10.24425/acs.2022.141720

Archives of Control Sciences
Volume 32(LXVIII), 2022
No. 2*, pages 451–484*

Linguistic q-rung orthopair fuzzy prioritized aggregation operators based on Hamacher t-norm and t-conorm and their applications to multicriteria group decision making

Nayana DEB, Arun SARKAR and Animesh BISWAS

The linguistic q-rung orthopair fuzzy (Lq-ROF) set is an important implement in the research area in modelling vague decision information by incorporating the advantages of q-rung orthopair fuzzy sets and linguistic variables. This paper aims to investigate the multicriteria decision group decision making (MCGDM) with Lq-ROF information. To do this, utilizing Hamacher t-norm and t-conorm, some Lq-ROF prioritized aggregation operators viz., Lq-ROF Hamacher prioritized weighted averaging, and Lq-ROF Hamacher prioritized weighted geometric operators are developed in this paper. The defined operators can effectively deal with different priority levels of attributes involved in the decision making processes. In addition, Hamacher parameters incorporated with the proposed operators make the information fusion process more flexible. Some prominent characteristics of the developed operators are also well-proven. Then based on the proposed aggregation operators, an MCGDM model with Lq-ROF context is framed. A numerical example is illustrated in accordance with the developed model to verify its rationality and applicability. The impacts of Hamacher and rung parameters on the achieved decision results are also analyzed in detail. Afterwards, a comparative study with other representative methods is presented in order to reflect the validity and superiority of the proposed approach.

Key words: linguistic q-rung orthopair fuzzy set, multicriteria group decision making, Hamacher operations, prioritized aggregation operator

1. Introduction

Multicriteria decision making (MCDM) has emerged as an important branch in modern decision science. It refers to find a suitable choice based on the

N. Deb (corresponding author, e-mail: nayanadeb1@gmail.com) and A. Biswas are with Department of Mathematics, University of Kalyani, Kalyani – 741235, India.

A. Sarkar is with Department of Mathematics, Heramba Chandra College, Kolkata – 700029, India.

Received 31.12.2021. Revised 10.5.2022.

evaluation information by a decision-maker (DM) from a collection of alternatives under a set of criteria. If the evaluation of alternatives against a certain criterion is performed under multiple DMs instead of a single DM, then the process is termed as multicriteria group decision making (MCGDM). With the increase in vagueness of environment day by day and the inherent fuzziness connected with human perception, decision information cannot always be provided using crisp numbers. In response to this issue Zadeh [1] first introduced the notion of fuzzy set. After that, several extensions of fuzzy set were developed, including intuitionistic fuzzy sets (IFSs) [2], interval-valued IFSs [3], Pythagorean fuzzy sets (PFSs) [4, 5], interval-valued PFS [6, 7], Fermatin fuzzy sets (FFSs) [8] etc. Since these extensions of fuzzy sets appear, they have received more and more attention in solving decision-making problems [9–15]. By enlarging the scope of IFS, PFS and FFS, recently, another variant of fuzzy set, q-rung orthopair fuzzy (q-ROF) set (q-ROFS) [16], has been developed as an efficient tool in terms of capturing uncertainty during the process of MCGDM. For q-ROFSs membership degree μ and non-membership degree ν satisfy the condition that sum of their q-th power is less than or equal to 1, i.e., $\mu^q + \nu^q \leqslant 1$. As a more generalized fuzzy set, q-ROFS include fuzzy sets, IFSs, PFSs, and FFSs as special cases with certain conditions. For instance, q-ROFS reduces to IFS, PFS, FFS by taking the value of rung parameter $q = 1, 2, 3$, respectively. So q-ROFS is the most valuable and focused extension of fuzzy sets in which DMs can modify the range of their judgement values by varying rung parameter q based on different indeterminate degrees.

So far, q-ROFSs have attracted many scholars attention. Liu and Wang [17] investigated multi-attribute decision making (MADM) problems with q-ROF information on developing q-ROF weighted averaging (WA) and weighted geometric (WG) operators. They [18] further extended Archimedean Bonferroni mean operators to q-ROF environment. Heronian mean was utilized to fuse q-ROF data, and thereby a MADM approach was developed by Wei et al. [19]. On the basis of the cosine function, Wang et al. [20] studied novel similarity measures for q-ROFSs. Further, a study on induced logarithmic distance measures for q-ROFSs was conducted by Zeng et al. [21]. In recent days, a variety of applications [22–28] on q-ROFSs have been developed by numerous researchers.

However, q-ROFS theory has successfully been applied in several decision-making processes, but in real-world issues, many attribute values are present that are often difficult to express quantitatively. In such cases, it seems suitable to express them using a qualitative form. To address such situations, Liu and Liu [29] invented linguistic q-ROF (Lq-ROF) set (Lq-ROFS), following the advantage of q-ROFS and linguistic variables [30], which is a generalization of linguistic intuitionistic fuzzy (LIF) set (LIFS) [31] and linguistic Pythagorean fuzzy (LPF) set (LPFS) [32]. In recent years, several significant researches on Lq-ROFS have been carried out, along with numerous decision-making theories.

In short, Lq-ROFS have been studied effectually from different perspectives, including information measures [33, 34], traditional decision techniques [35, 36], aggregation operators [29, 37–41]. Nevertheless, to generate the ranking of alternatives, aggregation operators usually can address decision making situations more effectively than conventional decision techniques because aggregation operators can produce a ranking of alternatives along with their collective evaluation values. In contrast, traditional techniques can be only able to produce ranking results. Liu and Liu [29] introduced some aggregation operators based on power Bonferroni mean and utilized them for MCGDM under Lq-ROF environment. An interactional partitioned Heronian mean based decision method with Lq-ROF information has been developed by Lin et al. [37]. Further, Liu and Liu [38] investigated Lq-ROF power Muirhead mean aggregation operators for MCGDM. Recently, Akram et al. [39] proposed an Einstein model in order to build a Lq-ROF group decision-making framework, and Liu et al. [40] developed some generalized point weighted aggregation operators for Lq-ROF group decision-making context as well.

It is important that in the process of MCDM, the required aggregation operators must be general and flexible enough to capture the relationship between the different criteria when aggregating the values of attributes. Assuming that the criteria are at the same priority level may lead to serious loss of information. Yager [42] introduced the prioritized averaging operator to overcome these issues, which may take into account various priority levels of criteria during the aggregating procedure. However, so far, the aggregated operators to fuse Lq-ROF information have not taken prioritization relation among criteria into account. Thus, introducing the concept prioritized aggregation (PA) operator in Lq-ROF environment for developing some MCGDM techniques would be a useful study in Literature. It is important to point out that among the existing aggregation operators for Lq-ROF numbers (Lq-ROFNs), most of the aggregation functions involve algebraic sum and product in order to carry the aggregation process. However, the operational rules play an important role in aggregating decision information. Hamacher operations [43], a generalized form of algebraic and Einstein operations [44], have significant importance in the aggregation process by means of a flexible parameter. Several achievements [45–47] have been discovered in the past decades employing Hamacher operational rules in the aggregation process. Therefore motivated by the idea of Hamacher t-norms and t-conorms with PA operators, some Lq-ROF aggregation operators, viz., Lq-ROF Hamacher prioritized WA (Lq-ROFHPWA), and Lq-ROF Hamacher prioritized WG (Lq-ROFHPWG) operators have been developed in this paper.

The paper is structured as follows. Section 2 reviews several fundamental concepts such as Lq-ROFSs, Hamacher t-norms and t-conorms and PA operators. Hamacher operational laws for Lq-ROFNs are proposed in Section 3. Section 4 introduces some newly Lq-ROF PA operators based on Hamacher operations, viz., Lq-ROFHPWA and Lq-ROFHPWG operators. Further, some characteristics of these developed operators are also exhibited in this section. Section 5 illustrates an MCGDM approach utilizing the proposed aggregation operators. A numerical example utilizing the developed approach has been provided in Section 6. Comparative and sensitivity analyses are discussed in Section 7. Finally, an overall summarization and scope for future studies have been demonstrated in Section 8.

2. Preliminaries

Some basic ideas of linguistic term set (LTS), Lq-ROFS, PA operator, and Hamacher t-norm and t-conorm are briefly discussed in this section.

2.1. LTS

Definition 1 [48] *Let $\mathfrak{S} = \{\mathfrak{S}_0, \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_t\}$ be a finite-ordered discrete set with odd cardinality and the terms $\mathfrak{S}_0, \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_t$ can be specified in terms of various real-world scenarios. Then $\mathfrak{S}$ is said to be a LTS if it satisfies the following conditions:*

(i) If $i > j$, then $\mathfrak{S}_i > \mathfrak{S}_j$, implies $\mathfrak{S}_i$ is superior than $\mathfrak{S}_j$ (Ordered);

(ii) $\neg(\mathfrak{S}_i) = \mathfrak{S}_j$, where $j = t - i$ (Negation);

(iii) If $i \leqslant j$, that is, $\mathfrak{S}_i \leqslant \mathfrak{S}_j$, then $\min\left(\mathfrak{S}_i, \mathfrak{S}_j\right) = \mathfrak{S}_i$ (Min operator);

(iv) If $i \geqslant j$, that is, $\mathfrak{S}_i \geqslant \mathfrak{S}_j$, then $\max\left(\mathfrak{S}_i, \mathfrak{S}_j\right) = \mathfrak{S}_i$ (Max operator).

For example, when an expert wants to evaluate the quality of comforts of a car, he/she may feel more convenient to assess it using LTS as

$$\begin{aligned}\mathfrak{S} &= \{\mathfrak{S}_0, \mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_6\} \\ &= \{\text{extreme low, very low, low, medium, high, very high, extreme high}\}.\end{aligned}$$

Further, Xu [49] prolonged the notion of discrete LTS $\mathfrak{S}$ to continuous LTS (CLTS) $\overline{\mathfrak{S}}$ such that $\overline{\mathfrak{S}} = \{\mathfrak{S}_\hbar | \mathfrak{S}_0 \leqslant \mathfrak{S}_\hbar \leqslant \mathfrak{S}_t, \hbar \in [0, t]\}$ and the components likewise meet all of the preceding requirements.

2.2. Lq-ROFS

Definition 2 [29] *An Lq-ROFS $\widetilde{\mathcal{B}}$ defined in a universe of discourse $X = \{x_1, x_2, \ldots, x_n\}$ is represented by*

$$\widetilde{\mathcal{B}} = \left\{\left\langle x, \mathfrak{S}_{\gamma_{\widetilde{\mathcal{B}}}}(x), \mathfrak{S}_{\zeta_{\widetilde{\mathcal{B}}}}(x)\right\rangle \middle| x \in X\right\}, \tag{1}$$

where $\mathfrak{S}_{\gamma_{\widetilde{\mathcal{B}}}}(x), \mathfrak{S}_{\zeta_{\widetilde{\mathcal{B}}}}(x) \in \mathfrak{S}_{[0,t]}$ *denote the linguistic membership and non-membership degrees, respectively satisfying the condition* $0 \leqslant \left(\gamma_{\widetilde{\mathcal{B}}}\right)^q + \left(\zeta_{\widetilde{\mathcal{B}}}\right)^q \leqslant t^q$ $(q \geqslant 1)$ *for every* $x \in X$*. For convenience, Liu and Liu [29] represents a Lq-ROFN as* $\widetilde{\beta} = \langle \mathfrak{S}_\gamma, \mathfrak{S}_\zeta \rangle$*. The linguistic indeterminacy degree of* x *to* $\widetilde{\beta}$ *is presented as* $\mathfrak{S}_{\pi_{\widetilde{\beta}}}(x) = \mathfrak{S}_{(t^q-\gamma^q-\zeta^q)^{\frac{1}{q}}}$.

Definition 3 [29] *Let* $\widetilde{\beta} = \langle \mathfrak{S}_\gamma, \mathfrak{S}_\zeta \rangle$ *be an Lq-ROFN, the score function,* $S\left(\widetilde{\beta}\right)$*, and accuracy function,* $A\left(\widetilde{\beta}\right)$*, of the Lq-ROFN can be defined as*

$$S\left(\widetilde{\beta}\right) = \left(\frac{t^q + \gamma^q - \zeta^q}{2}\right)^{\frac{1}{q}}, \tag{2}$$

and

$$A\left(\widetilde{\beta}\right) = \left(\gamma^q + \zeta^q\right)^{\frac{1}{q}}. \tag{3}$$

The following comparison method based on the score and accuracy functions is presented to compare any two Lq-ROFNs.

Definition 4 [29] *Let* $\widetilde{\beta}_1 = \langle \mathfrak{S}_{\gamma_1}, \mathfrak{S}_{\zeta_1} \rangle$*,* $\widetilde{\beta}_2 = \langle \mathfrak{S}_{\gamma_2}, \mathfrak{S}_{\zeta_2} \rangle$ *be any two Lq-ROFNs*

(i) If $S\left(\widetilde{\beta}_1\right) < S\left(\widetilde{\beta}_2\right)$*, then* $\widetilde{\beta}_1 \prec \widetilde{\beta}_2$*;*

(ii) If $S\left(\widetilde{\beta}_1\right) = S\left(\widetilde{\beta}_2\right)$*, then*

- *if* $A\left(\widetilde{\beta}_1\right) < A\left(\widetilde{\beta}_2\right)$*, then* $\widetilde{\beta}_1 \prec \widetilde{\beta}_2$ *which means* $\widetilde{\beta}_2$ *is better than* $\widetilde{\beta}_1$*;*
- *if* $A\left(\widetilde{\beta}_1\right) = A\left(\widetilde{\beta}_2\right)$*, then* $\widetilde{\beta}_1 \approx \widetilde{\beta}_2$*, which means* $\widetilde{\beta}_1$ *is equal to* $\widetilde{\beta}_2$*.*

2.3. PA operator

Yager [42] originally introduced the PA operator, which is presented in the following:

Definition 5 [42] *Consider* $\{C_i | i = 1, 2, \ldots, n\}$ *as a collection of criteria, the linear ordering* $C_1 \succ C_2 \succ \ldots \succ C_n$ *represents their priority. This ordering reveals that if* $j < k$ *then criteria* C_j *has a higher priority than* C_k*.* $C_j(x) \in [0, 1]$ *denotes the assessment value of any alternative* x *evaluated on the criteria* C_j*.*

If $PA\left(C_j(x)\right) = \sum_{j=1}^{n} w_j C_j(x)$*, where* $w_j = \frac{T_j}{\sum_{j=1}^{n} T_j}$*,* $T_j = \prod_{k=1}^{j-1} C_k(x)$

$(j = 2, \ldots, n)$*,* $T_1 = 1$*. Then* $PA\left(C_j(x)\right)$ *is called the PA operator.*

2.4. Hamacher t-norms and t-conorms

In 1978, Hamacher [43] introduced one of generalized t-norm and t-conorm, which is known as Hamacher t-norms and t-conorms, and expressed as ($\varsigma > 0$):

- Hamacher t-norm: $T_{\varsigma}^{H}(x,y) = \dfrac{xy}{\varsigma + (1-\varsigma)(x+y-xy)}$,
- Hamacher t-conorm: $S_{\varsigma}^{H}(x,y) = \dfrac{x+y-xy-(1-\varsigma)xy}{1-(1-\varsigma)xy}$.

3. Hamacher t-norms and t-conorms based operational laws for Lq-ROFNs

According to the Hamacher t-norms and t-conorms, the following operational rules of Lq-ROFNs are defined as follows.

Definition 6 *Let* $\overline{\mathfrak{S}} = \{\mathfrak{S}_{\hbar} : \hbar \in [0,t]\}$ *be a CLTS,* $\widetilde{\beta}_1 = \left\langle \mathfrak{S}_{\gamma_1}, \mathfrak{S}_{\zeta_1} \right\rangle$, $\widetilde{\beta}_2 = \left\langle \mathfrak{S}_{\gamma_2}, \mathfrak{S}_{\zeta_2} \right\rangle$ *and* $\widetilde{\beta} = \left\langle \mathfrak{S}_{\gamma}, \mathfrak{S}_{\zeta} \right\rangle$ *be three Lq-ROFNs. Then, the Hamacher operational laws of Lq-ROFNs are defined as* ($\lambda > 0$)

$$(i)\ \widetilde{\beta}_1 \oplus_H \widetilde{\beta}_2 = \left\langle \mathfrak{S}_{t\left(\frac{t^q\gamma_1^q + t^q\gamma_2^q - \gamma_1^q\gamma_2^q - (1-\varsigma)\gamma_1^q\gamma_2^q}{t^{2q} - (1-\varsigma)\gamma_1^q\gamma_2^q}\right)^{\frac{1}{q}}},\ \mathfrak{S}_{t\left(\frac{\zeta_1^q\zeta_2^q}{\varsigma t^{2q} + (1-\varsigma)\left(t^q\zeta_1^q + t^q\zeta_2^q - \zeta_1^q\zeta_2^q\right)}\right)^{\frac{1}{q}}} \right\rangle;$$

$$(ii)\ \widetilde{\beta}_1 \otimes_H \widetilde{\beta}_2 = \left\langle \mathfrak{S}_{t\left(\frac{\gamma_1^q\gamma_2^q}{\varsigma t^{2q} + (1-\varsigma)\left(t^q\gamma_1^q + t^q\gamma_2^q - \gamma_1^q\gamma_2^q\right)}\right)^{\frac{1}{q}}},\ \mathfrak{S}_{t\left(\frac{t^q\zeta_1^q + t^q\zeta_2^q - \zeta_1^q\zeta_2^q - (1-\varsigma)\zeta_1^q\zeta_2^q}{t^{2q} - (1-\varsigma)\zeta_1^q\zeta_2^q}\right)^{\frac{1}{q}}} \right\rangle;$$

$$(iii)\ \lambda\widetilde{\beta} = \left\langle \mathfrak{S}_{t\left(\frac{(t^q+\gamma^q(\varsigma-1))^\lambda - (t^q-\gamma^q)^\lambda}{(t^q+\gamma^q(\varsigma-1))^\lambda + (\varsigma-1)(t^q-\gamma^q)^\lambda}\right)^{\frac{1}{q}}},\ \mathfrak{S}_{t\left(\frac{\varsigma\zeta^{q\lambda}}{(t^q+(\varsigma-1)(t^q-\zeta^q))^\lambda + (\varsigma-1)\zeta^{q\lambda}}\right)^{\frac{1}{q}}} \right\rangle;$$

$$(iv)\ \widetilde{\beta}^{\lambda} = \left\langle \mathfrak{S}_{t\left(\frac{\varsigma\gamma^{q\lambda}}{(t^q+(\varsigma-1)(t^q-\gamma^q))^\lambda + (\varsigma-1)\gamma^{q\lambda}}\right)^{\frac{1}{q}}},\ \mathfrak{S}_{t\left(\frac{(t^q+\zeta^q(\varsigma-1))^\lambda - (t^q-\zeta^q)^\lambda}{(t^q+\zeta^q(\varsigma-1))^\lambda + (\varsigma-1)(t^q-\zeta^q)^\lambda}\right)^{\frac{1}{q}}} \right\rangle.$$

4. Development of Hamacher operations-based PA operators on Lq-ROF environment

In the following, utilizing Hamacher operations, the PA operator is extended into Lq-ROFNs and Lq-ROFHPWA and Lq-ROFHPWG operators are proposed.

Definition 7 *Let* $\left\{\widetilde{\beta}_1, \widetilde{\beta}_2, ..., \widetilde{\beta}_n\right\}$ *represents a collection of Lq-ROFNs, where* $\widetilde{\beta}_i = \left\langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i} \right\rangle$ *(*$i = 1, 2, \ldots, n$*) and* $q \geqslant 1$*. Then Lq-ROFHPWA operator is defined as*

$$Lq\text{-}ROFHPWA\left(\widetilde{\beta}_1, \widetilde{\beta}_2, ..., \widetilde{\beta}_n\right) = \bigoplus_{i=1}^{n}{}_H \left(\frac{T_i}{\sum\limits_{i=1}^{n} T_i} \widetilde{\beta}_i\right) \tag{4}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ *is the weight vectors of* $\widetilde{\beta}_i$ *with* $\omega_i \in [0,\ 1]$ *and* $\omega_i = \dfrac{T_i}{\sum\limits_{i=1}^{n} T_i}$*,* $T_i = \prod\limits_{k=1}^{i-1} \dfrac{S\left(\widetilde{\beta}_k\right)}{t}$ *(*$i = 2, \ldots, n$*),* $T_1 = 1$ *and* $S\left(\widetilde{\beta}_i\right)$ *is the score of* $\widetilde{\beta}_i$*.*

Theorem 1 *Let* $\widetilde{\beta}_i = \left\langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i} \right\rangle$ *(*$i = 1, 2, \ldots, n$*) represents a collection of Lq-ROFNs. Then, the aggregated result is also a Lq-ROFN based on Lq-ROFHPWA operator, and*

$$Lq\text{-}ROFHPWA\left(\widetilde{\beta}_1, \widetilde{\beta}_2, ..., \widetilde{\beta}_n\right) = \bigoplus_{i=1}^{n}{}_H \left(\frac{T_i}{\sum\limits_{i=1}^{n} T_i} \widetilde{\beta}_i\right)$$

$$= \left\langle \begin{array}{l} \mathfrak{S}_{t\left(\frac{\prod\limits_{i=1}^{n}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod\limits_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod\limits_{i=1}^{n}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\varsigma-1)\prod\limits_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}\right)^{\frac{1}{q}}}, \\ \mathfrak{S}_{t\left(\frac{\varsigma\prod\limits_{i=1}^{n}\zeta_i^{\frac{T_i}{\sum_{i=1}^{n} T_i}q}}{\prod\limits_{i=1}^{n}\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\varsigma-1)\prod\limits_{i=1}^{n}\zeta_i^{\frac{T_i}{\sum_{i=1}^{n} T_i}q}}\right)^{\frac{1}{q}}} \end{array} \right\rangle.$$

Proof. Based on Definition 6,

$$\frac{T_i}{\sum\limits_{i=1}^{n} T_i}\widetilde{\beta}_i = \left\langle \mathfrak{S}_t\left(\frac{\left(t^q+\gamma_i^q(\varsigma-1)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\left(t^q+\gamma_i^q(\varsigma-1)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\varsigma-1)\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}\right)^{\frac{1}{q}}, \mathfrak{S}_t\left(\frac{\varsigma\zeta_i^{q\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\varsigma-1)\zeta_i^{q\frac{T_i}{\sum_{i=1}^{n} T_i}}}\right)^{\frac{1}{q}} \right\rangle.$$

Then, it can be obtained that

$$\bigoplus_{H\, i=1}^{2}\left(\frac{T_i}{\sum\limits_{i=1}^{2} T_i}\widetilde{\beta}_i\right) = \frac{T_1}{\sum\limits_{i=1}^{2} T_i}\widetilde{\beta}_1 \oplus_H \frac{T_2}{\sum\limits_{i=1}^{2} T_i}\widetilde{\beta}_2 =$$

$$= \left\langle \mathfrak{S}_t\left(\frac{\left(t^q+\gamma_1^q(\varsigma-1)\right)^{\frac{T_1}{\sum_{i=1}^{2} T_i}} - \left(t^q-\gamma_1^q\right)^{\frac{T_1}{\sum_{i=1}^{2} T_i}}}{\left(t^q+\gamma_1^q(\varsigma-1)\right)^{\frac{T_1}{\sum_{i=1}^{2} T_i}} + (\varsigma-1)\left(t^q-\gamma_1^q\right)^{\frac{T_1}{\sum_{i=1}^{2} T_i}}}\right)^{\frac{1}{q}}, \mathfrak{S}_t\left(\frac{\varsigma\zeta_1^{\frac{T_1}{\sum_{i=1}^{2} T_i}q}}{\left(t^q+(\varsigma-1)\left(t^q-\zeta_1^q\right)\right)^{\frac{T_1}{\sum_{i=1}^{2} T_i}} + (\varsigma-1)\zeta_1^{\frac{T_1}{\sum_{i=1}^{2} T_i}q}}\right)^{\frac{1}{q}} \right\rangle \oplus_H$$

$$\left\langle \mathfrak{S}_t\left(\frac{\left(t^q+\gamma_2^q(\varsigma-1)\right)^{\frac{T_2}{\sum_{i=1}^{2} T_i}} - \left(t^q-\gamma_2^q\right)^{\frac{T_2}{\sum_{i=1}^{2} T_i}}}{\left(t^q+\gamma_2^q(\varsigma-1)\right)^{\frac{T_2}{\sum_{i=1}^{2} T_i}} + (\varsigma-1)\left(t^q-\gamma_2^q\right)^{\frac{T_2}{\sum_{i=1}^{2} T_i}}}\right)^{\frac{1}{q}}, \mathfrak{S}_t\left(\frac{\varsigma\zeta_2^{\frac{T_2}{\sum_{i=1}^{2} T_i}q}}{\left(t^q+(\varsigma-1)\left(t^q-\zeta_2^q\right)\right)^{\frac{T_2}{\sum_{i=1}^{2} T_i}} + (\varsigma-1)\zeta_2^{\frac{T_2}{\sum_{i=1}^{2} T_i}q}}\right)^{\frac{1}{q}} \right\rangle$$

$$= \left\langle \mathfrak{S}_t\left(\frac{\prod\limits_{i=1}^{2}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{2} T_i}} - \prod\limits_{i=1}^{2}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{2} T_i}}}{\prod\limits_{i=1}^{2}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{2} T_i}} + (\varsigma-1)\prod\limits_{i=1}^{2}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{2} T_i}}}\right)^{\frac{1}{q}}, \mathfrak{S}_t\left(\frac{\varsigma\prod\limits_{i=1}^{2}\zeta_i^{\frac{T_i}{\sum_{i=1}^{2} T_i}q}}{\prod\limits_{i=1}^{2}\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{2} T_i}} + (\varsigma-1)\prod\limits_{i=1}^{2}\zeta_i^{\frac{T_i}{\sum_{i=1}^{2} T_i}q}}\right)^{\frac{1}{q}} \right\rangle.$$

So, the theorem is true for $n = 2$.

Now let theorem is true for $n = m$, i.e.,

$$\mathrm{L}q\text{-ROFHPWA}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_m\right) = \bigoplus_{i=1}^{m}{}_{H}\left(\frac{T_i}{\sum\limits_{i=1}^{m} T_i}\widetilde{\beta}_i\right)$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{\prod\limits_{i=1}^{m}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m} T_i}} - \prod\limits_{i=1}^{m}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m} T_i}}}{\prod\limits_{i=1}^{m}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m} T_i}} + (\varsigma-1)\prod\limits_{i=1}^{m}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m} T_i}}}\right)^{\frac{1}{q}}}, \right.$$

$$\left. \mathfrak{S}_{t\left(\frac{\varsigma\prod\limits_{i=1}^{m}\zeta_i^{\frac{T_i}{\sum_{i=1}^{m} T_i}q}}{\prod\limits_{i=1}^{m}\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{m} T_i}} + (\varsigma-1)\prod\limits_{i=1}^{m}\zeta_i^{\frac{T_i}{\sum_{i=1}^{m} T_i}q}}\right)^{\frac{1}{q}}} \right\rangle.$$

Now would show that it is true for $n = m + 1$,

$$\mathrm{L}q\text{-ROFHPWA}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_m, \widetilde{\beta}_{m+1}\right)$$
$$= \left(\mathrm{L}q\text{-ROFHPWA}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_m\right)\right) \oplus_H \frac{T_{m+1}}{\sum\limits_{i=1}^{m+1} T_i}\widetilde{\beta}_{m+1}$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{\prod\limits_{i=1}^{m}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m} T_i}} - \prod\limits_{i=1}^{m}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m} T_i}}}{\prod\limits_{i=1}^{m}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m} T_i}} + (\varsigma-1)\prod\limits_{i=1}^{m}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m} T_i}}}\right)^{\frac{1}{q}}}, \right.$$

$$\left. \mathfrak{S}_{t\left(\frac{\varsigma\prod\limits_{i=1}^{m}\zeta_i^{\frac{T_i}{\sum_{i=1}^{m} T_i}q}}{\prod\limits_{i=1}^{m}\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{m} T_i}} + (\varsigma-1)\prod\limits_{i=1}^{m}\zeta_i^{\frac{T_i}{\sum_{i=1}^{m} T_i}q}}\right)^{\frac{1}{q}}} \right\rangle \oplus_H$$

$$
\left\langle \mathfrak{S}_t\left(\frac{\left(t^q+\gamma_i^q(\varsigma-1)\right)^{\frac{T_{m+1}}{\sum_{i=1}^{m+1}T_i}}-\left(t^q-\gamma_i^q\right)^{\frac{T_{m+1}}{\sum_{i=1}^{m+1}T_i}}}{\left(t^q+\gamma_i^q(\varsigma-1)\right)^{\frac{T_{m+1}}{\sum_{i=1}^{m+1}T_i}}+(\varsigma-1)\left(t^q-\gamma_i^q\right)^{\frac{T_{m+1}}{\sum_{i=1}^{m+1}T_i}}}\right)^{\frac{1}{q}},\ \mathfrak{S}_t\left(\frac{\varsigma\zeta_i^{q\frac{T_{m+1}}{\sum_{i=1}^{m+1}T_i}}}{\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_{m+1}}{\sum_{i=1}^{m+1}T_i}}+(\varsigma-1)\zeta_i^{q\frac{T_{m+1}}{\sum_{i=1}^{m+1}T_i}}}\right)^{\frac{1}{q}}\right\rangle
$$

$$
=\left\langle \mathfrak{S}_t\left(\frac{\prod_{i=1}^{m+1}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m+1}T_i}}-\prod_{i=1}^{m+1}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m+1}T_i}}}{\prod_{i=1}^{m+1}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m+1}T_i}}+(\varsigma-1)\prod_{i=1}^{m+1}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{m+1}T_i}}}\right)^{\frac{1}{q}},\right.
$$

$$
\left.\mathfrak{S}_t\left(\frac{\varsigma\prod_{i=1}^{m+1}\zeta_i^{\frac{T_i}{\sum_{i=1}^{m+1}T_i}q}}{\prod_{i=1}^{m+1}\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{m+1}T_i}}+(\varsigma-1)\prod_{i=1}^{m+1}\zeta_i^{\frac{T_i}{\sum_{i=1}^{m+1}T_i}q}}\right)^{\frac{1}{q}}\right\rangle.
$$

Since it is valid for $n = m + 1$, theorem is proved for all n. □

In the next, some particular cases, concerning parameter ς, for Lq-ROFHPWA operator are discussed.

- When $\varsigma = 1$, Lq-ROFHPWA operator reduces to the Lq-ROF weighted average (Lq-ROFPWA) operator as follows:

$$
\text{L}q\text{-ROFPWA}\left(\widetilde{\beta}_1,\widetilde{\beta}_2,\ldots,\widetilde{\beta}_n\right)=\left\langle \mathfrak{S}_t\left(\frac{t^q-\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{t^q}\right)^{\frac{1}{q}},\ \mathfrak{S}_t\left(\prod_{i=1}^{n}\left(\frac{\zeta_i}{t}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}q}\right)^{\frac{1}{q}}\right\rangle.
$$

- When $\varsigma = 2$, Lq-ROFHPWA operator reduces to the Lq-ROF Einstein weighted average (Lq-ROFEPWA) operator as follows:

$$
\begin{aligned}
&Lq\text{-ROFEPWA}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_n\right) \\
&= \left\langle \mathfrak{S}_{t\left(\frac{\prod_{i=1}^{n}\left(t^q+\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n}\left(t^q+\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}\right)^{\frac{1}{q}}},\ \mathfrak{S}_{t\left(\frac{2\prod_{i=1}^{n} \zeta_i^{\frac{T_i}{\sum_{i=1}^{n} T_i} q}}{\prod_{i=1}^{n}\left(t^q+\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + \prod_{i=1}^{n} \zeta_i^{\frac{T_i}{\sum_{i=1}^{n} T_i} q}}\right)^{\frac{1}{q}}} \right\rangle.
\end{aligned}
$$

Example 1 Let $\widetilde{\beta}_1 = \langle \mathfrak{S}_4, \mathfrak{S}_4 \rangle, \widetilde{\beta}_2 = \langle \mathfrak{S}_6, \mathfrak{S}_2 \rangle, \widetilde{\beta}_3 = \langle \mathfrak{S}_5, \mathfrak{S}_3 \rangle$ and $\widetilde{\beta}_4 = \langle \mathfrak{S}_7, \mathfrak{S}_2 \rangle$ be four Lq-ROFNs on LTS $\left\{S_i \middle| \, i = 0, 1, \ldots, 8\right\}$. Utilizing the score function of Lq-ROFNs, $S\left(\widetilde{\beta}_1\right) = 6.3496$, $S\left(\widetilde{\beta}_2\right) = 7.1138$, $S\left(\widetilde{\beta}_3\right) = 6.7313$ and $S\left(\widetilde{\beta}_4\right) = 7.5096$ are obtained. So, $T_1 = 1$, $T_2 = 0.7937$, $T_3 = 0.7058$ and $T_4 = 0.5939$. Then using Lq-ROFHPWA operator, the aggregated value of $\widetilde{\beta}_1$, $\widetilde{\beta}_2$, $\widetilde{\beta}_3$ and $\widetilde{\beta}_4$ is calculated as (Considering $\varsigma = 3$, $q = 3$)

$$
\begin{aligned}
&Lq\text{-ROFHPWA}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3, \widetilde{\beta}_4\right) = \left\langle \mathfrak{S}_{8\left(\frac{\prod_{i=1}^{4}\left(8^3+(3-1)\gamma_i^3\right)^{\frac{T_i}{\sum_{i=1}^{4} T_i}} - \prod_{i=1}^{4}\left(8^3-\gamma_i^3\right)^{\frac{T_i}{\sum_{i=1}^{4} T_i}}}{\prod_{i=1}^{4}\left(8^3+(3-1)\gamma_i^3\right)^{\frac{T_i}{\sum_{i=1}^{4} T_i}} + (3-1)\prod_{i=1}^{4}\left(8^3-\gamma_i^3\right)^{\frac{T_i}{\sum_{i=1}^{4} T_i}}}\right)^{\frac{1}{3}}}, \right. \\
&\left. \mathfrak{S}_{8\left(\frac{3\prod_{i=1}^{4} \zeta_i^{3\frac{T_i}{\sum_{i=1}^{4} T_i}}}{\prod_{i=1}^{4}\left(8^3+(3-1)\left(8^3-\zeta_i^3\right)\right)^{\frac{T_i}{\sum_{i=1}^{4} T_i}} + (3-1)\prod_{i=1}^{4} \zeta_i^{3\frac{T_i}{\sum_{i=1}^{4} T_i}}}\right)^{\frac{1}{3}}} \right\rangle = \left\langle \mathfrak{S}_{5.6021},\ \mathfrak{S}_{2.7571} \right\rangle.
\end{aligned}
$$

Furthermore, the proposed Lq-ROFHPWA operator meets certain important properties, which are stated as follows.

Theorem 2 (Idempotency) *Let $\widetilde{\beta}_i = \left\langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i} \right\rangle$ $(i = 1, 2, \ldots, n)$ be a collection of n Lq-ROFNs. If $\widetilde{\beta}_i = \widetilde{\beta} = \left\langle \mathfrak{S}_{\gamma}, \mathfrak{S}_{\zeta} \right\rangle$ for all $i = 1, 2, \ldots, n$, then Lq-ROFHPWA$\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_n\right) = \widetilde{\beta}$.*

Proof. Since $\widetilde{\beta}_i = \left\langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i} \right\rangle = \left\langle \mathfrak{S}_{\gamma}, \mathfrak{S}_{\zeta} \right\rangle = \widetilde{\beta}$ for all $i = 1, 2, \ldots, n$;
Then,

$$\text{L}q\text{-ROFHPWA}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_n\right) = \text{L}q\text{-ROFHPWA}\left(\widetilde{\beta}, \widetilde{\beta}, \ldots, \widetilde{\beta}\right)$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{\prod_{i=1}^{n}(t^q+(\varsigma-1)\gamma^q)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} - \prod_{i=1}^{n}(t^q-\gamma^q)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}(t^q+(\varsigma-1)\gamma^q)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} + (\varsigma-1)\prod_{i=1}^{n}(t^q-\gamma^q)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)^{\frac{1}{q}}}, \right.$$

$$\left. \mathfrak{S}_{t\left(\frac{\varsigma\prod_{i=1}^{n}\zeta^{\frac{T_i}{\sum_{i=1}^{n}T_i}q}}{\prod_{i=1}^{n}(t^q+(\varsigma-1)(t^q-\zeta^q))^{\frac{T_i}{\sum_{i=1}^{n}T_i}} + (\varsigma-1)\prod_{i=1}^{n}\zeta^{\frac{T_i}{\sum_{i=1}^{n}T_i}q}}\right)^{\frac{1}{q}}} \right\rangle$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{(t^q+(\varsigma-1)\gamma^q)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}} - (t^q-\gamma^q)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}}{(t^q+(\varsigma-1)\gamma^q)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}} + (\varsigma-1)(t^q-\gamma^q)^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}}}\right)^{\frac{1}{q}}}, \right.$$

$$\left. \mathfrak{S}_{t\left(\frac{\varsigma\zeta^{q\left(\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}\right)}}{(t^q+(\varsigma-1)(t^q-\zeta^q))^{\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}} + (\varsigma-1)\zeta^{q\left(\frac{T_1}{\sum_{i=1}^{n}T_i}+\frac{T_2}{\sum_{i=1}^{n}T_i}+\ldots+\frac{T_n}{\sum_{i=1}^{n}T_i}\right)}}\right)^{\frac{1}{q}}} \right\rangle$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{(t^q+(\varsigma-1)\gamma^q)-(t^q-\gamma^q)}{(t^q+(\varsigma-1)\gamma^q)+(\varsigma-1)(t^q-\gamma^q)}\right)^{\frac{1}{q}}}, \ \mathfrak{S}_{t\left(\frac{\varsigma\zeta^q}{(t^q+(\varsigma-1)(t^q-\zeta^q))+(\varsigma-1)\zeta^q}\right)^{\frac{1}{q}}} \right\rangle = \left\langle \mathfrak{S}_{\gamma}, \ \mathfrak{S}_{\zeta} \right\rangle.$$

Hence the theorem is proved. □

Theorem 3 (Boundedness) *Let $\widetilde{\beta}_i = \left\langle S_{\gamma_i}, S_{\zeta_i}\right\rangle$ $(i = 1,2,\ldots,n)$ be a collection of Lq-ROFNs, and $\gamma^- = \{\gamma_i\}$, $\gamma^+ = \{\gamma_i\}$, $\zeta^- = \{\zeta_i\}$, $\zeta^+ = \{\zeta_i\}$ then*

$$\widetilde{\beta}^- \leqslant Lq\text{-}ROFHPWA\left(\widetilde{\beta}_1,\widetilde{\beta}_2,\ldots,\widetilde{\beta}_n\right) \leqslant \widetilde{\beta}^+,$$

where $\widetilde{\beta}^- = \left\langle S_{\gamma^-}, S_{\zeta^+}\right\rangle$ and $\widetilde{\beta}^+ = \left\langle S_{\gamma^+}, S_{\zeta^-}\right\rangle$.

Proof. Let $f(x) = \dfrac{t^q + (\varsigma-1)x}{t^q - x}$, $x \in [0,t)$, then $f'(x) = \dfrac{t^q\varsigma}{(t^q-x)^2} > 0$, thus f is an increasing function. Since $\gamma^- \leqslant \gamma_i \leqslant \gamma^+$, for all $i = 1,2,\ldots,n$,

$$\frac{\left(t^q + (\varsigma-1)\left(\gamma^-\right)^q\right)}{\left(t^q - \left(\gamma^-\right)^q\right)} \leqslant \frac{\left(t^q + (\varsigma-1)\gamma_i^q\right)}{\left(t^q - \gamma_i^q\right)} \leqslant \frac{\left(t^q + (\varsigma-1)\left(\gamma^+\right)^q\right)}{\left(t^q - \left(\gamma^+\right)^q\right)}.$$

So,

$$\left(\frac{\left(t^q + (\varsigma-1)\left(\gamma^-\right)^q\right)}{\left(t^q - \left(\gamma^-\right)^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} \leqslant \left(\frac{\left(t^q + (\varsigma-1)\gamma_i^q\right)}{\left(t^q - \gamma_i^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} \leqslant \left(\frac{\left(t^q + (\varsigma-1)\left(\gamma^+\right)^q\right)}{\left(t^q - \left(\gamma^+\right)^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}}$$

$$\Leftrightarrow \prod_{i=1}^n \left(\frac{\left(t^q + (\varsigma-1)\left(\gamma^-\right)^q\right)}{\left(t^q - \left(\gamma^-\right)^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} \leqslant \prod_{i=1}^n \left(\frac{\left(t^q + (\varsigma-1)\gamma_i^q\right)}{\left(t^q - \gamma_i^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}}$$

$$\leqslant \prod_{i=1}^n \left(\frac{\left(t^q + (\varsigma-1)\left(\gamma^+\right)^q\right)}{\left(t^q - \left(\gamma^+\right)^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} \Leftrightarrow \frac{\left(t^q + (\varsigma-1)\left(\gamma^-\right)^q\right)}{\left(t^q - \left(\gamma^-\right)^q\right)} + (\varsigma-1)$$

$$\leqslant \prod_{i=1}^n \left(\frac{\left(t^q + (\varsigma-1)\gamma_i^q\right)}{\left(t^q - \gamma_i^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} + (\varsigma-1) \leqslant \frac{\left(t^q + (\varsigma-1)\left(\gamma^+\right)^q\right)}{\left(t^q - \left(\gamma^+\right)^q\right)} + (\varsigma-1)$$

$$\Leftrightarrow \frac{1}{\frac{\left(t^q+(\varsigma-1)\left(\gamma^-\right)^q\right)}{\left(t^q-\left(\gamma^-\right)^q\right)} + (\varsigma-1)} \geqslant \frac{1}{\prod_{i=1}^n \left(\frac{\left(t^q+(\varsigma-1)\gamma_i^q\right)}{\left(t^q-\gamma_i^q\right)}\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} + (\varsigma-1)}$$

$$\geqslant \frac{1}{\frac{\left(t^q+(\varsigma-1)\left(\gamma^+\right)^q\right)}{\left(t^q-\left(\gamma^+\right)^q\right)} + (\varsigma-1)} \Leftrightarrow \frac{\varsigma\left(t^q - \left(\gamma^-\right)^q\right)}{\varsigma t^q}$$

$$\geqslant \frac{\varsigma \prod_{i=1}^n \left(t^q - \gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^n T_i}}}{\prod_{i=1}^n \left(t^q + (\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^n T_i}} + (\varsigma-1)\prod_{i=1}^n \left(t^q - \gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^n T_i}}} \geqslant \frac{\varsigma\left(t^q - \left(\gamma^+\right)^q\right)}{\varsigma t^q}$$

$$\Leftrightarrow 1-\frac{\varsigma\left(t^q-(\gamma^-)^q\right)}{\varsigma t^q}\leqslant 1-\frac{\varsigma\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\varsigma-1)\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}$$

$$\leqslant 1-\frac{\varsigma\left(t^q-(\gamma^-)^q\right)}{\varsigma t^q}\Leftrightarrow\frac{(\gamma^-)^q}{t^q}$$

$$\leqslant 1-\frac{\varsigma\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\varsigma-1)\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\leqslant\frac{(\gamma^+)^q}{t^q}$$

i.e.,

$$\gamma^-\leqslant t\left(\frac{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\varsigma-1)\prod_{i=1}^{n}\left(t^q-\gamma_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)^{\frac{1}{q}}\leqslant\gamma^+. \tag{5}$$

Again let $g(y)=\frac{(t^q+(\varsigma-1)(t^q-y))}{y}$, $y\in(0,t]$, $\varsigma>0$, then $g'(y)=-\frac{\varsigma t^q}{y^2}<0$, thus $g(y)$ is a decreasing function.

Since for all i, $\zeta^+\geqslant\zeta_i\geqslant\zeta^-$, then

$$\frac{(t^q+(\varsigma-1)\left(t^q-(\zeta^+)^q\right))}{(\zeta^+)^q}\leqslant\frac{(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right))}{\zeta_i^q}\leqslant\frac{(t^q+(\varsigma-1)\left(t^q-(\zeta^-)^q\right))}{(\zeta^-)^q},$$

thus,

$$\left(\frac{(t^q+(\varsigma-1)\left(t^q-(\zeta^+)^q\right))}{(\zeta^+)^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\leqslant\left(\frac{(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right))}{\zeta_i^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}$$

$$\leqslant\left(\frac{(t^q+(\varsigma-1)\left(t^q-(\zeta^-)^q\right))}{(\zeta^-)^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}\Leftrightarrow\prod_{i=1}^{n}\left(\frac{(t^q+(\varsigma-1)\left(t^q-(\zeta^+)^q\right))}{(\zeta^+)^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}$$

$$\leqslant \prod_{i=1}^{n}\left(\frac{\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)}{\zeta_i^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}} \leqslant \prod_{i=1}^{n}\left(\frac{\left(t^q+(\varsigma-1)\left(t^q-(\zeta^-)^q\right)\right)}{(\zeta^-)^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}$$

$$\Leftrightarrow \frac{\left(t^q+(\varsigma-1)\left(t^q-(\zeta^+)^q\right)\right)}{(\zeta^+)^q} \leqslant \prod_{i=1}^{n}\left(\frac{\left(t^q+t(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)}{\zeta_i^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}$$

$$\leqslant \frac{\left(t^q+(\varsigma-1)\left(t^q-(\zeta^-)^q\right)\right)}{(\zeta^-)^q} \Leftrightarrow \frac{\varsigma t^q-(\varsigma-1)(\zeta^+)^q}{(\zeta^+)^q}+(\varsigma-1)$$

$$\leqslant \prod_{i=1}^{n}\left(\frac{\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)}{\zeta_i^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\varsigma-1) \leqslant \frac{\varsigma t^q-(\varsigma-1)(\zeta^-)^q}{(\zeta^-)^q}+(\varsigma-1)$$

$$\Leftrightarrow \frac{1}{\frac{\varsigma t^q}{(\zeta^+)^q}} \geqslant \frac{1}{\prod_{i=1}^{n}\left(\frac{\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)}{\zeta_i^q}\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\varsigma-1)} \geqslant \frac{1}{\frac{\varsigma t^q}{(\zeta^-)^q}}$$

$$\Leftrightarrow \zeta^+ \geqslant t\left(\frac{\varsigma\prod_{i=1}^{n}\zeta_i^{\frac{T_i}{\sum_{i=1}^{n}T_i}q}}{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\left(t^q-\zeta_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+(\varsigma-1)\prod_{i=1}^{n}\zeta_i^{\frac{T_i}{\sum_{i=1}^{n}T_i}q}}\right)^{\frac{1}{q}} \geqslant \zeta^-. \tag{6}$$

From (5) and (6), it is clear that

$$S\left(\widetilde{\beta}^-\right) \leqslant S\left(\text{L}q\text{-ROFHPWA}\left(\widetilde{\beta}_1,\widetilde{\beta}_2,...,\widetilde{\beta}_n\right)\right) \leqslant S\left(\widetilde{\beta}^+\right).$$

Therefore, $\widetilde{\beta}^- \leqslant \text{L}q\text{-ROFHPWA}\left(\widetilde{\beta}_1,\widetilde{\beta}_2,...,\widetilde{\beta}_n\right) \leqslant \widetilde{\beta}^+$.

Definition 8 *Let $\left\{\widetilde{\beta}_1,\widetilde{\beta}_2,...,\widetilde{\beta}_n\right\}$ be a set of Lq-ROFNs, where $\widetilde{\beta}_i = \left\langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i}\right\rangle$ $(i = 1,2,\ldots,n)$ and $q \geqslant 1$. Then Lq-ROFHPWG operator is defined as*

$$Lq\text{-}ROFHPWG\left(\widetilde{\beta}_1,\widetilde{\beta}_2,...,\widetilde{\beta}_n\right) = \bigotimes_{i=1}^{n}{}_H\left(\widetilde{\beta}_i\right)^{\omega_i}, \tag{7}$$

where $\omega = (\omega_1,\omega_2,\ldots,\omega_n)$ is the weight vectors of $\widetilde{\beta}_i$ with $\omega_i \in [0,1]$ and $\omega_i = \frac{T_i}{\sum_{i=1}^{n}T_i}$, $T_i = \prod_{k=1}^{i-1}\frac{S\left(\widetilde{\beta}_k\right)}{t}$ $(i = 2,\ldots,n)$, $T_1 = 1$ and $S\left(\widetilde{\beta}_i\right)$ is the score of $\widetilde{\beta}_i$.

Theorem 4 *Let $\widetilde{\beta}_i = \left\langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i} \right\rangle$ $(i = 1, 2, \ldots, n)$ be a set of Lq-ROFNs. Then, the aggregated result from the Lq-ROFHPWG operator is also a Lq-ROFN, where*

$$Lq\text{-}ROFHPWG\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_n\right) = \bigotimes_{i=1}^{n}{}_H \left(\widetilde{\beta}_i\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}$$

$$= \left\langle \mathfrak{S}_{t\left(\frac{\varsigma \prod_{i=1}^{n} \gamma_i^{\frac{T_i}{\sum_{i=1}^{n} T_i}q}}{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\left(t^q-\gamma_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\varsigma-1)\prod_{i=1}^{n} \gamma_i^{\frac{T_i}{\sum_{i=1}^{n} T_i}q}}\right)^{\frac{1}{q}}}, \mathfrak{S}_{t\left(\frac{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} - \prod_{i=1}^{n}\left(t^q-\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{\prod_{i=1}^{n}\left(t^q+(\varsigma-1)\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}} + (\varsigma-1)\prod_{i=1}^{n}\left(t^q-\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}\right)^{\frac{1}{q}}} \right\rangle.$$

Proof. Proof of this theorem is similar to the proof of Theorem 1.

Now, some particular cases of the Lq-ROFHPWG operator are discussed based on parameter ς.

- When $\varsigma = 1$, Lq-ROFHPWG operator reduces to the Lq-ROF prioritized weighted geometric (Lq-ROFPWG) operator as follows:

$$\text{L}q\text{-ROFPWG}\left(\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_n\right) = \left\langle \mathfrak{S}_{t\left(\prod_{i=1}^{n}\left(\frac{\gamma_i}{t}\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}q}\right)^{\frac{1}{q}}}, \mathfrak{S}_{t\left(\frac{t^q - \prod_{i=1}^{n}\left(t^q-\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n} T_i}}}{t^q}\right)^{\frac{1}{q}}} \right\rangle.$$

- When $\varsigma = 2$, Lq-ROFHPWG operator reduces to the Lq-ROF Einstein prioritized weighted geometric (Lq-ROFEPWG) operator as follows:

$$\text{L}q\text{-ROFEPWG}\left(\widetilde{\beta}_1,\widetilde{\beta}_2, ldots,\widetilde{\beta}_n\right)$$

$$=\left\langle \mathfrak{S}_{t\left(\frac{2\prod_{i=1}^{n}\gamma_i^{\frac{T_i}{\sum_{i=1}^{n}T_i}q}}{\prod_{i=1}^{n}\left(t^q+\left(t^q-\gamma_i^q\right)\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\gamma_i^{\frac{T_i}{\sum_{i=1}^{n}T_i}q}}\right)^{\frac{1}{q}}}, \mathfrak{S}_{t\left(\frac{\prod_{i=1}^{n}\left(t^q+\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}-\prod_{i=1}^{n}\left(t^q-\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}{\prod_{i=1}^{n}\left(t^q+\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}+\prod_{i=1}^{n}\left(t^q-\zeta_i^q\right)^{\frac{T_i}{\sum_{i=1}^{n}T_i}}}\right)^{\frac{1}{q}}} \right\rangle.$$

Theorem 5 (Idempotency) *Let* $\widetilde{\beta}_i = \langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i}\rangle$ *$(i = 1,2,\ldots,n)$ be a collection of n Lq-ROFNs. If* $\widetilde{\beta}_i = \widetilde{\beta} = \langle \mathfrak{S}_{\gamma}, \mathfrak{S}_{\zeta}\rangle$ *for all* $i = 1,2,\ldots,n$, *then* *Lq-ROFHPWG* $\left(\widetilde{\beta}_1,\widetilde{\beta}_2,\ldots,\widetilde{\beta}_n\right) = \widetilde{\beta}$.

Theorem 6 (Boundedness) *Let* $\widetilde{\beta}_i = \langle \mathfrak{S}_{\gamma_i}, \mathfrak{S}_{\zeta_i}\rangle$ *$(i = 1,2,\ldots,n)$ be a collection of Lq-ROFNs, and* $\gamma^- = \min_i\{\gamma_i\}$, $\gamma^+ = \max_i\{\gamma_i\}$, $\zeta^- = \min_i\{\zeta_i\}$, $\zeta^+ = \max_i\{\zeta_i\}$ *then*

$$\widetilde{\beta}^- \leqslant Lq\text{-}ROFHPWG\left(\widetilde{\beta}_1,\widetilde{\beta}_2,\ldots,\widetilde{\beta}_n\right) \leqslant \widetilde{\beta}^+.$$

The proofs of Theorem 5 and 6 are analogous to the previous.

5. An MCGDM approach based on Lq-ROF prioritized aggregation operators

In this section, a novel MCGDM approach have been propounded, in which the evaluation information is in the form of Lq-ROFNs.

For a group decision making problem, let $\mathcal{E} = \{\mathcal{E}^{(1)},\mathcal{E}^{(2)},\ldots,\mathcal{E}^{(u)}\}$ be the set of the DMs and the linear ordering $\mathcal{E}^{(1)} \succ \mathcal{E}^{(2)} \succ \ldots \succ \mathcal{E}^{(u)}$ represents the prioritization relationship among the DMs' in such a manner that DM, $\mathcal{E}^{(k)}$, has a higher priority than DM, $\mathcal{E}^{(l)}$, if $k < l$. Suppose $\mathcal{A} = \{A_1, A_2,\ldots,A_m\}$ be a discrete collection of alternatives. $\mathcal{G} = \{\mathcal{G}_1,\mathcal{G}_2,\ldots,\mathcal{G}_n\}$ represents the set of criteria with their prioritization as $\mathcal{G}_1 \succ \mathcal{G}_2 \succ \ldots \succ \mathcal{G}_n$, so that criteria $\mathcal{G}_j$ has a higher priority than $\mathcal{G}_i$, for $j < i$. DMs provide their evaluation values in terms of Lq-ROFNs based on LTS: $\mathfrak{S} = \{\mathfrak{S}_0,\mathfrak{S}_1,\mathfrak{S}_2,\ldots,\mathfrak{S}_t\}$. A L$q$-ROF decision matrix (L$q$-ROFDM) $\widetilde{X}^{(l)} = \left[\widetilde{\beta}_{ij}^{(l)}\right]_{m\times n} = \left[\left\langle \mathfrak{S}_{\gamma^{(l)}_{\widetilde{\beta}_{ij}}}, \mathfrak{S}_{\zeta^{(l)}_{\widetilde{\beta}_{ij}}}\right\rangle\right]_{m\times n}$, where $\left\langle \mathfrak{S}_{\gamma^{(l)}_{\widetilde{\beta}_{ij}}}, \mathfrak{S}_{\zeta^{(l)}_{\widetilde{\beta}_{ij}}}\right\rangle$ denotes a Lq-ROFN given by the DM $\mathcal{E}^{(l)}$ for the alternative A_i

under the criteria $\mathcal{G}_j$. Here corresponding to the DM $\mathcal{E}^{(l)}$, $\mathfrak{S}_{\gamma^{(l)}_{\widetilde{\beta}_{ij}}}$ indicates the satisfaction degree of the alternative A_i concerning the criteria $\mathcal{G}_j$; whereas $\mathfrak{S}_{\zeta^{(l)}_{\widetilde{\beta}_{ij}}}$ indicates that of dissatisfaction degree.

The purpose is to find the best suitable alternative(s) in light of the presented approach. The computational process is summarized step-by-step as follows.

Step 1. Normalize $\widetilde{\mathcal{X}}^{(l)}$, if required, into $\widetilde{R}^{(l)} = \left[\widetilde{r}^{(l)}_{ij}\right]_{m\times n}$ as follows:

$$\widetilde{r}^{(l)}_{ij} = \begin{cases} \left\langle \mathfrak{S}_{\gamma^{(l)}_{\widetilde{\beta}_{ij}}},\ \mathfrak{S}_{\zeta^{(l)}_{\widetilde{\beta}_{ij}}} \right\rangle & \text{if } \mathcal{G}_j \text{ is type of benefit criteria;} \\ \left\langle \mathfrak{S}_{\zeta^{(l)}_{\widetilde{\beta}_{ij}}},\ \mathfrak{S}_{\gamma^{(l)}_{\widetilde{\beta}_{ij}}} \right\rangle & \text{if } \mathcal{G}_j \text{ is type of cost criteria.} \end{cases}$$

Step 2. Calculate the value of $T^{(l)}_{ij}$ $(l = 1, 2, \ldots, u)$ with the following equations.

$$T^{(l)}_{ij} = \begin{cases} 1 & \text{for } l = 1, \\ \prod\limits_{k=1}^{l-1} \dfrac{S\left(\widetilde{r}^{(k)}_{ij}\right)}{t} & \text{for } l = 2, 3, \ldots, u. \end{cases} \tag{8}$$

Step 3. To aggregate all the individual Lq-ROFDM $\widetilde{R}^{(l)} = \left[\widetilde{r}^{(l)}_{ij}\right]_{m\times n}$ $(l = 1, 2, \ldots, u)$, using the Lq-ROFHPWA operator and obtain overall DM $\widetilde{R} = \left[\widetilde{r}_{ij}\right]_{m\times n}$ as

$$\widetilde{r}_{ij} = \text{L}q\text{-ROFHPWA}\left(\widetilde{r}^{(1)}_{ij}, \widetilde{r}^{(2)}_{ij}, \ldots, \widetilde{r}^{(u)}_{ij}\right)$$

$$= \left\langle \mathfrak{S}_{t\left(\dfrac{\prod\limits_{l=1}^{u}\left(t^q+(\varsigma-1)\left(\gamma^{(l)}_{ij}\right)^q\right)^{\frac{T^{(l)}_{ij}}{\sum_{l=1}^{u}T^{(l)}_{ij}}} - \prod\limits_{l=1}^{u}\left(t^q-\left(\gamma^{(l)}_{ij}\right)^q\right)^{\frac{T^{(l)}_{ij}}{\sum_{l=1}^{u}T^{(l)}_{ij}}}}{\prod\limits_{l=1}^{u}\left(t^q+(\varsigma-1)\left(\gamma^{(l)}_{ij}\right)^q\right)^{\frac{T^{(l)}_{ij}}{\sum_{l=1}^{u}T^{(l)}_{ij}}} + (\varsigma-1)\prod\limits_{l=1}^{u}\left(t^q-\left(\gamma^{(l)}_{ij}\right)^q\right)^{\frac{T^{(l)}_{ij}}{\sum_{l=1}^{u}T^{(l)}_{ij}}}} \right)^{\frac{1}{q}}}, \right.$$

$$\left.\mathfrak{S}_{t\left(\frac{\varsigma\prod_{l=1}^{u}\left(\zeta_{ij}^{(l)}\right)^{q\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}}{\prod_{l=1}^{u}\left(t^q+(\varsigma-1)\left(t^q-\left(\zeta_{ij}^{(l)}\right)^q\right)\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}+(\varsigma-1)\prod_{l=1}^{u}\left(\zeta_{ij}^{(l)}\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}q}}\right)^{\frac{1}{q}}}\right\rangle. \tag{9}$$

or, using the Lq-ROFHPWG operator

$$\widetilde{r}'_{ij}=\text{L}q\text{-ROFHPWG}\left(\widetilde{r}_{ij}^{(1)},\widetilde{r}_{ij}^{(2)},\ldots,\widetilde{r}_{ij}^{(u)}\right)$$

$$=\left\langle\mathfrak{S}_{t\left(\frac{\varsigma\prod_{l=1}^{u}\left(\gamma_{ij}^{(l)}\right)^{q\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}}{\prod_{l=1}^{u}\left(t^q+(\varsigma-1)\left(t^q-\left(\gamma_{ij}^{(l)}\right)^q\right)\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}+(\varsigma-1)\prod_{l=1}^{u}\left(\gamma_{ij}^{(l)}\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}q}}\right)^{\frac{1}{q}}},\right.$$

$$\left.\mathfrak{S}_{t\left(\frac{\prod_{l=1}^{u}\left(t^q+(\varsigma-1)\left(\zeta_{ij}^{(l)}\right)^q\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}-\prod_{l=1}^{u}\left(t^q-\left(\zeta_{ij}^{(l)}\right)^q\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}}{\prod_{l=1}^{u}\left(t^q+(\varsigma-1)\left(\zeta_{ij}^{(l)}\right)^q\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}+(\varsigma-1)\prod_{l=1}^{u}\left(t^q-\left(\zeta_{ij}^{(l)}\right)^q\right)^{\frac{T_{ij}^{(l)}}{\sum_{l=1}^{u}T_{ij}^{(l)}}}}\right)^{\frac{1}{q}}}\right\rangle. \tag{10}$$

Step 4. Calculate the values of T_{ij} as

$$T_{ij}=\begin{cases}1 & \text{for } j=1\\ \prod_{k=1}^{j-1}\frac{S\left(\widetilde{r}_{ik}\right)}{t} & \text{for } j=2,3,\ldots,n.\end{cases} \tag{11}$$

Step 5. Aggregate the Lq-ROFNs $\widetilde{r}_{ij}$ for each alternative A_i using the Lq-ROFHPWA (or Lq-ROFHPWG) operators as follows:

$$
\begin{aligned}
\widetilde{r}_i &= \text{L}q\text{-ROFHPWA}\left(\widetilde{r}_{i1},\widetilde{r}_{i2},\ldots,\widetilde{r}_{in}\right)\\
&= \left\langle \mathfrak{S}\left(t\frac{\prod\limits_{j=1}^{n}\left(t^q+(\varsigma-1)\gamma_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}-\prod\limits_{j=1}^{n}\left(t^q-\gamma_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\prod\limits_{j=1}^{n}\left(t^q+(\varsigma-1)\gamma_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}+(\varsigma-1)\prod\limits_{j=1}^{n}\left(t^q-\gamma_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}\right)^{\frac{1}{q}},\right.\\
&\qquad\left.\mathfrak{S}\left(t\frac{\varsigma\prod\limits_{j=1}^{n}\left(\zeta_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}q}}{\prod\limits_{j=1}^{n}\left(t^q+(\varsigma-1)\left(t^q-\zeta_{ij}^q\right)\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}+(\varsigma-1)\prod\limits_{j=1}^{n}\left(\zeta_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}q}}\right)^{\frac{1}{q}}\right\rangle,
\end{aligned}
\tag{12}
$$

or

$$
\begin{aligned}
\widetilde{r}'_i &= \text{L}q\text{-ROFHPWG}\left(\widetilde{r}_{i1},\widetilde{r}_{i2},\ldots,\widetilde{r}_{in}\right)\\
&= \left\langle \mathfrak{S}\left(t\frac{\varsigma\prod\limits_{j=1}^{n}\left(\gamma_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}q}}{\prod\limits_{j=1}^{n}\left(t^q+(\varsigma-1)\left(t^q-\gamma_{ij}^q\right)\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}+(\varsigma-1)\prod\limits_{j=1}^{n}\left(\gamma_{ij}\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}q}}\right)^{\frac{1}{q}},\right.\\
&\qquad\left.\mathfrak{S}\left(t\frac{\prod\limits_{j=1}^{n}\left(t^q+(\varsigma-1)\zeta_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}-\prod\limits_{j=1}^{n}\left(t^q-\zeta_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}{\prod\limits_{j=1}^{n}\left(t^q+(\varsigma-1)\zeta_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}+(\varsigma-1)\prod\limits_{j=1}^{n}\left(t^q-\zeta_{ij}^q\right)^{\frac{T_{ij}}{\sum_{j=1}^{n}T_{ij}}}}\right)^{\frac{1}{q}}\right\rangle.
\end{aligned}
\tag{13}
$$

Step 6. Calculate the score values for each $\widetilde{r}_i$ (or $\widetilde{r}'_i$) $(i = 1, 2, \ldots, m)$ using Eq. (2).

Step 7. Rank the alternatives A_i $(i = 1, 2, \ldots, m)$ based on the comparison rule presented in Definition 4.

Based on the methodology developed in this paper, the following illustrative example is considered and solved.

6. Illustrative example

In this section, a numerical example, previously studied by Arora and Garg [50], has been illustrated from the field of global suppliers with Lq-ROF context.

Following notations are used to represent the MCGDM problem relating to the selection of the best global suppliers by a manufacturing company to utilize in their assembling process.

Suppose there are four alternatives A_1, A_2, A_3 and A_4 which are considered for evaluating over the five criteria $\{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5\}$. The prioritization relationship for the criterion is $\mathcal{G}_1 \succ \mathcal{G}_2 \succ \mathcal{G}_3 \succ \mathcal{G}_4 \succ \mathcal{G}_5$. The different alternatives A_i $(i = 1, 2, 3, 4)$ are evaluated by the four DMs, $\mathcal{E}^{(l)}$ $(l = 1, 2, 3, 4)$ with priority levels $\mathcal{E}^{(1)} \succ \mathcal{E}^{(2)} \succ \mathcal{E}^{(3)} \succ \mathcal{E}^{(4)}$ on the basis of the criteria $\mathcal{G}_i$ $(i = 1, 2, 3, 4, 5)$. DMs $\mathcal{E}^{(l)}$ $(l = 1, 2, 3, 4)$ provide his/her decision preferences in terms of Lq-ROFNs using the linguistic term set: $\mathfrak{S} = \{\mathfrak{S}_0$ = extremely poor, $\mathfrak{S}_1$ = very poor, $\mathfrak{S}_2$ = poor, $\mathfrak{S}_3$ = slightly poor, $\mathfrak{S}_4$ = fair, $\mathfrak{S}_5$ = slightly good, $\mathfrak{S}_6$ = good, $\mathfrak{S}_7$ = very good, $\mathfrak{S}_8$ = extremely good$\}$. In Tables 1, 2, 3 and 4, the decision information provided by the four DMs, $\mathcal{E}^{(1)}$, $\mathcal{E}^{(2)}$, $\mathcal{E}^{(3)}$ and $\mathcal{E}^{(4)}$ are presented in terms of Lq-ROFNs, respectively.

Table 1: Lq-ROFDM $\widetilde{X}^{(1)}$ provided by the DM $\mathcal{E}^{(1)}$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
A_1	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_4, \mathfrak{S}_3)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$
A_2	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$
A_3	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_3)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_1)$	$(\mathfrak{S}_3, \mathfrak{S}_4)$
A_4	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_4, \mathfrak{S}_3)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_4, \mathfrak{S}_4)$

The procedure of selecting the most desirable alternative(s) utilizing the above-proposed operators are presented in the following steps.

Step 1. Since all the criteria are of the same type, the normalization process is not needed for this problem, i.e., $\widetilde{X}^{(l)} = \widetilde{R}^{(l)} = \left[\widetilde{r}_{ij}^{(l)}\right]_{m \times n}$ $(l = 1, 2, 3, 4)$.

Table 2: Lq-ROFDM $\widetilde{\mathcal{X}}^{(2)}$ provided by the DM $\mathcal{E}^{(2)}$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
A_1	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_4, \mathfrak{S}_4)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_3, \mathfrak{S}_5)$
A_2	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_1)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_4, \mathfrak{S}_3)$
A_3	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_3)$	$(\mathfrak{S}_4, \mathfrak{S}_4)$
A_4	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_4, \mathfrak{S}_3)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_3)$

Table 3: Lq-ROFDM $\widetilde{\mathcal{X}}^{(3)}$ provided by the DM $\mathcal{E}^{(3)}$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
A_1	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_3, \mathfrak{S}_4)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$
A_2	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_5, \mathfrak{S}_1)$
A_3	$(\mathfrak{S}_5, \mathfrak{S}_3)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_4, \mathfrak{S}_3)$	$(\mathfrak{S}_3, \mathfrak{S}_1)$
A_4	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_4, \mathfrak{S}_4)$

Table 4: Lq-ROFDM $\widetilde{\mathcal{X}}^{(4)}$ provided by the DM $\mathcal{E}^{(4)}$

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
A_1	$(\mathfrak{S}_5, \mathfrak{S}_3)$	$(\mathfrak{S}_4, \mathfrak{S}_4)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_1)$	$(\mathfrak{S}_4, \mathfrak{S}_2)$
A_2	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_7, \mathfrak{S}_1)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_6, \mathfrak{S}_1)$
A_3	$(\mathfrak{S}_5, \mathfrak{S}_2)$	$(\mathfrak{S}_3, \mathfrak{S}_4)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_3, \mathfrak{S}_3)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$
A_4	$(\mathfrak{S}_4, \mathfrak{S}_3)$	$(\mathfrak{S}_5, \mathfrak{S}_1)$	$(\mathfrak{S}_4, \mathfrak{S}_2)$	$(\mathfrak{S}_6, \mathfrak{S}_2)$	$(\mathfrak{S}_5, \mathfrak{S}_2)$

Step 2. Utilizing Eq. (8), the values of T_{ij} are obtained as:

$$T_{ij}^1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad T_{ij}^2 = \begin{bmatrix} 0.9413 & 0.8892 & 0.8124 & 0.9413 & 0.8501 \\ 0.8892 & 0.8501 & 0.8921 & 0.8892 & 0.9413 \\ 0.8921 & 0.8414 & 0.9413 & 0.8532 & 0.7741 \\ 0.8501 & 0.9413 & 0.8124 & 0.8921 & 0.7937 \end{bmatrix},$$

$$T_{ij}^3 = \begin{bmatrix} 0.8860 & 0.7058 & 0.7224 & 0.8001 & 0.6286 \\ 0.8370 & 0.7253 & 0.7958 & 0.7559 & 0.7647 \\ 0.7583 & 0.7506 & 0.8860 & 0.7179 & 0.6144 \\ 0.7559 & 0.7647 & 0.6906 & 0.8397 & 0.6678 \end{bmatrix},$$

$$T_{ij}^4 = \begin{bmatrix} 0.8860 & 0.7058 & 0.7224 & 0.8001 & 0.6286 \\ 0.8370 & 0.7253 & 0.7958 & 0.7559 & 0.7647 \\ 0.7583 & 0.7506 & 0.8860 & 0.7179 & 0.6144 \\ 0.7559 & 0.7647 & 0.6906 & 0.8397 & 0.6678 \end{bmatrix}.$$

Step 3. Based on the DMs' information provided in Tables 1, 2, 3 and 4, the proposed Lq-ROFHPWA operator, presented in Eq. (9), is utilized to aggregate them into a collective matrix. The result obtained is summarized in Table 5.

Table 5: Collective Lq-ROFDM $\widetilde{R}$ based on Lq-ROFHPWA operator

	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	$\mathcal{G}_5$
A_1	$(\mathfrak{S}_{6.4811}, \mathfrak{S}_{1.2739})$	$(\mathfrak{S}_{5.0036}, \mathfrak{S}_{2.7792})$	$(\mathfrak{S}_{5.3178}, \mathfrak{S}_{2.3785})$	$(\mathfrak{S}_{6.2793}, \mathfrak{S}_{1.2059})$	$(\mathfrak{S}_{4.4125}, \mathfrak{S}_{2.6195})$
A_2	$(\mathfrak{S}_{6.5691}, \mathfrak{S}_{1.2187})$	$(\mathfrak{S}_{5.7933}, \mathfrak{S}_{1.4508})$	$(\mathfrak{S}_{6.2900}, \mathfrak{S}_{1.0000})$	$(\mathfrak{S}_{5.5805}, \mathfrak{S}_{2.0000})$	$(\mathfrak{S}_{5.8300}, \mathfrak{S}_{1.3641})$
A_3	$(\mathfrak{S}_{5.3499}, \mathfrak{S}_{1.7823})$	$(\mathfrak{S}_{5.0564}, \mathfrak{S}_{2.1833})$	$(\mathfrak{S}_{6.6134}, \mathfrak{S}_{1.1640})$	$(\mathfrak{S}_{4.5227}, \mathfrak{S}_{2.1264})$	$(\mathfrak{S}_{3.7615}, \mathfrak{S}_{2.6698})$
A_4	$(\mathfrak{S}_{5.4024}, \mathfrak{S}_{2.1750})$	$(\mathfrak{S}_{6.1455}, \mathfrak{S}_{1.3558})$	$(\mathfrak{S}_{4.5377}, \mathfrak{S}_{1.9576})$	$(\mathfrak{S}_{6.1452}, \mathfrak{S}_{1.3678})$	$(\mathfrak{S}_{4.4942}, \mathfrak{S}_{3.2912})$

Step 4. Using Eq. (11), the values of T_{ij} are calculated as:

$$T_{ij} = \begin{bmatrix} 1.0000 & 0.9141 & 0.7716 & 0.6627 & 0.5995 \\ 1.0000 & 0.9186 & 0.8105 & 0.7338 & 0.6395 \\ 1.0000 & 0.8636 & 0.7348 & 0.6767 & 0.5646 \\ 1.0000 & 0.8635 & 0.7755 & 0.6482 & 0.5820 \end{bmatrix}.$$

Step 5. The collective value $\widetilde{r}_i$ of each alternative A_i is obtained based on Lq-ROFHPWA operator using Eq. (12).

$$\widetilde{r}_1 = (\mathfrak{S}_{5.6888}, \mathfrak{S}_{1.9095}), \quad \widetilde{r}_2 = (\mathfrak{S}6.0830, \mathfrak{S}_{1.3555}),$$
$$\widetilde{r}_3 = (\mathfrak{S}_{5.3168}, \mathfrak{S}_{1.8840}), \quad \widetilde{r}_4 = (\mathfrak{S}_{5.4816}, \mathfrak{S}_{1.8903}).$$

Step 6. The score values for each $\widetilde{r}_i$ $(i = 1, 2, 3, 4)$ are calculated based on Eq. (2) as:

$$S(\widetilde{r}_1) = 7.0107, \quad S(\widetilde{r}_2) = 7.1615, \quad S(\widetilde{r}_3) = 6.8951, \quad S(\widetilde{r}_4) = 6.9450.$$

Step 7. The rank of the alternatives A_i $(i = 1, 2, 3, 4)$ based on the comparison rule presented in Definition 4 is found as $A_2 \succ A_1 \succ A_4 \succ A_3$.

On the other hand, if the above MCGDM problem is solved with Lq-ROFHPWG operator, the score values of four different alternatives are obtained as:

$$S(\widetilde{r}'_1) = 6.8368, \quad S(\widetilde{r}'_2) = 7.0992, \quad S(\widetilde{r}'_3) = 6.7596, \quad S(\widetilde{r}'_4) = 6.8349.$$

Thus the ordering of the alternatives are found as $A_2 \succ A_1 \succ A_4 \succ A_3$.

6.1. Influence of rung parameter q on decision making results

The proposed methodology allows DMs to flexibly change their range of evaluation information with the use of rung parameter q. The parameter q plays a significant role in the decision results. In solving the above numerical problem, the parameter $q = 3$ is considered. To investigate the impact of rung parameter q on the decision result, the above problem is further solved based on different values of the parameter q from 1 to 10. For convenience, the Hamacher parameter is kept fixed at $\varsigma = 3$ in the computational process.

Table 6: Influence of rung parameter q with Lq-ROFHPWA operator on ranking results

Parameter q	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	Ranking
$q = 1$	5.878	6.3438	5.7277	5.7425	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 2$	6.7813	7.0339	6.6403	6.703	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 3$	7.0107	7.1615	6.8951	6.945	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 4$	7.1366	7.2364	7.0445	7.0817	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 5$	7.2323	7.3004	7.1590	7.1871	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 6$	7.3109	7.3578	7.2523	7.274	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 7$	7.3767	7.4091	7.3296	7.3466	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 8$	7.4322	7.4546	7.3941	7.4077	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 9$	7.4794	7.4949	7.4485	7.4594	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 10$	7.5199	7.5307	7.4947	7.5035	$A_2 \succ A_1 \succ A_4 \succ A_3$

The obtained score values for each alternative are listed in Tables 6 and 7 using Lq-ROFHPWA and Lq-ROFHPWG operators, respectively. From the ranking results as viewed from Table 7, it is inferred that slight differences in the ranking results using Lq-ROFHPWG operator are found when parameter q changes. Whereas, based on using Lq-ROFHPWA operator in Table 6, the ranking of alternatives is consistent with the rung parameter q. However, in all the cases, A_2 is the optimal choice. This indicates that the parameter q has a steadiness in the decision results in terms of generating the best choice.

Further, in Figs. 1 and 2, a clear view of the impact of rung parameters utilizing Lq-ROFHPWA and Lq-ROFHPWG operators, respectively, have been depicted. From Figs.1 and2, it is observed that when the parameter $q \in [1, 10]$ changes, the score values for the alternatives changes accordingly. It reveals from Fig. 1 that different alternatives do not change their ordered positions. Thus for Lq-ROFHPWA operator, the ranking of alternatives is stable. On the other hand, in Fig. 2, there is a change in the ordered position of the alternatives A_1 and A_4 is noticed. As a consequence, the ranking of alternatives slightly differs based on the Lq-ROFHPWG operator.

Finally, it is important to note that DMs can change the value of q according to their preferences for expressing their evaluation values in a wider range, which makes the proposed methodology a flexible method.

Table 7: Influence of rung parameter q with Lq-ROFHPWG operator on ranking results

Parameter q	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	Ranking
$q = 1$	5.6188	6.2362	5.5312	5.5679	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 2$	6.5615	6.9549	6.4661	6.5596	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 3$	6.8368	7.0992	6.7596	6.8349	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 4$	6.9933	7.1799	6.9371	6.9914	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 5$	7.1117	7.2468	7.0731	7.1104	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 6$	7.2096	7.3073	7.1840	7.2089	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 7$	7.2923	7.3624	7.2757	7.2920	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 8$	7.3625	7.4124	7.3519	7.3624	$A_2 \succ A_1 \succ A_4 \succ A_3$
$q = 9$	7.4222	7.4575	7.4155	7.4223	$A_2 \succ A_4 \succ A_1 \succ A_3$
$q = 10$	7.4730	7.4979	7.4689	7.4731	$A_2 \succ A_4 \succ A_1 \succ A_3$

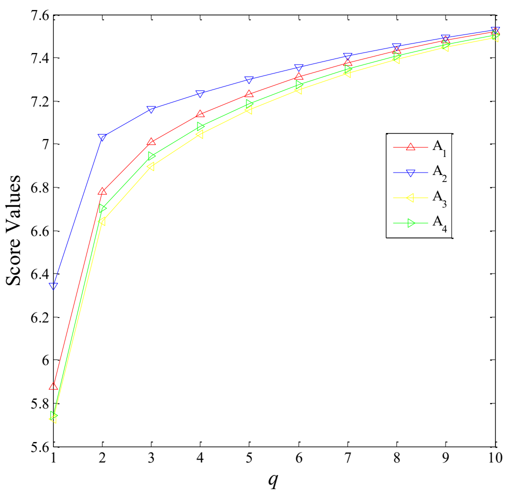

Figure 1: Score values of alternative for $q \in [1, 10]$ based on Lq-ROFHPWA operator ($\varsigma = 3$)

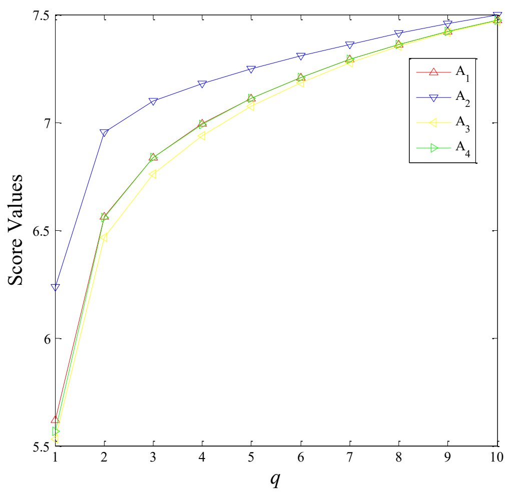

Figure 2: Score values of alternative for $q \in [1, 10]$ based on Lq-ROFHPWG operator ($\varsigma = 3$)

6.2. Influence of Hamacher parameter on decision making results

The proposed method carries the robustness of the Hamacher parameter ς. Varying the Hamacher parameter ς in (0, 10] the impact of the parameter on decision results is investigated. For convenience, the rung parameter is kept fixed at $q = 3$ in the computational process.

In Tables 8 and 9, the achieved results based on Lq-ROFHPWA and Lq-ROFHPWG operators are presented, respectively. The score of the alternatives varies accordingly with different parameters ς using Lq-ROFHPWA and Lq-ROFHPWG operators.

Table 8: Ranking results for varying ς by using Lq-ROFHPWA operator

Parameter ς	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	Ranking
$\varsigma = 1$	7.0628	7.1870	6.9333	6.9859	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 2$	7.0299	7.1705	6.9094	6.9598	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 3$	7.0107	7.1615	6.8951	6.9450	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 4$	6.9976	7.1558	6.8852	6.9352	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 5$	6.9880	7.1518	6.8778	6.9282	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 6$	6.9807	7.1488	6.8720	6.9229	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 7$	6.9748	7.1465	6.8673	6.9187	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 8$	6.9699	7.1446	6.8634	6.9153	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 9$	6.9659	7.1431	6.8601	6.9125	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 10$	6.9624	7.1418	6.8572	6.9101	$A_2 \succ A_1 \succ A_4 \succ A_3$

Table 9: Ranking results for varying ς by using Lq-ROFHPWG operator

Parameter ς	$S(A_1)$	$S(A_2)$	$S(A_3)$	$S(A_4)$	Ranking
$\varsigma = 1$	6.7963	7.0769	6.7349	6.8109	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\varsigma = 2$	6.8232	7.0920	6.7516	6.8270	$A_2 \succ A_4 \succ A_1 \succ A_3$
$\varsigma = 3$	6.8368	7.0992	6.7596	6.8349	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 4$	6.8455	7.1036	6.7646	6.8399	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 5$	6.8516	7.1065	6.7681	6.8433	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 6$	6.8563	7.1086	6.7708	6.8459	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 7$	6.8600	7.1102	6.7730	6.8480	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 8$	6.8631	7.1115	6.7748	6.8497	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 9$	6.8657	7.1125	6.7763	6.8511	$A_2 \succ A_1 \succ A_4 \succ A_3$
$\varsigma = 10$	6.8679	7.1133	6.7777	6.8523	$A_2 \succ A_1 \succ A_4 \succ A_3$

To visualize in effect in a better way, Figs. 3 and 4 are provided based on different values of $\varsigma \in (0, 10]$. In light of Fig. 3, the presented results reveal that no change in ranking order is found while using Lq-ROFHPWA operator. On the other hand, from Fig. 4, it is perceived that $\varsigma \in (0, 2.6050)$ the ranking is $A_2 \succ A_4 \succ A_1 \succ A_3$ and for $\varsigma \in [2.605, 10]$ the ranking is $A_2 \succ A_1 \succ A_4 \succ A_3$ based on Lq-ROFHPWG operator. But it is interesting to mention here that the optimal choice remains the same as A_2 for each case.

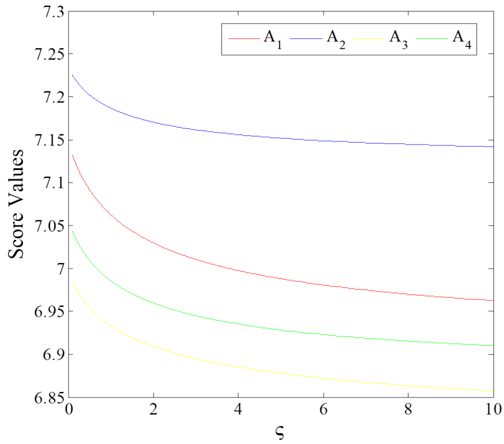

Figure 3: Score values of alternative for $\varsigma \in (0, 10]$ based on Lq-ROFHPWA operator ($q = 3$)

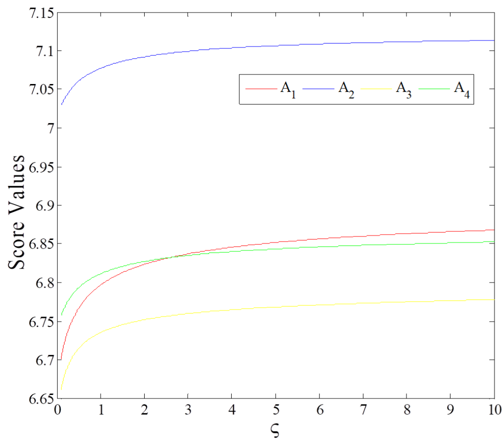

Figure 4: Score values of alternative for $\varsigma \in (0, 10]$ based on Lq-ROFHPWG operator ($q = 3$)

Moreover, an optimistic or pessimistic view of DMs can be reflected through the achieved outcomes. Because when the parameter ς becomes larger, the fused results based on Lq-ROFHPWA operator become smaller, while using Lq-ROFHPWG operator, the fused results become larger. Hence DMs can select appropriate Hamacher parameter values according to their needs while making decisions.

7. Comparative analysis

Arora and Garg [50] investigated MCGDM problems under LIF environment. They solved the problem presented in Section 6 using LIF prioritized WA operator, and a similar ranking result is found in the present paper. This shows the validity of the proposed method in dealing with MCGDM problems. However, the present method is more general and flexible than that of Arora and Garg [50]. Since the proposed MCGDM method is based on Lq-ROF environment, it can capture more fuzzy assessment information provided by the DMs. Also, Hamacher operations are considered in the present method that can easily replace the traditional algebraic operations by taking exact parameter values. So, the method proposed by Arora and Garg [50], which is basically developed on the basis of algebraic operations, becomes a particular case of the proposed method.

To prove the effectiveness of the developed operators more significantly, another comparative analysis by applying some existing operators, viz., LIFWA and LIFWG [31], LIFEWA and LIFEWG [51], LIFHWA and LIFHWG [52], LPFWA and LPFWG [32], LPFEWA and LPFEWG [53], LPFHWA and LPFHWG [54], and Lq-ROFWA and Lq-ROFWG [37] operators on the same numerical example considering the equal importance of the DMs and as well as for the criteria. The overall score values and the ranking of the alternatives by means of those existing operators are collected in Table 10.

Table 10: Score values and ranking results compared with existing methods

Operators	Score values	Ranking
LIFWA [31]	$S(A_1) = 5.8554,\ S(A_2) = 6.3633,$ $S(A_3) = 5.6240,\ S(A_4) = 5.7772$	$A_2 \succ A_1 \succ A_4 \succ A_3$
LIFWG [31]	$S(A_1) = 5.3657,\ S(A_2) = 6.1800,$ $S(A_3) = 5.2144,\ S(A_4) = 5.4596$	$A_2 \succ A_4 \succ A_1 \succ A_3$
LIFEWA [51]	$S(A_1) = 5.8012,\ S(A_2) = 6.3463,$ $S(A_3) = 5.5745,\ S(A_4) = 5.7396$	$A_2 \succ A_1 \succ A_4 \succ A_3$
LIFEWG [51]	$S(A_1) = 5.4446,\ S(A_2) = 6.2069,$ $S(A_3) = 5.2784,\ S(A_4) = 5.5066$	$A_2 \succ A_4 \succ A_1 \succ A_3$
LIFHWA ($\varsigma = 3$) [52]	$S(A_1) = 5.7773,\ S(A_2) = 6.3394,$ $S(A_3) = 5.5526,\ S(A_4) = 5.7237$	$A_2 \succ A_1 \succ A_4 \succ A_3$
LIFHWG ($\varsigma = 3$) [52]	$S(A_1) = 5.4922,\ S(A_2) = 6.2248,$ $S(A_3) = 5.3169,\ S(A_4) = 5.5362$	$A_2 \succ A_4 \succ A_1 \succ A_3$
LPFWA [32]	$S(A_1) = 6.8143,\ S(A_2) = 7.0517,$ $S(A_3) = 6.6335,\ S(A_4) = 6.7371$	$A_2 \succ A_1 \succ A_4 \succ A_3$
LPFWG [32]	$S(A_1) = 6.4446,\ S(A_2) = 6.9138,$ $S(A_3) = 6.3316,\ S(A_4) = 6.5013$	$A_2 \succ A_4 \succ A_1 \succ A_3$
LPFEWA [53]	$S(A_1) = 6.7718,\ S(A_2) = 7.0345,$ $S(A_3) = 6.5980,\ S(A_4) = 6.7062$	$A_2 \succ A_1 \succ A_4 \succ A_3$
LPFEWG [53]	$S(A_1) = 6.4929,\ S(A_2) = 6.9336,$ $S(A_3) = 6.3662,\ S(A_4) = 6.5289$	$A_2 \succ A_4 \succ A_1 \succ A_3$
LPFHWA ($\varsigma = 3$) [54]	$S(A_1) = 6.7498,\ S(A_2) = 7.0262,$ $S(A_3) = 6.5789,\ S(A_4) = 6.6909$	$A_2 \succ A_1 \succ A_4 \succ A_3$
LPFHWG ($\varsigma = 3$) [54]	$S(A_1) = 6.5203,\ S(A_2) = 6.9446,$ $S(A_3) = 6.3852,\ S(A_4) = 6.5448$	$A_2 \succ A_4 \succ A_1 \succ A_3$
Lq-ROFWA ($q = 3$) [37]	$S(A_1) = 7.0435,\ S(A_2) = 7.1812,$ $S(A_3) = 6.9049,\ S(A_4) = 6.9754$	$A_2 \succ A_1 \succ A_4 \succ A_3$
Lq-ROFWG ($q = 3$) [37]	$S(A_1) = 6.7687,\ S(A_2) = 7.0681,$ $S(A_3) = 6.6906,\ S(A_4) = 6.8007$	$A_2 \succ A_4 \succ A_1 \succ A_3$
Lq-ROFHPWA operator	$S(A_1) = 7.0107,\ S(A_2) = 7.1615,$ $S(A_3) = 6.8951,\ S(A_4) = 6.9450$	$A_2 \succ A_1 \succ A_4 \succ A_3$
Lq-ROFHPWG operator	$S(A_1) = 6.8368,\ S(A_2) = 7.0992,$ $S(A_3) = 6.7596,\ S(A_4) = 6.8349$	$A_2 \succ A_1 \succ A_4 \succ A_3$

Abbreviations: LIF WA (LIFWA), LIF WG (LIFWG), LIF Einstein WA (LIFEWA), LIF Einstein WG (LIFEWG), LIF Hamacher WA (LIFHWA), LIF Hamacher WG (LIFHWG), LPF WA (LPFWA), LPF WG (LPFWG), LPF Einstein WA (LPFEWA) and LPF Einstein WG (LPFEWG), LPF Hamacher WA (LPFHWA), LPF Hamacher WG (LPFHWG), Lq-ROF WA (Lq-ROFWA), Lq-ROF WA (Lq-ROFWG).

From the above analysis, it is seen that all the operators have the same optimal alternatives. Nevertheless, ranking results differ using averaging and geometric operators for the existing methods. However, in the case of the present method, the ranking is consistent for both the Lq-ROFHPWA and Lq-ROFHPWG operators. The possible reason for this is the fact that method proposed operators can consider the priority over criteria, but all the existing methods [31,32,37,51–54]] fail to incorporate this important characteristic. Hence the proposed method is more reasonable and effective in dealing with real-life MCGDM problems.

8. Conclusion

This paper investigates MCGDM under Lq-ROF environment. For this purpose, two novel Lq-ROFHPWA and Lq-ROFHPWG operators are proposed in this paper. The proposed Lq-ROF operators combine Hamacher operations with prioritized aggregation functions. For this, the proposed operators can consider the prioritized relationship between the input arguments as well as they have the ability to make the aggregation process flexible and general by incorporating Hamacher parameter. Further, the newly developed operators are utilized to develop an MCGDM approach with Lq-ROF context. Subsequently, a numerical example is provided to verify the practicality and effectiveness of the developed approach. Figures and tables have also been delivered to describe the influences of rung parameter q and Hamacher parameter ς on the decision results in detail. In addition, a comparative analysis is also presented to analyze the superiority of the proposed method. In the future research, it would be meaningful to apply the proposed method to other decision-making fields, viz., fuzzy cluster analysis, image pattern recognition, supplier selection, pattern recognition and so forth.

References

[1] L.A. Zadeh: Fuzzy sets. *Information and Control*, **8**(3), (1965), 338–353. DOI: 10.1016/S0019-9958(65)90241-X.

[2] K.T. Atanassov: Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*. **20** (1986), 87–96. DOI: 10.1016/S0165-0114(86)80034-3.

[3] K.T. Atanassov and G. Gargov: Interval-valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, **31**(3), (1989), 343–349. DOI: 10.1016/0165-0114(89)90205-4.

[4] R.R. YAGER: Pythagorean fuzzy subsets. In: *Proceeding of The Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada*, (2013), 57–61. DOI: 10.1109/IFSA-NAFIPS.2013.6608375.

[5] R.R. YAGER: Pythagorean membership grades in multicriteria decision making. *IEEE Transactions on Fuzzy Systems*, **22**(4), (2014), 958–965. DOI: 10.1109/TFUZZ.2013.2278989.

[6] X. PENG and Y. YANG: Fundamental properties of interval-valued Pythagorean fuzzy aggregation operators. *International Journal of Intelligent Systems*, **31**(5), (2016), 444–487. DOI: 10.1002/int.21790.

[7] K. RAHMAN and S. ABDULLAH: Some induced generalized geometric aggregation operators based on interval-valued Pythagorean fuzzy numbers. *Mathematical Sciences*, **14**(4), (2020), 397-407. DOI: 10.1007/s40096-020-00350-9.

[8] T. SENAPATI and R.R. YAGER: Fermatean fuzzy sets. *Journal of Ambient Intelligence and Humanized Computing*, 11 (2020), 663-674. DOI: 10.1007/s12652-019-01377-0.

[9] S. KUMAR and A. BISWAS: A unified TOPSIS approach to MADM problems in interval-valued intuitionistic fuzzy environment. In: N. Verma and A. Ghosh (Eds), Computational Intelligence: Theories, Applications and Future Directions – Volume II, *Advances in Intelligent Systems and Computing Springer, Singapore*, (2019), 799. DOI: 10.1007/978-981-13-1135-2_33.

[10] A. BISWAS and S. KUMAR: Generalization of extent analysis method for solving multicriteria decision making problems involving intuitionistic fuzzy numbers. *OPSEARCH*, **56**(2019), 1142–1166. DOI: 10.1007/s12597-019-00413-z.

[11] A. BISWAS and B. SARKAR: Interval-valued Pythagorean fuzzy TODIM approach through point operator-based similarity measures for multicriteria group decision making. *Kybernetes*, **48**(3), (2019), 496–519. DOI: 10.1108/K-12-2017-0490.

[12] A. SARKAR, and A. BISWAS: Multicriteria decision-making using Archimedean aggregation operators in Pythagorean hesitant fuzzy environment. *International Journal of Intelligent Systems*, **34**(7), (2019), 1361–1386. DOI: 10.1002/int.22099.

[13] A. BISWAS and N. DEB: Pythagorean fuzzy Schweizer and Sklar power aggregation operators for solving multi-attribute decision-making problems. *Granular Computing*, **6**(1), (2021), 991–1007. DOI: 10.1007/s41066-020-00243-1.

[14] A. Sarkar and A. Biswas: Interval-valued dual hesitant fuzzy prioritized aggregation operators based on Archimedean t-conorm and t-norm and their applications to multi-criteria decision making. *Archives of Control Sciences*, **31**(1), (2021), 213–247. DOI: 10.24425/acs.2021.136887.

[15] G. Wei and M. Lu: Dual hesitant Pythagorean fuzzy Hamacher aggregation operators in multiple attribute decision making. *Archives of Control Sciences*, **27**(3), (2017), 365–395. DOI: 10.1515/acsc-2017-0024.

[16] R.R. Yager: Generalized orthopair fuzzy sets. *IEEE Transactions on Fuzzy Systems*, 25(5), (2017), 1222-1230. DOI: 10.1109/TFUZZ.2016.2604005.

[17] P. Liu and P. Wang: Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making. *International Journal of Intelligent Systems*, **33**(2), (2018), 259–280. DOI: 10.1002/int.21927.

[18] P. Liu and P. Wang: Multiple-attribute decision-making based on Archimedean Bonferroni Operators of q-rung orthopair fuzzy numbers. *IEEE Transactions on Fuzzy Systems*, **27**(5), (2018), 834–848. DOI: 10.1109/TFUZZ.2018.2826452.

[19] G. Wei, H. Gao and Y. Wei: Some q-rung orthopair fuzzy Heronian mean operators in multiple attribute decision making. *International Journal of Intelligent Systems*, **33**(7), (2018), 1426–1458. DOI: 10.1002/int.21985.

[20] P. Wang, J. Wang, G. Wei and C. Wei: Similarity measures of q-rung orthopair fuzzy sets based on cosine function and their applications. *Mathematics*, 7(4), (2019), 740. DOI: 10.3390/math7040340.

[21] S. Zeng, Y. Hu and X. Xie: Q-rung orthopair fuzzy weighted induced logarithmic distance measures and their application in multiple attribute decision making. *Engineering Applications of Artificial Intelligence*, **100** (2021), 104167. DOI: 10.1016/j.engappai.2021.104167.

[22] M. Riaz, W. Sałabun, H.M. Athar Farid, N. Ali and J. Wątróbski: A robust q-rung orthopair fuzzy information aggregation using Einstein operations with application to sustainable energy planning decision management. *Energies*, **13**(9), (2020), 2155. DOI: 10.3390/en13092155.

[23] A. Pinar and F.E. Boran: A q-rung orthopair fuzzy multicriteria group decision making method for supplier selection based on a novel distance measure. *International Journal of Machine Learning and Cybernetics*, **11** (2020), 1749–1780. DOI: 10.1007/s13042-020-01070-1.

[24] H. Garg and S.M. Chen: Multiattribute group decision making based on neutrality aggregation operators of q-rung orthopair fuzzy sets. *Information Sciences*, **517** (2020), 427–447. DOI: 10.1016/j.ins.2019.11.035.

[25] M. Akram, G. Shahzadi and X. Peng: Extension of Einstein geometric operators to multi-attribute decision making under q-rung orthopair fuzzy information. *Granular Computing*, **6** (2021), 779–795. DOI: 10.1007/s41066-020-00233-3.

[26] M.J. Khan, P. Kumam and M. Shutaywi: Knowledge measure for the q-rung orthopair fuzzy sets. *International Journal of Intelligent Systems*, 36(8), (2021), 628–655. DOI: 10.1002/int.22313.

[27] N. Alkan and C. Kahraman: Evaluation of government strategies against COVID-19 pandemic using q-rung orthopair fuzzy TOPSIS method. *Applied Soft Computing*, **110** (2021), 107653. DOI: 10.1016/j.asoc.2021.107653.

[28] B. Zhao, R. Zhang and Y. Xin: Evaluation of medical service quality based on a novel multi-criteria decision-making method with unknown weighted information. *Archives of Control Sciences*, **31**(3), (2021), 365–395. DOI: 10.24425/acs.2021.138696.

[29] P. Liu and W. Liu: Multiple-attribute group decision-making based on power Bonferroni operators of linguistic q-rung orthopair fuzzy numbers. *International Journal of Intelligent Systems*, **34**(12), (2019), 652–689. DOI: 10.1002/int.22071.

[30] L.A. Zadeh: The concept of a linguistic variable and its application to approximate reasoning – I, *Information sciences*, 8(4), (1975), 199–249. DOI: 10.1016/0020-0255(75)90046-8.

[31] Z. Chen, P. Liu and Z. Pei: An approach to multiple attribute group decision making based on linguistic intuitionistic fuzzy numbers. *International Journal of Computational Intelligence Systems*, **8**(4), (2015), 747–760. DOI: 10.1080/18756891.2015.1061394.

[32] H. Garg: Linguistic Pythagorean fuzzy sets and its applications in multi-attribute decision-making process. *International Journal of Intelligent Systems*, **33**(6), (2018), 1234–1263. DOI: 10.1002/int.21979.

[33] D. Peng, J. Wang, D. Liu and Z. Liu: The similarity measures for linguistic q-rung orthopair fuzzy Mmulti-criteria group decision making using projection method. In *IEEE Access*, **7** (2019), 176732–176745. DOI: 10.1109/ACCESS.2019.2957916.

[34] R. Verma: Generalized similarity measures under linguistic q-rung orthopair fuzzy environment with application to multiple attribute decision-making. *Granular Computing*, **7** (2021), 253–275. DOI: 10.1007/s41066-021-00264-4.

[35] D. Liu and A. Huang: Consensus reaching process for fuzzy behavioral TOPSIS method with probabilistic linguistic q-rung orthopair fuzzy set based on correlation measure. *International Journal of Intelligent Systems*, **35**(11), (2020), 494–528. DOI: 10.1002/int.22215.

[36] D. Liu, Y. Liu and L. Wang: The reference ideal TOPSIS method for linguistic q-rung orthopair fuzzy decision making based on linguistic scale function. *Journal of Intelligent & Fuzzy Systems*, **39**(3), (2020), 4111–4131. DOI: 10.3233/JIFS-200244.

[37] M. Lin, X. Li and L. Chen: Linguistic q-rung orthopair fuzzy sets and their interactional partitioned Heronian mean aggregation operators. *International Journal of Intelligent Systems*, **35**(2), (2019), 217–249. DOI: 10.1002/int.22136.

[38] P. Liu and W. Liu: Multiple-attribute group decision-making method of linguistic q-rung orthopair fuzzy power Muirhead mean operators based on entropy weight. *International Journal of Intelligent Systems*, **34**(3), (2019), 1755–1794. DOI: 10.1002/int.22114.

[39] M. Akram, S. Naz, S.A. Edalatpanah and R. Mehreen: Group decision-making framework under linguistic q-rung orthopair fuzzy Einstein models. *Soft Computing*, **25** (2021), 10309–10334. DOI: 10.1007/s00500-021-05771-9.

[40] P. Liu, S. Naz, M. Akram and M. Muzammal: Group decision-making analysis based on linguistic q-rung orthopair fuzzy generalized point weighted aggregation operators. *International Journal of Machine Learning and Cybernetics*, **13** (2022), 883–906. DOI: 10.1007/s13042-021-01425-2.

[41] J. Wang, X. Shang, X. Feng and M. Sun: A novel multiple attribute decision making method based on q-rung dual hesitant uncertain linguistic sets and Muirhead mean. *Archives of Control Sciences*, **30**(2), (2020), 233–272. DOI: 10.24425/acs.2020.133499.

[42] R.R. Yager: Prioritized aggregation operators, *International Journal of Approximate Reasoning*, **48**(1), (2008), 263–274. DOI: 10.1016/j.ijar.2007.08.009.

[43] H. Hamachar: Uber logische verknunpfungenn unssharfer aussagen undderen zugen horige bewertungsfunktione, *Progress in Cybernetics and Systems Research*, III; Trappl, R., Klir, G.J., Ricciardi, L., Eds, **3** (1978), 276–288.

[44] A. Fahmi, S. Abdullah and F. Amin: Cubic uncertain linguistic powered Einstein aggregation operators and their application to multi-attribute

group decision making. *Mathematical Sciences*, **13** (2019), 129–152. DOI: 10.1007/s40096-019-0285-5.

[45] H. Gao: Pythagorean fuzzy Hamacher prioritized aggregation operators in multiple attribute decision making. *Journal of Intelligent & Fuzzy Systems*, **35**(2), (2018), 2229–2245. DOI: 10.3233/JIFS-172262.

[46] G. Wei, M. Lu, X. Tang and Y. Wei: Pythagorean hesitant fuzzy hamacher aggregation operators and their application to multiple attribute decision making. *International Journal of Intelligent Systems*, **33**(6), (2018), 1197–1233. DOI: 10.1002/int.21978.

[47] A. Jan, A. Khan, W. Khan and M. Afridi: A novel approach to MADM problems using Fermatean fuzzy Hamacher prioritized aggregation operators. *Soft Computing*, **25** (2021), 13897–13910. DOI: 10.1007/s00500-021-06308-w.

[48] F. Herrera, E. Herrera-Viedma and J.L. Verdegay: A model of consensus in group decision making under linguistic assessments. *Fuzzy Sets and Systems*, **78**(1), (1996), 73–87. DOI: 10.1016/0165-0114(95)00107-7.

[49] Z.S. Xu: A method based on linguistic aggregation operators for group decision making under linguistic preference relations. *Information Sciences*, **166**(1-4), (2004), 19–30. DOI: 10.1016/j.ins.2003.10.006.

[50] R. Arora and H. Garg: Group decision-making method based on prioritized linguistic intuitionistic fuzzy aggregation operators and its fundamental properties. *Computational and Applied Mathematics*, **38** (2019), 1–32. DOI: 10.1007/s40314-019-0764-1.

[51] H. Garg, and K. Kumar: Group decision making approach based on possibility degree measures and the linguistic intuitionistic fuzzy aggregation operators using Einstein norm operations. *Journal of Multiple-Valued Logic & Soft Computing*, **31** (2018), 175–209. DOI: 10.3934/jimo.2018162.

[52] J. Tang and F. Meng: Linguistic intuitionistic fuzzy Hamacher aggregation operators and their application to group decision making. *Granular Computing*, **4** (2019), 109–124. DOI: 10.1007/s41066-018-0089-2.

[53] Y. Rong, Z. Pei and Y. Liu: Linguistic Pythagorean Einstein operators and their application to decision making. *Information*, **11**(1), (2020), 46. DOI: 10.3390/info11010046.

[54] Y. Liu, Y. Qin, L. Xu, H.B. Liu and J. Liu: Multiattribute group decision-making approach with linguistic Pythagorean fuzzy information. *IEEE Access*, **7** (2019), 143412–143430. DOI: 10.1109/ACCESS.2019.2945005.