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Construction of Generalized Rademacher Functions in Terms of Ternary Logic: Solving the Problem of Visibility of Using Galois Fields for Digital Signal Processing

Elizaveta S. Vitulyova, Dinara K. Matrassulova, and Ibragim E. Suleimenov

***Abstract*—Generalized Rademacher functions, constructed as a sequence of elements of Galois fields are intended to find the spectral representation of signals with levels. These functions form a complete basis on the interval corresponding to -1 discrete time intervals and for passing into the classical Rademacher functions. The advantage of such spectra obtained using Galois Fields Fourier Transform is that the range of variation of the spectrum amplitudes remains the same as the range of variation of the original signal, which is modeled on discrete time functions taking values in the Galois field.**

***Keywords*—digital signal processing; non-binary Galois fields; Fourier transform; Rademacher functions; Walsh function; multivalued logic; visibility problem; algebraic extensions; ternary representation of a number**

I. Introduction

WALSH functions, which form a complete set of "digital" orthogonal functions, are one of the main tools for spectral analysis of digital signals [1-3]. Their construction was originally an attempt to implement a "digital" analogue of harmonic functions most widely used for spectral analysis of analog signals.

Various options for improving digital signal processing tools based on the use of the Walsh function are well known in the current literature [4-6]. Various modifications of the Rademacher functions are also known, for example [7,8], on the basis of which the Walsh function is constructed. However, the approaches proposed in the cited works do not allow to completely overcome the main drawback of these "digital bases" (more precisely, orthogonal systems of piecewise constant functions), which is associated with the problem of their completeness. "Digital" analogs of harmonic functions, which would be used as widely as the classical Walsh function, have not yet been proposed, which, among other things, is associated with methodological problems. We emphasize once again that the Walsh functions can only partially be regarded as an analogue of harmonic functions, which, in particular, follows from their aperiodic nature.

At the same time, the need to develop "digital" functions that could play the same role as harmonic functions has long been ripe, which does not require extensive evidence - the vast majority of signals that are exchanged between modern devices are precisely digital.

In various branches of information theory, in particular, in the theory of coding and decoding, as well as for the purposes of pattern recognition, Galois fields are widely used, both binary [9,10] and non-binary [11-13], and based on the latter have constructed and applied analogs of the Fourier transform [14-16].

Nonbinary Galois fields can also be used to algebraize multivalued logic, for example, the Galois field GF (3) containing three elements can be used to construct algebraic operations, to which operations performed on variables of ternary logic are reduced - in the same way as classical the binary logic variables correspond to the Galois field GF (2).

It is pertinent to emphasize that interest in multivalued logics, which goes back to the logic of Lukasiewicz [17], has recently been growing again [18-20]. This is due, among other things, to the problem of artificial intelligence [21, 22], since it is the multi-valued logics, which since the time of Lukasiewicz have been considered and are considered as an alternative to the logic of Aristotle, that allow revealing many features of the functioning of intelligence, which obviously turn out to be irreducible to binary logic.

Establishing the relationship between multivalued logics and digital signal processing methods using non-binary Galois fields is of considerable interest, at least from a methodological point of view. Indeed, such a wide distribution of binary variables is determined, among other things, by the fact that there is a rigid connection between the methods underlying computing systems (digital electronics), methods of digital signal processing (for example, methods of error-correcting coding, the most famous of which is the code Hamming [23]) and the actual tools of mathematical logic, which also widely use binary variables.

The literature contains works [24-26], which provide very strong arguments in favor of using ternary logic for applied purposes, but it has not yet received as widespread distribution as it deserves. This is largely due to, among other things, methodological problems. Constructions using Galois fields are

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distinguished by a high degree of abstraction, i.e., providing visualization of the corresponding methods is also an important task, especially when we consider the possibilities of using ternary codes for artificial intelligence systems.

This paper shows that such concepts of abstract algebra as "algebraic extension", "primitive element", etc. can be made quite descriptive if we introduce the concept of "logical imaginary unit". We emphasize that the functions of a complex variable have long become one of the main tools of radio engineering, therefore, the above concept creates, among other things, the prerequisites for overcoming interdisciplinary barriers that often arise when using abstract algebra tools in digital signal processing.

This paper presents a method for spectral analysis of digitized signals, based on the use of the proposed interpretation of the concept of "logical imaginary unit" and ternary logic, which makes it possible to visualize the construction of generalized Rademacher functions.

The advantage of this approach is that the spectrum components also correspond to logical variables that can be assigned to a specific Galois field. This provides a significant reduction in the amount of information required for the transmission of spectra over communication channels, and also creates the preconditions for the use of methods of error-correcting coding in artificial intelligence systems using ternary logic (the connection between operations performed by neural networks and algorithms for error-correcting coding is shown in [27, 28].

II. Generalization of Rademacher functions and prerequisites for using the term "logical imaginary unit"

The considered version of the generalized Rademacher functions can be constructed based on the fact that for any element ζ of an arbitrary Galois field containing $n+1$ elements,

$$\zeta^n = 1 \tag{1}$$

The meaning of relation (1) can be illustrated by the example of the field $GF(3^2)$, which can be obtained, for example, by an algebraic extension of the field $GF(3)$ using the polynomial

$$f(x) = x^2 + 1, \tag{2}$$

which is irreducible over the field $GF(3)$.

All Galois fields $GF(3)$ are isomorphic, but for the purposes of this work, it is convenient to use the representation [29] through three elements 1, 0, and -1, for which the following rules of addition and multiplication are satisfied.

$$1+1=-1;\ -1-1=1;\ -1+1=1-1=0, \tag{3}$$

$$-1\cdot -1 = 1;\ a\cdot 0 = 0; \mathrm{a}+0=\mathrm{a} \tag{4}$$

The equation

$$x^2+1=0 \tag{5}$$

has no solutions in the field $GF(3)$, therefore, it is permissible to introduce into consideration its formal root, which can be interpreted as a logical imaginary unit i due to the fact that

$$i^2=-1 \tag{6}$$

The algebraic extension of the field $GF(3)$ to the field $GF(3^2)$ when representing the elements of $GF(3)$ through a triple (-1,0,1) in this case is a collection of elements represented in the form

$$A = a_0 + a_1 i, \tag{7}$$

which formally coincides with the notation for complex numbers, with the difference, however, that in the notation (5) the coefficients a_i belong to the field $GF(3)$ represented through the elements (-1,0,1).

Table I explicitly indicates the powers of the element $(1-i)$, calculated in accordance with formula (4) and the rules of operating with the elements of the field $GF(3) = (-1{,}0{,}1)$. Correctly element$(1-i)$ should be interpreted as a primitive root of one, i.e., an element whose degrees are given all elements of the considered field $GF(3^2)$.

TABLE I
Degrees of primitive element $(1-i)$

n	2	3	4	5	6	7	8
$(1-i)^n$	i	$1+i$	-1	$-(1-i)$	$-i$	$-(1+i)$	1

It can be seen that the 8-th power of the element $(1-i)$ is indeed equal to 1, which corresponds to the record (1), since the $GF(3^2)$ field contains 9 elements. The same result is also true for the 8-th power of any other nonzero element of the field $GF(3^2)$, since an arbitrary nonzero element of this field can be represented in the form

$$A = (1-i)^k; 0 \le k \le 7, \tag{8}$$

Correctly, formula (7) is a consequence of the general conclusion about the structure of Galois fields, according to which its multiplicative group is cyclic. In particular, the eighth power of the element A in representation (8) is

$$A = ((1-i)^k)^8 = ((1-i)^8)^k = 1^k = 1 \tag{9}$$

There is also a general theorem for the sum of degrees.

$$1+\zeta+\zeta^2+\dots+\zeta^{n-1} = \begin{cases} \text{"n"}, \zeta = 1 \\ 0, \zeta \ne 1 \end{cases} \tag{10}$$

where n - is the number of nonzero elements in the given Galois field.

This theorem is applicable to any nonzero element from any Galois field, since there is a relation that follows from the formula for the geometric progression

$$1+\zeta+\zeta^2+\dots+\zeta^{n-1} = \frac{1-\zeta^n}{1-\zeta} \tag{11}$$

We emphasize that in the formula (10) the number "n" appears only formally, since the summation should be performed precisely in the sense of addition in this particular field, and "n" is far from necessarily its element. The number "n" in the formula (10), accordingly, is nothing more than a symbol implying the summation of "n" ones.

Formula (11) can be directly used to construct generalized Rademacher functions. Let's show it.

We construct the following sequences, starting from the powers of the element $\theta = (1-i)$. Similar sequences, of course, can be built starting from any other primitive element, i.e., element whose degree up to the seventh inclusive is given by all elements of $GF(3^2)$ except for zero. We have

$$\begin{gathered} w_1 = (1, \theta, \theta^2, \theta^3, \dots, \theta^7) \\ w_2 = (1, \theta^2, \theta^{2\cdot 2}, \theta^{2\cdot 3}, \dots, \theta^{2\cdot 7}) \\ \dots \\ w_7 = (1, \theta^7, \theta^{7\cdot 2}, \theta^{7\cdot 3}, \dots, \theta^{7\cdot 7}) \end{gathered} \tag{12}$$

As follows from (12), there are exactly seven such sequences, more generally, - n - 1. Let us supplement the set of these sequences with the sequence

$$w_0 = (1,1,1,1,\dots,1), \quad (13)$$

consisting only of units.

When (13) is included in set (12), obviously, the number of sequences of the considered form will be equal to n - the number of nonzero elements of the considered Galois field (in the case under consideration, $n = 8$).

We emphasize that, by virtue of (1), all degrees appearing in (12) de facto do not exceed 7. Otherwise, the products of integers (degrees) included in them are calculated by $mod8$.

The above is illustrated by Table II, which shows the sequences formed according to the rules (12) and (13). It can be seen that, as follows from the general theory of Galois fields, only 8 elements appear in this table, which are the powers of element $1 - i$ listed in Table I.

TABLE II
Sequences formed according to rules (12) and (13), for the case $\theta = 1 - i$

	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	$1-i$	i	$1+i$	-1	$-1+i$	$-i$	$-1-i$
2	1	i	-1	$-i$	1	i	-1	$-i$
3	1	$1+i$	$-i$	$1-i$	-1	$-1-i$	i	$-1+i$
4	1	-1	1	-1	1	-1	1	-1
5	1	$-1+i$	i	$-1-i$	-1	$1-i$	$-i$	$1+i$
6	1	$-i$	-1	i	1	$-i$	-1	i
7	1	$-1-i$	$-i$	$-1+i$	-1	$1+i$	i	$1-i$

Note that in Table II, in addition to the sequence with number 0, there are three sequences whose period differs from 8. These are sequences with numbers 2, 4 and 6. This feature is associated with the fact that the multiplicative group of the field under consideration has subgroups, the number of elements in which is a divisor of the number of elements in this group (2 and 4).

For each of the sequences w_k appearing in Table II, it is possible to choose a sequence $w_{\tilde{k}}$ from the same list that will hold

$$(w_k, w_{\tilde{k}}) = (1,1,1,1,\dots,1), \quad (14)$$

where (a, b) – direct product of two sequences a and b.

$$(a, b) = (a_1 b_1, a_2 b_2, \dots, a_n b_n), \quad (15)$$

Specifically, the value of $\tilde{k}$ is determined from the condition

$$k \equiv \tilde{k}(mod8), \quad (16)$$

Formula (15) also shows that in the list Table II, the number $\tilde{k}$ is determined by k uniquely, i.e., the sequence $w_{\tilde{k}}$ can be considered as conjugate to the sequence w_k, in the same way as, when using spectral representations in terms of harmonic functions, the function $exp(ikt)$ is considered as the conjugate with respect to the function $exp(-ikt)$. The uniqueness of the choice of the number $\tilde{k}$ with respect to the number k also follows from the fact that, in the Galois field, each nonzero element has an inverse element, which is unique. Accordingly, the second element of the sequence $w_{\tilde{k}}$ is the inverse element to the second element of the sequence w_k, which is uniquely determined; the same is true for the other elements by the construction of these sequences.

The numbers of the original sequences and the numbers of the sequences conjugated to them are presented in Table III.

TABLE III
Numbers of $\tilde{k}$ of sequences conjugate to sequences numbered k

k	0	1	2	3	4	5	6	7
$\tilde{k}$	0	7	6	5	4	3	2	1

It can be seen that for the case under consideration the correspondence between the sequences $w_{\tilde{k}}$ and the sequences w_k can be considered as an analogue of the correspondence between the complex representations of harmonic functions related to each other by the operation of complex conjugation

$$exp(i\omega t) \leftrightarrow exp(-i\omega t), \quad (17)$$

where ω – harmonic frequency, t – time variable.

Applying formula (11) to the sequences under consideration, one can see that

$$\sum_{j=0}^{j=7} w_k^{(j)} w_{\tilde{k}}^{(j)} = \begin{cases} 1, k = \tilde{k} \\ 0, k \neq \tilde{k} \end{cases} \quad (18)$$

In other words, in the interval containing 8 measures, the sequences shown in Table II form a complete basis.

Based on formula (18), one can immediately go to the spectral representation of the signal in the form

$$\vec{u} = \sum_{j=0}^{j=7} z_j \vec{w}_j \quad (19)$$

where $\vec{u}$ – the sequence of 8 elements of the Galois field $GF(3^2)$, which can also be interpreted as a piecewise constant function that takes values in this field with the number of values equal to 8.

Of course, this representation is valid only for functions that take a value in the Galois field $GF(3^2)$ and are specified on an interval of 8 clock cycles, but, as will be clear from what follows, it admits a generalization that provides the possibility of real practical use.

The amplitudes of the spectral components, which are also elements of the Galois field, are expressed in terms of the function $\vec{u}$ as follows.

$$z_k = (\vec{u}, \vec{w}_{\tilde{k}}) = \sum_{i=0}^{i=7} z_i (\vec{w}_i, \vec{w}_{\tilde{i}}) \quad (20)$$

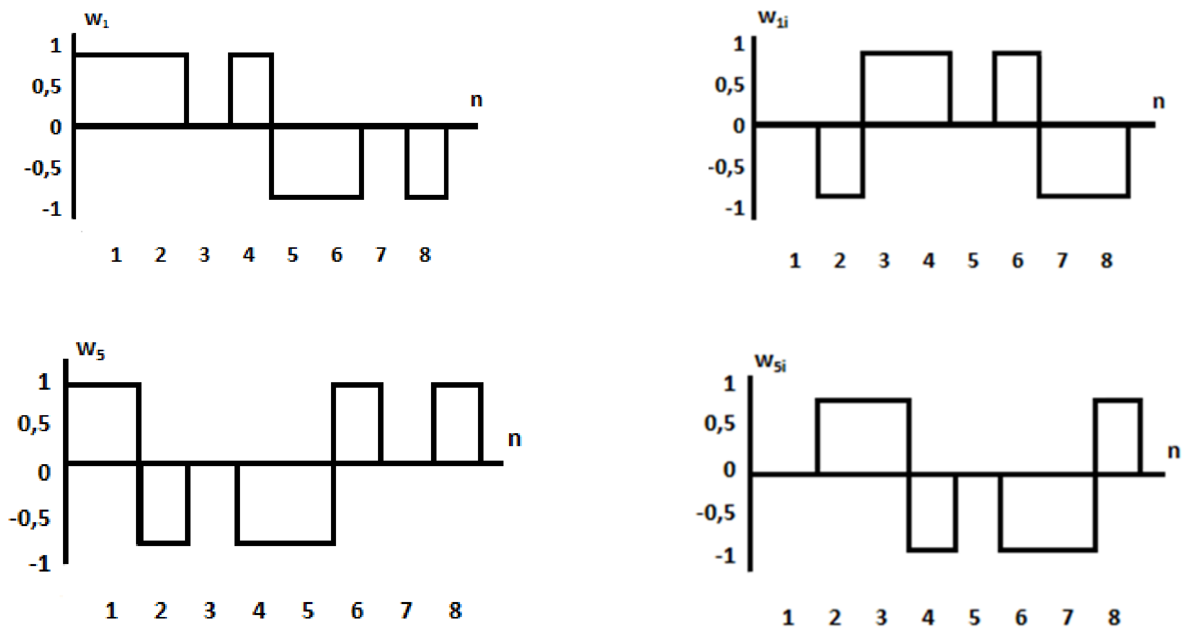
which follows directly from the formula (18).

Thus, the obtained sequences can indeed be interpreted as generalized Rademacher functions, which constitute a complete basis on an interval of 8 clock cycles.

We emphasize that the above procedure can also be applied to the Galois field $GF(3)$. In this case, the number of sequences of the form (12) is reduced to one, in which there are only 3-1 = 2 values equal to -1 and 1, i.e., in this limit, the functions under consideration indeed transform into the classical Rademacher functions, which justifies the term "generalized Rademacher functions" used.

Since all elements of the field under consideration are representable in the form $a = a_1 + ia_2$, it is permissible to speak about the real and imaginary parts of the functions $\vec{u}$, which can be depicted graphically.

Diagrams illustrating the behavior of these functions using such graphs are shown in Fig. 1 (the left column is the real part, the right one is imaginary).



In table II you can see that one of the sequences under consideration are constant, one has period 2, two have period 4. The period of other sequences can be considered equal to 8. This corresponds to the fact that the field under consideration contains only four primitive elements, the degrees of which are given by all elements fields. The same construction can be used to develop algorithms using fields $GF(p^2)$, where p is a prime number. Namely, the construction shown in Table II was based on a primitive element, the degree of which is given by the entire field. Finding such an element, generally speaking, is a nontrivial task. However, in the construction of generalized Rademacher functions, de facto, all elements of the field are used, more precisely, each of these functions is a power of one of the elements of the field. Therefore, it is not necessary to find such an element analytically; it can also be done by software, determining the frequency of the obtained functions.

Thus, the use of the concept of "logical imaginary unit" is really expedient. At a minimum, the functions of a complex variable have been used in radio engineering for more than a long time. Against this background, the concept of algebraic extension looks much more abstract; it has not entered widespread use, which is especially true for specialists in the field of applied electronics. Consequently, the degree of clarity will certainly be higher if the theoretical constructions are closer to the conceptual apparatus that has already become commonly used.

Let's take a look at how exactly these functions can be applied to digital signal processing.

III. Application of the Proposed Generalization of Rademacher Functions to Digital Signal Processing

First of all, we note that real digital signals always change in a finite range of amplitudes; moreover, they correspond to a finite number of levels. Consequently, it is always possible to choose the division of the range of variation of the amplitudes so that the discrete signal levels correspond to a certain Galois field. It is common practice to split the amplitude range into subranges that correspond to the binary representation of the signal, but this is no more than a matter of convention.

Galois fields $GF(3^n)$ orrespond to the ternary representation of numbers, just as the fields $GF(2^n)$ correspond to their binary representation. Indeed, any integer can be represented in the form

$$a \dots bc \leftrightarrow a \cdot 3^n + \dots + b \cdot 3^1 + c \cdot 3^0 \tag{21}$$

where the letter designations correspond to one of the elements of the field GF (3), more precisely, to its mapping to a triple (-1,0,1).

According to rule (21), a ternary number is converted to decimal. An example of such a conversion for a specific combination of symbols of ternary logic is given by the following entry:

$$1\tilde{1}01 \leftrightarrow 1 \cdot 3^3 - 1 \cdot 3^2 + 0 \cdot 3^1 + 1 \cdot 3^0 = \\ = 27 - 9 + 1 = 35 \tag{22}$$

where for convenience we have introduced the notation $\tilde{1}$ for the field element «-1».

Such a number record can be associated with an element of the Galois field, formed according to the rule:

$$a \dots bc \leftrightarrow a + \dots + b \cdot \theta^{n-2} + c \cdot \theta^{n-1} \tag{23}$$

where is a primitive element of the field $GF(3^n)$, the degrees of which are generated by all elements of the given field.

In particular, ternary numbers containing two digits are represented by the elements of the Galois field $GF(3^2)$, discussed above.

$$ab \leftrightarrow a + ib \tag{24}$$

Such ternary numbers correspond to the division of the amplitude scale into nine discrete levels. This, of course, is not enough for solving practical problems, but the proposed approach can be easily generalized to the fields $GF(3^n)$. Moreover, the number $n = 6$ already gives 729 levels, which is sufficient for many practical applications.

For clarity, we will restrict ourselves to considering a model signal corresponding to a division into 9 levels. Figure 2 shows the original time series (model "signal"), the values of which represent the number of deaths from coronavirus infection per month in Illinois in 2020 (according to the website https://data.cdc.gov/). In accordance with the above, a time period of 8 months is considered.

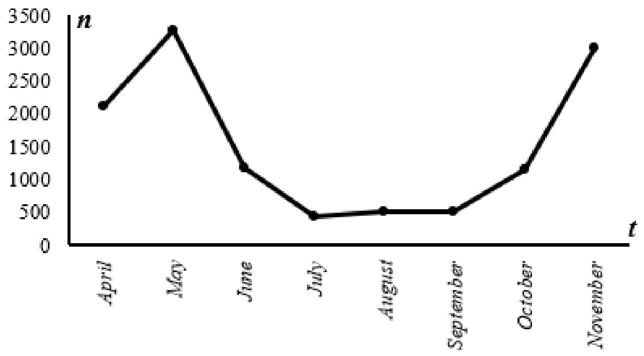


Fig. 2. The initial model signal is the dependence of the number of deaths from coronavirus infection per month in Illinois for 2020, the dates are indicated on the time axis

In fig. 3 shows, the same time series reduced to a rough scale containing 9 levels.

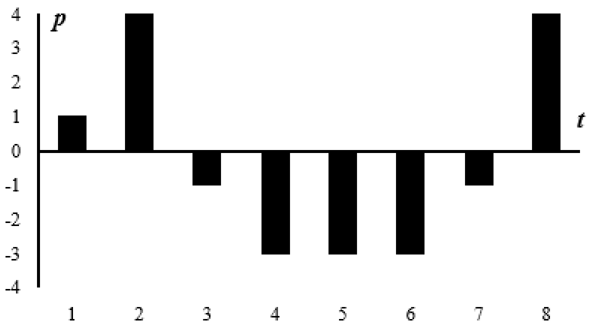


Fig. 3. Reducing the model signal to a scale of amplitudes corresponding to a division into 9 levels

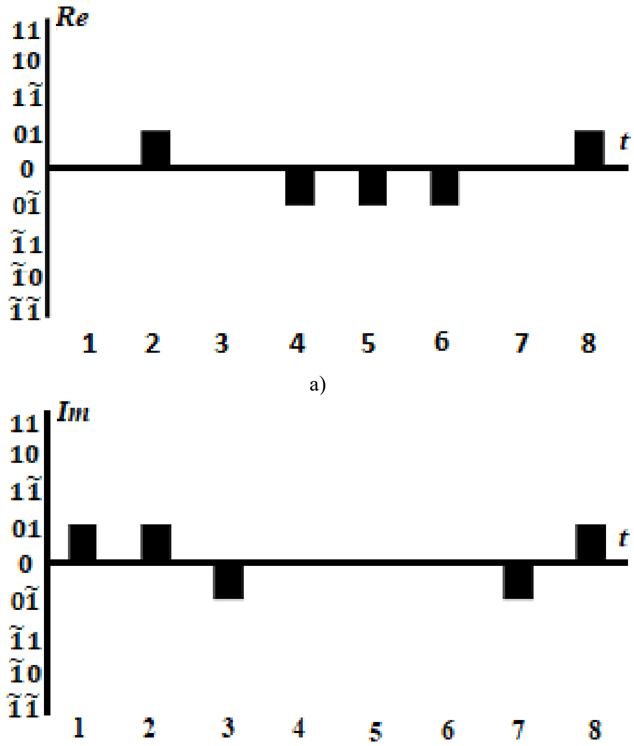
This signal can be represented through an algebraic form corresponding to formula (24), where the real part corresponds to the most significant bit of a two-digit ternary number, and the imaginary part corresponds to the lower one.

The corresponding dependences, actually showing the dependence of the values of the most significant and least significant bits on discrete time, are shown in Fig. 4.

Figure 4 shows the components of its "complex" spectrum in the Galois field $GF(3^2)$, obtained using formula (20).

This figure clearly demonstrates the main advantage of spectra in Galois fields. Namely, unlike the spectra obtained using the Walsh function (or its analogs), the actual range of variation of the amplitudes of the spectral components remains the same as the range of variation of the original signal.

Consequently, both the real and imaginary parts can take on only three possible values, i.e., the amount of information for each component of the spectrum is exactly 2 trit (a unit of the amount of information formed by analogy with the concept of a bit when using the binary representation of numbers).



a) b)

Fig. 4. Real (a) and imaginary (b) components of the model signal corresponding to its representation through the elements of the Galois field $GF(3^2)$

From the point of view of communication theory, this means that when transmitting information about a signal in a spectral representation using the classical Walsh function, the amount of transmitted information increases significantly (compared to the amount of information contained in the original signal).

Thus, the amount of information (expressed in trit) contained in the model signal shown in Fig. 3 is

$$I = 8\log_3 9 = 16 \tag{25}$$

The same amount of information is contained in the aggregate and in the spectral representations (Fig. 5). On the contrary, taking into account the variation of the possible values of the amplitudes of the spectral components, the amount of information contained in the spectrum calculated using the classical Walsh function (Fig. 6) is approximately

$$I \approx 8\log_3 27 = 24 \tag{26}$$

In a similar way, spectra can be constructed in the fields $GF(3^n)$, the transition to which makes it possible to work with sequences of length $3^n - 1$, which for sufficiently large $n \geq 6$ can already find practical application.

Indeed, any element of the Galois field $GF(3^n)$, which is an algebraic extension of the ground field $GF(3)$, can be represented in the form

$$a = a_0 + \cdots + a_{n-2}\theta^{n-2} + a_{n-1}\theta^{n-1} \tag{27}$$

where a_i are elements of the field $GF(3)$, θ is the primitive root of unity.

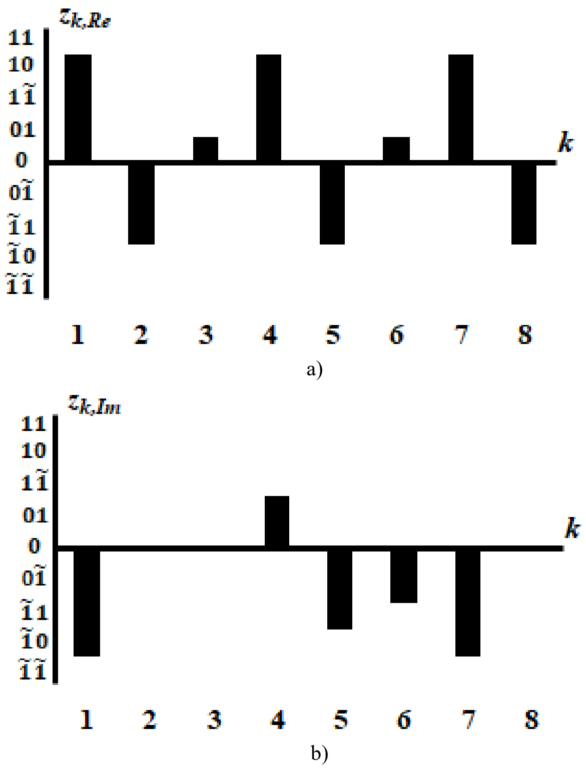


Fig. 5. Real (a) and imaginary (b) components of the spectrum of the reduced model signal corresponding to the signal representation in the Galois field $GF(3^2)$

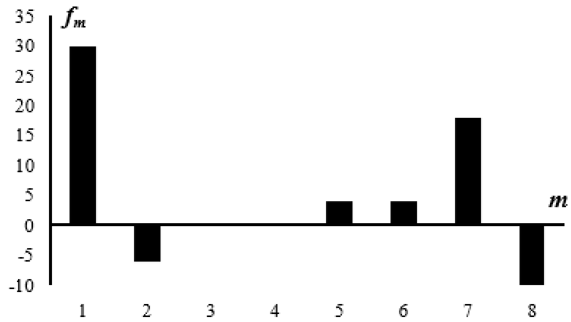


Fig.6. The spectrum of the reduced model signal, calculated using the classical Walsh function

The element θ is also the root of an irreducible equation, specifically the equations of the division field of a circle, which are obtained by expanding the polynomial

$$f = \theta^{p^n-1} - 1 \tag{28}$$

by factors.

In particular, for the field $GF(3^2)$ the equation that the primitive root of unity satisfies has the form

$$\theta^3 = \theta + 1 \tag{29}$$

This equation, among other things, allows expressing the powers of the elements θ^k, $k > 3$ in terms of linear combinations of the form

$$a = a_0 + a_1\theta + a_2\theta^2, \tag{30}$$

which makes it possible to construct sequences similar to those presented in Table II, according to the scheme described above.

We emphasize that with the representation of signal amplitudes through Galois fields in the form (23), it is impossible to perform calculations as with real numbers, in particular, the product of numbers presented in this way will not be equal to their product in the classical sense. However, this circumstance does not negate the above advantage.

Indeed, quite often transmission of a signal in a spectral form is preferred. In particular, this applies to a situation when a telemetry signal is recorded by several receivers spaced apart in space, and it is required to form a single frame (or a single video image). If the signal is transmitted directly (that is, without conversion to a spectral representation), then the obvious problem arises of splitting the image into frames and then combining them.

This problem is far from as simple as it seems at first glance, especially if the image recorders are located at different distances from the base optical plane. In this case, it will be necessary to provide at least a homothetic transformation of one or several image fragments.

The task is greatly simplified when information is recorded (or at least only transmitted) not about individual image fragments, but about fragments of their spectral representation (the digital spatial spectrum of a signal can be determined by the same methods as the spectrum of a time-dependent signal). This is due to the fact that each spectral component is calculated based on data about the entire image as a whole.

However, if you use the transmission of information about the spectra obtained using the Walsh basis (or its modifications), then this advantage is lost due to the above amount of transmitted information. It is fully realized only when the transition to the spectral representation is not associated with an increase in the amount of transmitted information, which ensures the transition to spectra in Galois fields.

IV. OTHER OPTIONS FOR CONSTRUCTING GENERALIZED RADEMACHER FUNCTIONS AND SOME METHODOLOGICAL ASPECTS OF THEIR APPLICATION

Above, a generalization of the Rademacher functions was constructed starting from the main Galois field $GF(3^n)$. This, of course, is not required, sequences similar to those presented in Table II can be carried out for any other simple Galois field $GF(p)$, where p is a prime number.

The extension of any such field to $GF(p^2)$ can be constructed according to the same scheme, and the elements of such a field can be represented through a record that is no different from the representation of complex numbers.

$$a = a_1 + ia_2 \tag{31}$$

The only difference is that operations on the elements a_1 and a_2 are performed in the sense of the Galois field $GF(p)$.

For example, for the case p = 5, the following multiplication (Table IV) is also valid. Examples of generalized Rademacher functions constructed with its help using the same technique as above are shown in Fig. 7.

TABLE IV

Multiplication table for elements of the Galois field $GF(5)$ in the representation $GF(5) = (-2, -1, 0, 1, 2)$

	1	2
1	1	2
2	2	-1

This representation is not the only one, for example, one can use mod5 multiplication. The advantage of the representation given by Table IV is that it naturally allows the use of negative numbers, which is essential for further expanding the term «logical imaginary unit».

However, it should be emphasized that there is an important nuance here. Table IV shows that the equation by which complex numbers are introduced into classical mathematics

$$i^2 = -1 \qquad (32)$$

in this field is resolvable. Specifically, its solution is the field element "2" and also "- 2".

Consequently, it is impossible to construct an algebraic field extension in this way.

However, this difficulty is easily overcome if, instead of the field element "-1" in equation (32), another field element is used, at which this equation becomes unsolvable. Then it becomes possible to use the standard procedure of algebraic extension, which allows us to proceed further to the construction of generalized Rademacher functions.

In particular, we can start from the unsolvable equation in the field GF (5)

$$i^2 = -2 \qquad (33)$$

the undecidability of which directly follows from Table IV.

In this case, generalized Rademacher functions can indeed be constructed (Fig. 7).

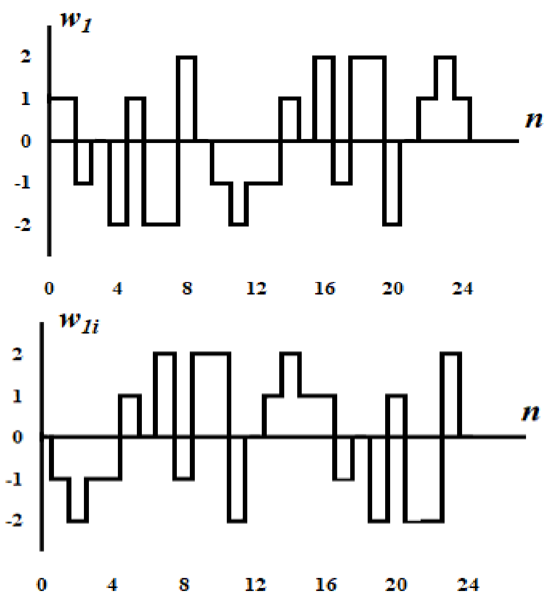


Fig. 7. Examples of generalized Rademacher functions for the field $GF(5^2)$

Conclusion

Thus, the representation of digital signals in terms of functions that take values in non-binary Galois fields is of interest from the point of view of digital signal processing by spectral methods. In this case, the function describing the spectrum of the signal also takes values in the same Galois field as the original signal. This provides, in particular, a reduction in the amount of information required for signal transmission in spectral representation.

The use of Galois fields $GF(p^3)$ in this case allows interpretation through the concept of a logical imaginary unit, which is advisable to use in order to increase the degree of clarity when using the methods of abstract algebra for applied purposes.

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