WARSAW UNIVERSITY OF TECHNOLOGY Index 351733
FACULTY OF CIVIL ENGINEERING
COMMITTEE FOR CIVIL AND WATER ENGINEERING
POLISH ACADEMY OF SCIENCES ISSN 1230-2945

DOI: 10.24425/ace.2022.141872

ARCHIVES OF CIVIL ENGINEERING

Vol. LXVIII ISSUE 3 2022

pp. 37 –47

Research paper

A nonlinear statistical empirical model for transversely isotropic rocks under uniaxial compression condition

Yansheng Deng[1], Chenjie Shen[2], Baoping Zou[3]

Abstract: The mechanical characteristics of transversely isotropic rocks are significantly different under various levels of inclination, and it is difficult to describe exactly the mechanical behaviour of transversely isotropic rocks. Assuming that rock consists of a great deal of microelements, and the microelement strength controlled by Mohr–Coulomb criterion follows the log normal distribution. The elastic modulus is used to reflect the anisotropy of rock, and the weak patches stiffness model is verified and employed to depict the variation of elastic modulus with different inclination angle. Based on basic damage mechanics theory and statistical method, a nonlinear statistical empirical model for transversely isotropic rocks is proposed under uniaxial compressive condition. In order to verify the correctness of the proposed model, comparison analyses between predicted results and experimental data taken from published literature are carried out, which have good consistency. Finally, the discussions on the influences of the distribution parameters a, c and elastic modulus with different inclination angle, E_θ, on proposed model is offered.

Keywords: transversely isotropic rocks, statics damage, uniaxial compression

[1]PhD., School of Civil Engineering and Architecture, Zhejiang University of Science and Technology, Hangzhou 310023, China, e-mail: ysdeng@zust.edu.cn, ORCID: 0000-0002-4329-1827
[2]Graduate student, School of Civil Engineering and Architecture, Zhejiang University of Science and Technology, Hangzhou 310023, China, e-mail: 212102814027@zust.edu.cn, ORCID: 0000-0002-5392-4974
[3]Prof. PhD., School of Civil Engineering and Architecture, Zhejiang University of Science and Technology, Hangzhou 310023, China, e-mail: zoubp@zust.edu.cn, ORCID: 0000-0003-3729-2425

1. Introduction

Rock is formed by long-term geological effects with lots of defects such as fine cracks and voids, which is different from metal material or polymers. With the increase of external load, more and more faults inside rock appear. The mechanical properties of rock deteriorate gradually; finally, the failure or yield occur. This process from damage to fracture is very similar to damage mechanics. Therefore, according to continuous damage theory and statistical method, Krajcinovic and Silva [1] derived a statistical damage constitutive model in the early research. Numerous researches about statistics damage constitutive model have been published since Lemaitre [2] proposed the equivalent strain assumption. Tang [3] and Cao et al. [4] supposed the strength of microelement follows statistical distribution, like Weibull distribution. Drucker–Prager (D–P) failure criterion was employed as the failure criterion of microelement. Wang [5] assumed the microelement strength follows the Weibull distribution, studied the effects of Mohr–Coulomb (M–C) and D–P on statistical model of rock, and introduced a coefficient to represent the residual strength of rock. To determine the microelement strength, Yu [6] introduced the four-parameter criterion for quasi-brittle materials, like rock, concrete, etc, which can degenerate to the common criteria, such as Hsieh-Ting-Chen (H-T-C) criterion, von Mises, D–P, M–C criteria, when the four parameters values are equal. Due to the existence of voids or pores in natural rocks, Cao et al. [7] employed the statistical damage model to investigate the influences of voids and volume changes on the stain softening and hardening.

In recent years, a number of researchers have mainly focused on the three aspects study of statistical damage constitutive model: the probabilistic distribution of microelement strength in rocks [8–10], the failure criterion of rock [11–13] and the application of statistical damage constitutive model to different rock [14–16]. In addition, some developed statistical damage constitutive models have been proposed, like Zhao et al. [17] introduced the damage tolerance principal to improve constitutive law for quasi-brittle rocks; Peng et al. [18] proposed the strength degradation index (SDI) and a negative exponential function, by which the relationship between SDI and confining pressure can be described.

Most of statistical damage constitutive models mentioned above are suitable for nearly isotropic rocks. For transversely isotropic rocks such as phyllites, shales, schists and slates, as well as gneisses, the elastic and strength behaviour are substantially different from that of isotropic rock due to the presence of banding [19–23]. In this paper, the probabilistic distributions of microelement strength in rocks are analysed at first, and the variation of elastic modules with the orientation of the sample with respect to the principal stresses is studied. Then, the statistical empirical model for transversely isotropic rock is proposed considering the M–C criterion as the microelement strength criterion in transversely isotropic rock under uniaxial compressive condition. Finally, comparisons between calculated results and experimental studies are displayed to illustrate the feasibility and validity of the proposed model.

2. Statistical empirical model

2.1. Statistical damage variable

It is assumed that rock consists of lots of microelements. With the increase of external loads, the microelements fail gradually. The statistical damage variable, D, can be given as

$$D = \frac{N_d}{N} \tag{2.1}$$

where: N is the total number of microelements; N_d is the number of damaged microelements.

The microelement fails when its stress S reaches the critical strength F, and assuming that the microelement critical strength F follows a certain probability distribution. When the stress S increases to S+dS, the number of damaged microelements enhances by

$$\mathrm{d}N_d = Np(S)\,\mathrm{d}S \tag{2.2}$$

where: p denotes the density function of the probability distribution for the microelement strength F.

If the stress level increases from 0 to S, the number of damaged microelements is

$$N_d = \int_0^s Np(x)\,\mathrm{d}x = NP(S) \tag{2.3}$$

Subscribed Equation (2.3) into Equation (2.1), the statistical damage variable D is

$$D = P(S) \tag{2.4}$$

According to the above literatures, the Weibull distribution is most popular probability distribution for the microelement strength. However, the Weibull distribution still has some drawbacks when it is adopted to describe statistical strength distribution for rock [24–26]. Basu et al. [25] proposed that the gamma or log-normal distribution function may reflect more appropriately statistical strength distribution for brittle materials. Ji et al. [27] used the non-normal distribution function and statistics damage theory to investigate deformation and strength anisotropy of layer rocks. So, based on [27], the log normal distribution is employed as an alternative in this paper, and its probability density function is

$$P(S) = F\left[\frac{\ln(S/S_0)}{\eta}\right] = F(a\ln S + b) \tag{2.5}$$

where: S_0 and η are distribution parameters, $a = 1/\eta$, $b = -(\ln S_0)/\eta$, F represents standard normal distribution function

$$\begin{cases} F(x) = \displaystyle\int_{-\infty}^{x} f(s)\,\mathrm{d}s \\ f(x) = \dfrac{1}{\sqrt{2\pi}} \exp\left(\dfrac{x^2}{2}\right) \end{cases} \tag{2.6}$$

So, the statistical damage variable D for transversely isotropic rocks is

$$D = F(a \ln S + b) \tag{2.7}$$

Fig. 1 represents the variation of D with S/S_0 under different η in log normal distribution. With the increase of S/S_0, D rises gradually from 0 to 1. η reflects the uniformity of the material to a certain extent. The smaller η is, the faster damage develops, which means the more uniform the material is.

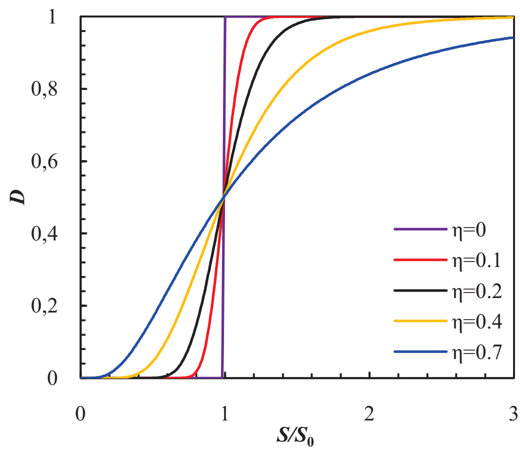

Fig. 1. Variation of damage variable D with S/S_0 for different η

2.2. The equivalent elastic modulus E_θ

Considering the inclination angle θ (Fig. 2) plays a significant role in mechanical behaviour of the transversely isotropic rock. The equivalent elastic modulus E_θ is introduced to express the effect of different θ on mechanical behaviour, which denotes the elastic modulus at different θ. Therefore, the key is to determine the relationship between E_θ and θ. $F_{\phi\rho}$ and $N_{\varepsilon\sigma}$ [29] proposed the equivalent elastic modulus model below to consider the

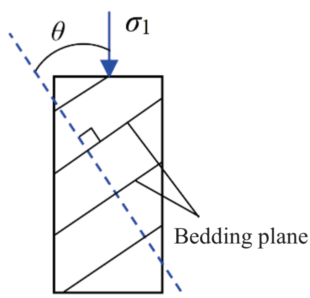

Fig. 2. Definition of the inclination angle θ

bedding plane-induced anisotropy based on the anisotropic stiffness prediction in the plane of a patchy weakness model.

$$E_\theta = E_{90^\circ}\left(1 - \alpha \sin^2 2\theta - \beta \cos^4 \theta\right) \tag{2.8}$$

where: E_{90° means the Young's modulus with $\theta = 90^\circ$, α means the number of weak patches in the weak plane, β is represents the degree of excessive normal compliance related to the weak patches.

Both α and β are dimensionless parameters that can be determined experimentally.

To verify the validity of expression for transversely isotropic rock, comparisons are carried out between experimental results taken from the published literatures and Equation (2.8) with different rocks. Fig. 3 plots these comparisons of experimental and predicted results of different rocks such as artificial layered rocks, AS gneiss, BR shale, YC schist, Mancos shale and Longmaxi shale. In the figures, the solid lines represent the calculated results obtained from the Equation (2.8), while data points denote the experimental results. From Fig. 3, the correlation between experimental and predicted results of E_θ and θ corresponded well for different rocks.

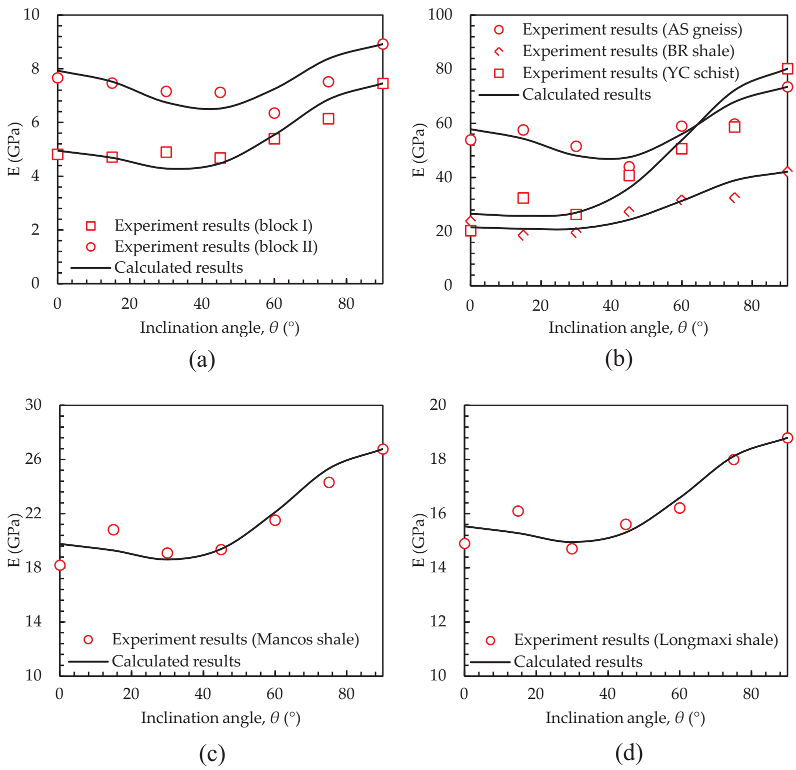

Fig. 3. Comparison of experimental data and calculated results with different rocks: (a) Artificial transversely isotropic rock [30]; (b) AS gneiss, BR shale and YC schist [19]; (c) Mancos shale [29]; (d) Longmaxi shale [15]

2.3. Statistics empirical model establishment

Based on the strain equivalent hypothesis [2], the effective principal stress σ_i^* ($i = 1, 2, 3$) is

$$\sigma_i^* = \frac{\sigma_i}{1 - D}, \quad i = 1, 2, 3 \tag{2.9}$$

where: σ_i ($i = 1, 2, 3$) represents the apparent stress, D is the damage variable, which varies from 0 (intact states) to 1 (entire damaged states).

According to Hooke's law, the axial strain ε_1 under uniaxial compressive condition for transversely isotropic rock can be expressed by

$$\varepsilon_1 = \frac{\sigma_1^*}{E_\theta} \tag{2.10}$$

To M–C criterion under uniaxial compressive condition, there is

$$S = \sigma_1^* - \sigma_1^* \sin\varphi = c\cos\varphi \tag{2.11}$$

where: c and φ represent cohesive force and frictional angle, respectively.

Substituting Equations (2.4), (2.10)–(2.11) into Equation (2.9), the following equation based on M–C criterion can be given:

$$\sigma_1 = E_\theta \varepsilon_1 \left(1 - P[E_\theta \varepsilon_1 (1 - \sin\varphi)]\right) \tag{2.12}$$

Incorporating Equation (2.7) and Equation (2.8) into Equation (2.12), the statistical empirical model for transversely isotropic rock under uniaxial compressive condition as follows:

$$\sigma_1 = E_{90^\circ} \varepsilon_1 F(-a \ln \varepsilon_1 - c) \left(1 - \alpha \sin^2 2\theta - \beta \cos^4 \theta\right) \tag{2.13}$$

where: $c = E_\theta \varepsilon_1 (1 - \sin\varphi)$.

2.4. Determination of parameters

Based on Equation (2.8) and experimental data, the parameters α and β can be determined by the least square method for E_θ. The extreme value method is employed considering the peak value characteristics of stress-strain curves to solve the parameter a and c in Equation (2.13).

Differentiating Equation (2.12) gets

$$\frac{\partial \sigma_1}{\partial \varepsilon_1} = E_\theta F(-a \ln \varepsilon_1 - c) - a E_\theta f(-a \ln \varepsilon_1 - c) = 0 \tag{2.14}$$

Rearranging of Equation (2.14) obtains

$$\sigma_1 = a E_{90^\circ} \varepsilon_1 f(-a \ln \varepsilon_1 - c)(1 - \alpha \sin^2 2\theta - \beta \cos^4 \theta) \tag{2.15}$$

Hence, a and c can be obtained when the values of σ_1 and ε_1 at the peak. Finally, the relationship between a, c and θ can be obtained by 3-order polynomial fitting method.

3. Validation and discussion

3.1. Model verification

To verify the accuracy of the proposed model for transversely isotropic rock, experimental data for layer shale made by Jia et al. [30] are adopted. According to Section 2.4 and experimental data, the parameters α, β, a and c can be calculated in Table 1.

Table 1. Parameters for verification example

Parameters	Inclination angle, θ [°]			
	0	30	60	90
α	–0.125			
β	–0.722			
a	16.7	37.5	83.6	17.4
c	85.6	226	415	80

Fig. 4 presents the comparisons of experiment and calculated results for E_θ. Fig. 5 plots the 3-order polynomial fitting of a, c and θ. It can be seen that the calculated results are in good agreement with the experimental results.

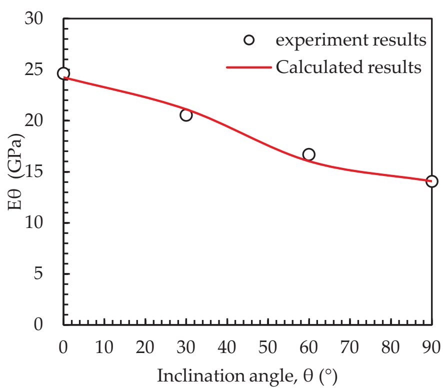

Fig. 4. Comparisons of experiment and calculated results for E_θ

Put the parameters in Table 1 and fitting equation in Fig. 5 into Equation (2.12), the stress-strain curves under uniaxial compressive condition with different inclination angle, θ, can be obtained. Comparison of experiment and calculated results with different θ are shown in Fig. 6. From Fig. 6, the proposed model of transversely isotropic rock can preferably depict the experimental results. With the increase of axial strain, the stress increases linearly to peak strength and then decreases. In particular, the peak strength is the smallest when $\theta = 30°$, while the peak strength reaches the maximum when $\theta = 0°$. It is noticed that when $\theta = 90°$, the peak strength is also high, which is similar to the maximum.

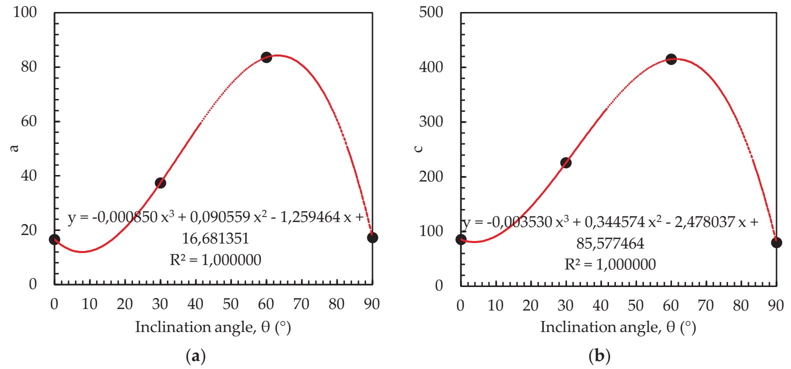

Fig. 5. 3-order polynomial fitting of a, c and θ: (a) Fitting of a and θ; (b) Fitting of c and θ

That's probably because when $\theta = 20{-}50°$, the failure is controlled by the sliding mode, while the non-sliding mode in which the material strength dominated occurs with the others inclination angle.

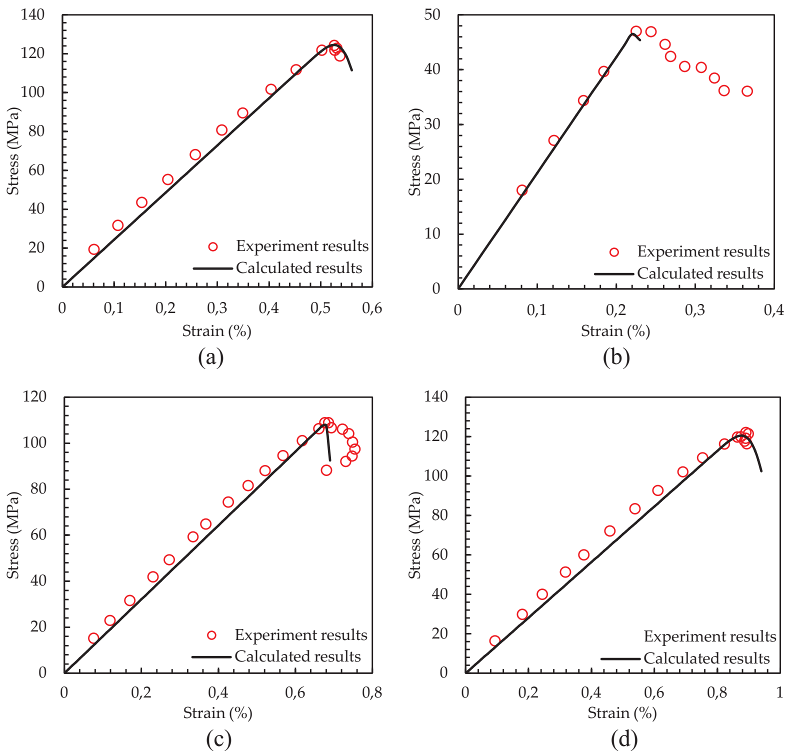

Fig. 6. Comparisons of calculated results with experiment data under different θ:
(a) $\theta = 0°$; (b) $\theta = 30°$; (c) $\theta = 60°$; (d) $\theta = 90°$

3.2. Influence of the parameters on the model

Taking the uniaxial experiment data with $\theta = 90°$ as an example, the related parameters $a = 17.4$, $c = 80$, and $E_{90°} = 14.1$ GPa. The other parameters are constant when one of them is changed to research its effect on the model. Fig. 7 plots the stress-strain curves with different a, c and E_θ.

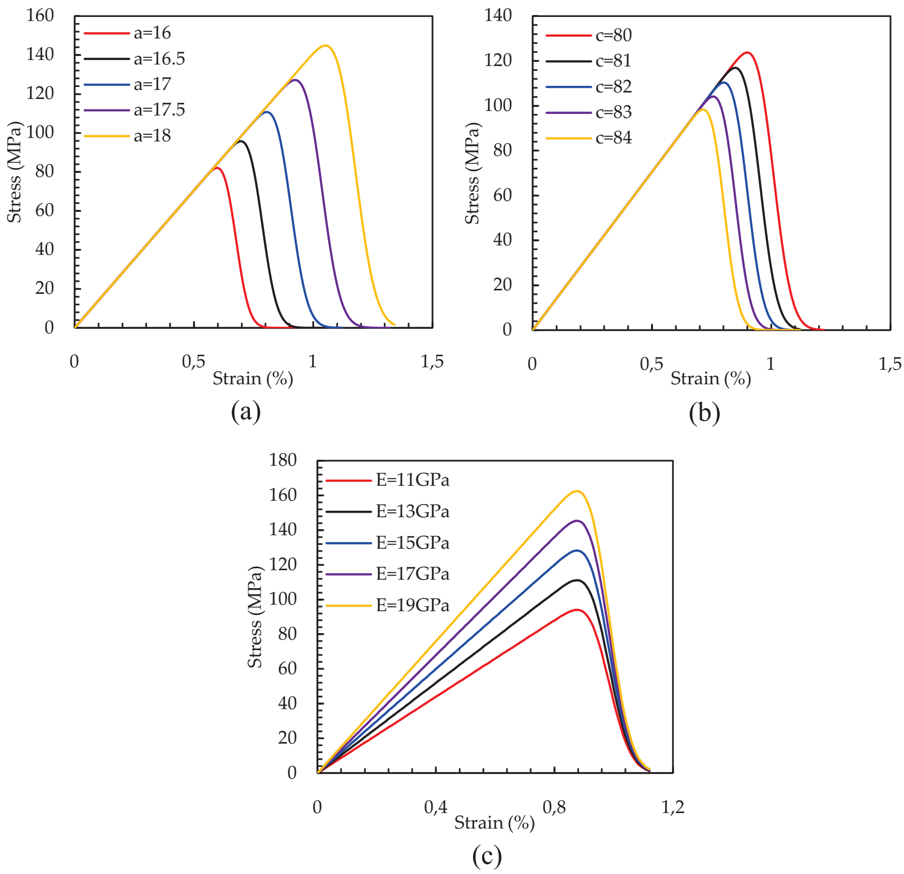

Fig. 7. Influence of a, c and E_θ on model: (a) a; (b) c; (c) E_θ

From Fig. 7, parameters a, c influence the uniaxial compression strength (UCS) of rock, but they have no effect on the overall trend of the stress-strain curves. With the increase of parameter, a, the UCS increases, while with the increase of parameter c, the strength decreases. It is noted that elastic modulus, E_θ, affects not only the compression strength of rock but also curves form. Hence, it is significant to predict accurately the elastic modulus for different inclination angle, θ.

4. Conclusions

A nonlinear statistical empirical model for transversely isotropic rock under uniaxial compressive condition was proposed in the present study. A log normal distribution is employed to depict the distribution of microelement strength, which obey M–C criterion. The proposed model is checked by compare the experimental data and predicted results. The main conclusions are as follows:

1. The current nonlinear statistical empirical model can be effective to express well stress-strain relationship of transversely isotropic rock with different inclination angle under uniaxial compressive condition. All of parameters can be determined conveniently by using uniaxial compression testing.
2. By comparing artificial layered rocks, AS gneiss, BR shale, YC schist, Mancos shale and Longmaxi shale, the weak patches stiffness model proposed by Fjær and Nes (2014) can predict precisely the variation of elastic modulus for transversely isotropic rock with different inclination angle and can be used in the model.
3. Elastic modulus influences both the UCS and stress-strain curve formation, while a and c just relate to the compression strength.

References

[1] D. Krajcinovic, M.A. Silva, "Statistical aspects of the continuous damage theory", *International Journal of Solids and Structures*, 1982, no. 18, no. 7, pp. 551–562, DOI: 10.1016/0020-7683(82)90039-7.

[2] J. Lemaitre, "How to use damage mechanics", *Nuclear Engineering and Design*, 1984, vol. 80, no. 2, pp. 233–245, DOI: 10.1016/0029-5493(84)90169-9.

[3] C.A. Tang, *Disaster during Rock Fracture*. Coal Industry Press, Beijing, 1993.

[4] W.G. Cao, Z.L. Fang, X.J. Tang, "A study of statistical constitutive model for softening and damage rocks", *Chinese Journal of Rock Mechanics and Engineering*, 1998, no. 17, pp. 628–633.

[5] Z.L. Wang, Y.C. Li, J.G. Wang, "A damage-softening statistical constitutive model considering rock residual strength", *Computers and Geosciences*, 2007, vol. 33, no. 1, pp. 1–9, DOI: 10.1016/j.cageo.2006.02.011.

[6] T.T. Yu, "Statistical damage constitutive model of quasi-brittle materials", *Journal of Aerospace Engineering*, 2009, vol. 22, no. 1, pp. 95–100, DOI: 10.1061/(ASCE)0893-1321(2009)22:1(95).

[7] W.G. Cao, H. Zhao, X. Li, Y.J. Zhang, "Statistical damage model with strain softening and hardening for rocks under the influence of voids and volume changes", *Canadian Geotechnical Journal*, 2010, vol.47, no. 8, pp. 857–871.

[8] J. Deng, D.S. Gu, "On a statistical damage constitutive model for rock materials", *Computers & Geosciences*, 2011, vol. 37, no. 2, pp. 122–128, DOI: 10.1016/j.cageo.2010.05.018.

[9] J.B. Wang, Z.P. Song, B.Y. Zhao, et al. "A study on the mechanical behavior and statistical damage constitutive model of sandstone", *Arabian Journal for Science and Engineering*, 2018, vol. 43, pp. 5179–5192, DOI: 10.1007/s13369-017-3016-y.

[10] S. Chen, C.S. Qiao, "Composite damage constitutive model of jointed rock mass considering crack propagation length and joint friction effect", *Arabian Journal of Geosciences*, 2018, vol. 11, no. 11, DOI: 10.1007/s12517-018-3643-y.

[11] C.B. Li, L.Z. Xie, L. Ren, J. Wang, "Progressive failure constitutive model for softening behavior of rocks based on maximum entropy theory", *Environmental Earth Sciences*, 2015, vol. 73, pp. 5905–5915, DOI: 10.1007/s12665-015-4228-7.

[12] H.Z. Li, H.J. Liao, G.D. Xiong, et al., "A three-dimensional statistical damage constitutive model for geomaterials", *Journal of Mechanical Science and Technology*, 2015, vol. 29, pp. 71–77.

[13] X.S. Liu, J.G. Ning, Y.L. Tan, Q.H. Gu, "Damage constitutive model based on energy dissipation for intact rock subjected to cyclic loading", *International Journal of Rock Mechanics and Mining Sciences*, 2016, vol. 85, pp. 27–32, DOI: 10.1016/j.ijrmms.2016.03.003.

[14] H.Y. Liu, L.M. Zhang, "A damage constitutive model for rock mass with nonpersistently closed joints under uniaxial compression", *Arabian Journal for Science and Engineering*, 2015, vol. 40, pp. 3107–3117, DOI: 10.1007/s13369-015-1777-8.

[15] C. Gao, L.Z. Xie, H.P. Xie, et al., "Estimation of the equivalent elastic modulus in shale formation: Theoretical model and experiment", *Journal of Petroleum Science and Engineering*, 2017, vol. 151, pp. 468–479, DOI: 10.1016/j.petrol.2016.12.002.

[16] S. Chen, C.S. Qiao, C.S. Ye, M.U. Khan, "Comparative study on three-dimensional statistical damage constitutive modified model of rock based on power function and Weibull distribution", *Environmental Earth Sciences*, 2018, vol. 77, art. ID 108, DOI: 10.1007/s12665-018-7297-6.

[17] H. Zhao, C. Zhang, W.G. Cao, M.H. Zhao, "Statistical meso-damage model for quasi-brittle rocks to account for damage tolerance principle", *Environmental Earth Sciences*, 2016, vol. 75, art. ID 862, DOI: 10.1007/s12665-016-5681-7.

[18] J. Peng, M. Cai, G. Rong, et al., "Determination of confinement and plastic strain dependent post-peak strength of intact rocks", *Engineering Geology*, 2017, vol. 2018, pp. 187–196, DOI: 10.1016/j.enggeo.2017.01.018.

[19] J.W. Cho, H. Kim, S. Jeon, K.B. Min, "Deformation and strength anisotropy of Asan gneiss, Boryeong shale, and Yeoncheon schist", *International Journal of Rock Mechanics and Mining Sciences*, 2012, vol. 50, pp. 158–169, DOI: 10.1016/j.ijrmms.2011.12.004.

[20] L.X. Xiong, H.Y. Yuan, Y. Zhang, et al., "Experimental and numerical study of the uniaxial compressive stress-strain relationship of a rock mass with two parallel joints", *Archives of Civil Engineering*, 2019, vol. 65, no. 2, pp. 67–80, DOI: 10.2478/ace-2019-0019.

[21] N. Yao, Y.C. Ye, B. Hu, et al., "Particle flow code modeling of the mechanical behavior of layered rock under uniaxial compression", *Archives of Mining Sciences*, 2019, vol. 64, no. 1, pp. 181–196, DOI: 10.24425/ams.2019.126279.

[22] X.J. Zhang, H.M. An, "Analysis of positive elevation effect and prediction of vibration velocity of bench blasting vibration", *Archives of Civil Engineering*, 2021, vol. 67, no. 1, pp. 599–618, DOI: 10.24425/ace.2021.136492.

[23] A. Nowakowski, J. Nurkowski, "About some problem related to determination of the E.G. boit coefficient for rocks", *Archives of Mining Sciences*, 2021, vol. 66, no. 1, pp. 133–150, DOI: 10.24425/ams.2021.136697.

[24] R. Danzer, P. Supancic, J. Pascual, T. Lube, "Fracture statistics of ceramics – Weibull statistics and deviations from Weibull statistics", *Engineering Fracture Mechanics*, 2007, vol. 74, no. 18, pp. 2919–2932, DOI: 10.1016/j.engfracmech.2006.05.028.

[25] B. Basu, D. Tiwari, D. Kundu, R. Prasad, "Is Weibul distribution the most appropriate statistical strength distribution for brittle materials", *Ceramics International*, 2009, vol. 35, no. 1, pp. 237–246, DOI: 10.1016/j.ceramint.2007.10.003.

[26] M.T. Todinov, "Is Weibull distribution the correct model for predicting probability of failure initiated by non-interacting flaws", *International Journal of Solids and Structures*, 2009, vol. 46, no. 3-4, pp. 887–901, DOI: 10.1016/j.ijsolstr.2008.09.033.

[27] M. Ji, K. Chen, H.J. Guo, "Constitutive model of rock uniaxial damage based on rock strength statistics", *Advances in Civil Engineering*, 2018, vol. 2018, art. ID 5047834, pp. 1–8, DOI: 10.1155/2018/5047834.

[28] E. Fjær, O.M. Nes, "The impact of heterogeneity on the anisotropic strength of an outcrop shale", *Rock Mechanics and Rock Engineering*, 2014, vol. 47, pp. 1603–1611, DOI: 10.1007/s00603-014-0598-5.

[29] Y.M. Tien, P.F. Tsao, "Preparation and mechanical properties of artificial transversely isotropic rock", *International Journal of Rock Mechanics and Mining Sciences*, 2000, vol. 37, no. 6, pp. 1001–1012, DOI: 10.1016/S1365-1609(00)00024-1.

[30] C.G. Jia, J.H. Chen, Y.T. Guo, et al., "Research on mechanical behaviors and failure modes of layer shale", *Rock and Soil Mechanics*, 2013, vol. 34, pp. 57–61.

Received: 23.07.2021, Revised: 15.03.2022