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Observers of fractional linear continuous-time systems

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The concepts of full-order and reduced-order observers are extended to the fractional linear continuous-time systems. Necessary and sufficient conditions for the existence of the observers for fractional linear systems are established. Procedures for designing of the observers are given and illustrated by examples.

Key words: design, fractional, linear, continuous-time, system, full-order, reduced-order, observer.

1. Introduction

Mathematical fundamentals of the fractional calculus are given in the monographs [8, 10, 14, 15]. The fractional linear systems have been investigated in [1–5, 7–10, 13–18]. Positive linear systems with different fractional orders have been addressed in [4, 5, 8, 10]. The stabilization problem of linear systems has been investigated in [17, 18].

The concept of observers has been introduced by Luenberger in [11, 12] and has been applied to reconstruct the estimate of the state vectors of linear systems. The reduced-order perfect observers for singular systems have been analyzed in [6] and the nonlinear observers of fractional descriptor discrete-time nonlinear systems in [7]. The observers found many practical industrial applications.

In this paper the design methods of full-order and reduced-order observers will be extended to the fractional linear continuous-time systems.

The paper is organized as follows. The design technique of full-order observers has been extended to the fractional linear continuous-time systems in section 2 and the design technique of reduce-order observers in section 3. Concluding remarks are given in section 4.

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The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n \times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n \times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, I_n – the $n \times n$ identity matrix.

2. Full-order fractional observers

Consider the fractional continuous-time linear system

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = Ax(t) + Bu(t), \qquad 0 < \alpha < 1, \tag{1a}$$

$$y(t) = Cx(t), \tag{1b}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$,$C \in \Re^{p\times n}$,

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau, \qquad \dot{x}(\tau) = \frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} \tag{1c}$$

is the Caputo fractional derivative and

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t, \qquad Re(z) > 0 \tag{1d}$$

is the gamma function [8, 10].

Definition 1 *The fractional linear system (1) is called observable if knowing its input $u(t)$ and output $y(t)$ in some given interval $[0,\ t]$, $t > 0$ it is possible to find its unique initial condition $x(0)$.*

Theorem 1 *[8, 10] The fractional linear system (1) is observable if and only if*

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{2}$$

Definition 2 *The fractional linear system*

$$\frac{\mathrm{d}^\alpha \hat{x}(t)}{\mathrm{d}t^\alpha} = A\hat{x}(t) + Bu(t) + H\left[y(t) - C\hat{x}(t)\right], \tag{3}$$

where $\hat{x}(t) \in \Re^n$, $u(t) \in \Re^m$, $y(t) \in \Re^p$ *are the state, input and output vectors, is called full order fractional asymptotic observer of the system (1) if*

$$\lim_{t\to\infty} [x(t) - \hat{x}(t)] = 0 \tag{4}$$

for all initial conditions $x(0)$ *and* $\hat{x}(0)$.

The matrix $H \in \Re^{n\times p}$ will be determined during the design procedure of the observer.

Let

$$e(t) = x(t) - \hat{x}(t). \tag{5}$$

Taking the fractional derivative of (5) and using (1a) and (3) we obtain

$$\begin{aligned}\frac{\mathrm{d}^\alpha e(t)}{\mathrm{d}t^\alpha} &= \frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} - \frac{\mathrm{d}^\alpha \hat{x}(t)}{\mathrm{d}t^\alpha}\\ &= Ax(t) + Bu(t) - [A\hat{x}(t) + Bu(t) + H(y(t) - C\hat{x}(t))]\\ &= A\,[x(t) - \hat{x}(t)] - HC\,[x(t) - \hat{x}(t)] = Fe(t),\end{aligned} \tag{6}$$

where

$$F = A - HC. \tag{7}$$

The solution of the equation (6) has the form

$$e(t) = \Phi_0(t)e(0), \qquad \Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{8}$$

where $e(0) = x(0) - \hat{x}(0)$.

From (8) it follows that

$$\lim_{t\to\infty} [e(t)] = 0 \tag{9}$$

if and only if all eigenvalues $s_1, \ldots, s_n$ of the matrix F are in the left-hand part of the complex plane

$$\mathrm{Re}\, s_k < 0 \quad \text{for} \;\; k = 1, \ldots, n. \tag{10}$$

It is well-known [7, 8] that we may choose the matrix H so that all eigenvalues of the matrix F satisfied the condition (10) if and only if the pair (A, C) is observable, i.e.

$$\operatorname{rank}\begin{bmatrix} I_n s - A \\ C \end{bmatrix} = n, \qquad s \in C \quad \text{or} \quad \operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n. \tag{11}$$

Therefore, the following theorem has been proved.

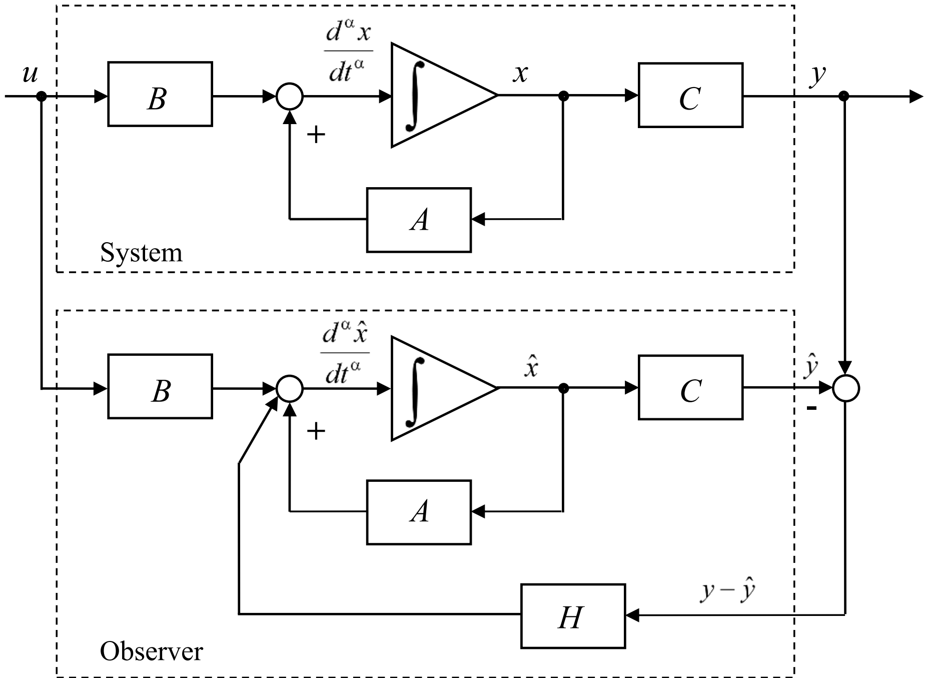

Figure 1: The system with observer

Theorem 2 *For the fractional linear system (1) there exists full order fractional asymptotic observer (3) if and only if the pair $(A,\ C)$ is observable.*

To design the fractional observer (3) for the system (1) we have to find the matrix H satisfying the condition

$$\det\left[I_n s - F\right] = \det\left[I_n s - A + HC\right] = p(s), \tag{12}$$

where $p(s)$ is a polynomial satisfying the condition (10).

The dynamic of the observer should be faster than the dynamic of the system.

If the condition (11) is satisfied then the following procedure can be used to design the fractional observer (3).

Procedure 1

Step 1. *Transform the pair $(A,\ C)$ to the canonical observer form [8, 10] using the similarity transformation: $A = QAQ$, $C = CQ$.*

Step 2. *Choose the desired eigenvalues $s_1,\ \ldots,\ s_n$ of the matrix F of observer (3).*

Step 3. *Using one of the known procedures [6, 7] compute the matrix H of the observer (3).*

Example 1

For the fractional linear system (1) with the matrices

$$A = \begin{bmatrix} 0 & 0.5 & 1 \\ 0 & 0 & -1 \\ 2 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \tag{13}$$

find the fractional observer (3).

The pair (A, C) given by (13) is observable since

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & -1 \\ -2 & 1 & 3 \end{bmatrix} = 3 = n. \tag{14}$$

Therefore, by Theorem 2 there exist the observer (3) for the fractional system with (13).

Using Procedure 1 we obtain the following.

Step 1. The matrix Q transforming the pair (A, C) to its canonical form is given by

$$Q = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{15}$$

and

$$\begin{aligned} \overline{A} &= QAQ^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0.5 & 1 \\ 0 & 0 & -1 \\ 2 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}, \\ \overline{C} &= CQ^{-1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \end{aligned} \tag{16}$$

Step 2. We choose as the eigenvalues of the matrix F, $s_1 = s_2 = s_3 = -4$. In this case the characteristic polynomial of F has the form

$$\det\left[I_3 s - F\right] = (s+4)^3 = s^3 + 12s^2 + 48s + 64 \tag{17}$$

and

$$\overline{F} = \begin{bmatrix} 0 & 0 & -64 \\ 1 & 0 & -48 \\ 0 & 1 & -12 \end{bmatrix}. \tag{18}$$

Step 3. Knowing the matrix $\overline{F}$ given by (18) and $\overline{A}$, $\overline{C}$ given by (16) from the equation

$$\begin{bmatrix} 0 & 0 & -64 \\ 1 & 0 & -48 \\ 0 & 1 & -12 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} \tag{19}$$

we may find the matrix $\overline{H}$ and

$$H = Q^{-1}\overline{H} = \begin{bmatrix} 0 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 63 \\ 50 \\ 11 \end{bmatrix} = \begin{bmatrix} 25 \\ 63 \\ 11 \end{bmatrix}. \tag{20}$$

The equation of the fractional observer has the form

$$\begin{aligned} \frac{\mathrm{d}^\alpha \hat{x}(t)}{\mathrm{d}t^\alpha} &= F\hat{x}(t) + Bu(t) + H(y(t) \\ &= \begin{bmatrix} 0 & 0 & -64 \\ 1 & 0 & -48 \\ 0 & 1 & -12 \end{bmatrix} \hat{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) + \begin{bmatrix} 25 \\ 63 \\ 11 \end{bmatrix} y(t). \end{aligned} \tag{21}$$

3. Reduced order fractional observers

Assume that the matrix C of the fractional system (1) has full row rank

$$\operatorname{rank} C = p \tag{22}$$

and the submatrix C_1 is nonsingular of $C = [C_1,\ C_2]$

$$\det C_1 \neq 0. \tag{23}$$

Note that if (22) holds than by permutation of the columns of the matrix C it is possible to obtain (23).

It is easy to check that the nonsingular matrix

$$Q = \begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-p} \end{bmatrix} \tag{24}$$

satisfies the equality

$$\overline{C} = CQ = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-p} \end{bmatrix} = \begin{bmatrix} I_p & 0 \end{bmatrix}. \tag{25}$$

Define the new state vector

$$\bar{x} = Q^{-1}x = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \qquad \bar{x}_1 \in \Re^p, \quad \bar{x}_2 \in \Re^{n-p}. \tag{26}$$

Using (26) we decompose the equations (1) into the subsystems

$$\frac{\mathrm{d}^\alpha \bar{x}_1(t)}{\mathrm{d}t^\alpha} = A_{11}\bar{x}_1(t) + A_{12}\bar{x}_2(t) + B_1 u(t), \tag{27a}$$

$$\frac{\mathrm{d}^\alpha \bar{x}_2(t)}{\mathrm{d}t^\alpha} = A_{21}\bar{x}_1(t) + A_{22}\bar{x}_2(t) + B_2 u(t), \tag{27b}$$

$$y = \overline{C}\bar{x} = \bar{x}_1\,, \tag{27c}$$

where

$$Q^{-1}AQ = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad A_{11} \in \Re^{p\times p}, \quad A_{22} \in \Re^{(n-p)\times(n-p)}, \tag{27d}$$

$$Q^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad B_1 \in \Re^{p\times m}, \quad B_2 \in \Re^{(n-p)\times m}. \tag{27e}$$

Note that

$$\bar{u}(t) = B_2 u(t) + A_{21} y(t) \tag{28a}$$

and

$$\bar{y}(t) = \frac{\mathrm{d}^\alpha y(t)}{\mathrm{d}t^\alpha} - A_{11} y(t) - B_1 u(t) \tag{28b}$$

are known and from the equations (27a) and (27b) we have

$$\frac{\mathrm{d}^\alpha \bar{x}_2(t)}{\mathrm{d}t^\alpha} = A_{22}\bar{x}_2(t) + \bar{u}(t), \tag{29}$$

$$\bar{y}(t) = A_{12}\bar{x}_2(t). \tag{30}$$

Therefore, the problem of finding the estimate $\hat{x}(t)$ of the state vector has been reduced to finding the estimate $\hat{x}_2(t)$ of the subvector $\bar{x}_2$. By Theorem 2 there exists a reduced order fractional observer estimating the subvector $\bar{x}_2$ if and only if the pair $(A_{22},\ A_{12})$ is observable.

We shall show that the pair $(A_{22},\ A_{12})$ is observable if and only if the pair $(A,\ C)$ is observable.

Applying the condition (11) to the system (27) and taking into account that the similarity transformation does not change the observability of the system we obtain

$$\operatorname{rank}\begin{bmatrix} I_p s - A_{11} & -A_{12} \\ -A_{21} & I_{n-p}s - A_{22} \\ I_p & 0 \end{bmatrix} = n. \tag{31}$$

Note that the condition (31) is satisfied if and only if

$$\operatorname{rank}\begin{bmatrix} -A_{12} \\ I_{n-p}s - A_{22} \end{bmatrix} = n - p. \tag{32}$$

Therefore, the pair (A_{22}, A_{12}) is observable if and only if the pair (A, C) is observable.

Theorem 3 *There exists a reduced-order fractional observer of the system (29), (30) if and only if the pair (A, C) of the system (1) is observable.*

Note that to design the reduced-order fractional observer of the system (29), (30) we may apply the Procedure 1.

Therefore, the reduced-order observer of the fractional system (1) can be design by the use of the following procedure.

Procedure 2

Step 1. *Compute the nonsingular matrix (24) and find the matrix $\overline{C}$ given by (25).*

Step 2. *Find the vectors $\bar{u}(t)$ and $\bar{y}(t)$ defined by (28) and the equations (29), (30) of the subsystem.*

Step 3. *Apply Procedure 2.1 to compute the matrices of the reduced-order observer.*

Example 2

For the fractional linear system (1) with the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & -2 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0.5 & 0 & -0.5 \end{bmatrix} \tag{33}$$

find the reduced-order observer of the system.

The pair (A, C) given (33) is observable since

$$\det \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & 0.5 & 0 & -0.5 \\ 0 & 1 & -1 & -2 \\ -0.5 & -1 & 0 & 1 \end{vmatrix} = -0.25 \neq 0. \tag{34}$$

Using the Procedure 2 for (33) we obtain

Step 1. In this case $C_1 = -C_2$ and the matrix (24) has the form

$$Q = \begin{bmatrix} C_1^{-1} & -C_1^{-1}C_2 \\ 0 & I_{n-p} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{35}$$

and

$$\overline{C} = CQ = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0.5 & 0 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Step 2. Using (27) and (35) we compute

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 & -2 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & -1 & -1 \\ -0.5 & -2 & -0.5 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix} \tag{36a}$$

(36b)

and

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = Q^{-1}B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{36c}$$

In this case the equations (27) have the forms

$$\begin{bmatrix} \dfrac{\mathrm{d}^\alpha \bar{x}_1(t)}{\mathrm{d}t^\alpha} \\ \dfrac{\mathrm{d}^\alpha \bar{x}_2(t)}{\mathrm{d}t^\alpha} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$$

$$= \begin{bmatrix} 0 & 2 & -1 & -1 \\ -0.5 & -2 & -0.5 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} u(t), \tag{37a}$$

where

$$\bar{x}_1 = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix}, \qquad \bar{x}_2 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}. \tag{37b}$$

Therefore, we have

$$\frac{\mathrm{d}^\alpha \bar{x}_2(t)}{\mathrm{d}t^\alpha} = A_{22}\bar{x}_2(t) + \bar{u}(t), \qquad \bar{y}(t) = A_{12}\bar{x}_2(t), \tag{38a}$$

where

$$\bar{u}(t) = B_2 u(t) + A_{21} y(t), \qquad \bar{y}(t) = \frac{\mathrm{d}^\alpha y(t)}{\mathrm{d}t^\alpha} - A_{11} y(t) - B_1 u(t). \tag{38b}$$

Step 3.To design the reduced-order fractional observer of the system (38) the Procedure 1 is used.

Assuming the eigenvalues $s_1 = s_2 = -5$ of the matrix F we obtain

$$F = \begin{bmatrix} 0 & 1 \\ -25 & -10 \end{bmatrix} \tag{39}$$

and

$$F = A_{22} - HA_{12} = \begin{bmatrix} 0 & 1 \\ -25 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - H \begin{bmatrix} -1 & -1 \\ -0.5 & 0 \end{bmatrix}. \tag{40}$$

From (40) we have

$$H = [A_{22} - F]\, A_{12}^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -25 & -10 \end{bmatrix} \right) \begin{bmatrix} -1 & -1 \\ -0.5 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -4 \\ -10 & 34 \end{bmatrix}. \tag{41}$$

The reduced-order observer is given by

$$\begin{aligned} \frac{\mathrm{d}^\alpha \hat{\hat{x}}_2(t)}{\mathrm{d}t^\alpha} &= F\hat{\hat{x}}_2(t) + (B_2 - HB_1)u(t) + (A_{21} + HA_{11})y(t) + H\frac{\mathrm{d}^\alpha y(t)}{\mathrm{d}t^\alpha} \\ &= \begin{bmatrix} 0 & 1 \\ -25 & -10 \end{bmatrix} \hat{\hat{x}}_2(t) + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u(t) + \begin{bmatrix} 2 & 10 \\ 19 & 50 \end{bmatrix} y(t) + \begin{bmatrix} 1 & -4 \\ -10 & 34 \end{bmatrix} \frac{\mathrm{d}^\alpha y(t)}{\mathrm{d}t^\alpha}. \end{aligned} \tag{42}$$

4. Concluding remarks

The concepts of full-order and reduced-order observers have been extended to the fractional linear continuous-time systems. Necessary and sufficient conditions for the existence of the observers for fractional linear systems have been established (Theorems 2 and 3). Procedures for designing of the observers have been given (Procedures 1 and 2). The procedures have been illustrated by numerical examples. An open problem is an extension of the considerations to the different orders fractional linear systems.

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