Wykorzystując tę ideę w niniejszej pracy zaproponowano

procedurę wyznaczania przemieszczeń pionowych w sytuacjach, gdy ze względów technicznych lub ekonomicznych nie można wykonać konstrukcji, której warunki geometryczne zapewniają obserwacje nadliczbowe, a tym samym możliwości wyrównania rezultatów obserwacji. W takich sytuacjach najczęściej uzyskujemy jednoznaczne wyniki obliczeń. Przykładem mogą być wyniki obserwacji uzyskane z systemów niwelacji hydrostatycznej lub wiszące ciągi niwelacji geometrycznej. W rozdziale 2 przedstawiono podstawy teoretyczne metody. Analizę dokładności uzyskanych parametrów przedstawiono w rozdziale 3. Praktyczne obliczenia wykonane na symulowanej osnowie i omówienie uzyskanych wyników są treścią rozdziału 4.

Obliczenia realizowano w dwóch następujących wariantach: wariant I: obliczenia wykonano dla przewyższeń uzyskanych pomiędzy kolejnymi sensorami, wariant II: obliczenia wykonano dla przewyższeń uzyskanych pomiędzy kolejnymi sensorami, a sensorem referencyjnym.

Analizując przedstawione w Tabeli 10 (Rozdział 5) rezultaty przemieszczeń $\hat{d}$ widzimy, że proponowaną w niniejszej pracy metodą (M (K), Wariant II) uzyskano takie same wyniki, jak metodami: M (KM) [3] i M (RD) (z bezpośrednich wyników pomiaru). Uzyskane z estymacji wartości przemieszczeń punktu referencyjnego (RS) w obu wariantach wynoszą $\hat{d}_{RS} = -2.5$ [mm]. Możliwość obliczenia przemieszczenia pionowego sensora referencyjnego jest niewątpliwą zaletą zgłoszonej propozycji. Taka informacja umożliwia pełniejszą interpretację uzyskanych z niwelacji hydrostatycznej wyników przemieszczeń pionowych. Tym samym uzyskujemy większą wiedzę o zachowaniu się całego badanego obiektu traktowanego jako bryła sztywna. Zaletą zgłoszonej propozycji, jest także możliwość oceny dokładności ostatecznych wartości przemieszczeń $\hat{d}$ reprezentowanych błędami średnimi $m_{\hat{d}}$. Warto dodać, że przedstawione w niniejszej pracy macierze kowariancji umożliwiają ocenę dokładności otrzymanych parametrów na wszystkich etapach estymacji realizowanego algorytmu. Należy także stwierdzić, że zaproponowany sposób opracowania wyników obserwacji uzyskanych z niwelacji hydrostatycznej znalazłswoje praktyczne potwierdzenie.

Przedstawione rezultaty zachęcają do dalszych bardziej szczegółowych badań na rzeczywistych obiektach inżynierskich.

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