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Generalized observer design of index one for descriptor systems with unknown inputs

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Generalized observers are proposed to relax the existing conditions required to design Luenberger observers for rectangular linear descriptor systems with unknown inputs. The current work is focused on designing index one generalized observers, which can be naturally extended to higher indexes. Sufficient conditions in terms of system operators for the existence of generalized observers are given and proved. Orthogonal transformations are used to derive the results. A physical model is presented to show the usefulness of the proposed theory.

Key words: descriptor systems (DAEs), unknown inputs, state estimation, generalized observer, index

1. Introduction

Descriptor systems – also known as differential algebraic equations (DAEs) or singular systems – are combinations of differential and algebraic equations implicitly. Many real life plants are naturally modelled as descriptor systems, such as constrained mechanical systems [1, 2], chemical control processes [3, 4], electrical circuits [5, 6], and secure communications [7], for instance.

This work discusses the linear time invariant descriptor system with unknown inputs as follows

$$\bar{E}\dot{x} = \bar{A}x + \bar{B}u + \bar{F}v, \tag{1a}$$

$$\bar{y} = \bar{C}x + \bar{G}v, \tag{1b}$$

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where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^k$, and $\bar{y} \in \mathbb{R}^p$ are the state vector, the control input vector, and the output (measurement) vector, respectively. $v \in \mathbb{R}^q$ is the vector of unknown inputs. $\bar{E} \in \mathbb{R}^{m\times n}$, $\bar{A} \in \mathbb{R}^{m\times n}$, $\bar{B} \in \mathbb{R}^{m\times k}$, $\bar{F} \in \mathbb{R}^{m\times q}$, $\bar{C} \in \mathbb{R}^{p\times n}$, and $\bar{G} \in \mathbb{R}^{p\times q}$ are constant matrices. System (1) is called regular descriptor system if matrices E and A are square and $\exists\ \lambda \in \mathbb{C}$ such that $\det(\lambda\bar{E} - \bar{A}) \neq 0$. Regularity is the condition for the existence and uniqueness of the solution for descriptor systems. A descriptor system that is not regular is called irregular. In the case, where $\bar{E}$ and $\bar{A}$ are square and $\bar{E}$ is nonsingular, the system is well known as state space representation. In this article, we study rectangular descriptor systems where the number of equations is not necessarily equal to the number of states. It is evident that rectangular descriptor systems, often referred as over- or under-determined systems [8], are the most general form of irregular descriptor systems. Therefore, the system (1) is sufficiently general and covers enormous types of linear control systems.

In the literature, unknown inputs are also referred as noises or disturbances [9]. In many practical situations, descriptor systems are modeled with presence of unknown inputs. Contrary to state space systems, solutions of descriptor systems contain higher ordered input derivatives, see [2, 10]. As a result, descriptor systems are exceptionally delicate to slight input changes. Thus, considering unknown inputs in observers design problems for descriptor systems are more important compared to observer design for state space systems. In this work, we are considering unknown inputs in the most general form by considering their presence in the both dynamic and output equations. Considering the presence of unknown inputs in the measurement equation (1b) is essential because contrary to known inputs, it is not possible to take out unknown inputs from the output equation without loss of generality.

The problem of state estimation for dynamical systems is well established in the control theory since it is utilized for various purposes *e.g.* feedback control and synchronization [11]. Various kinds of observers have been designed for system (1) for state estimation. For a comparison, we enlist a few types of the observers given in the previous works.

(O1) $\dot{z} = L_1 z + L_2 y + L_3 u,$
$\hat{x} = F_1 z + F_2 y + F_3 u.$

(O2) $\dot{z} = L_1 z + L_2 y + L_3 u,$

$$\hat{x} = F_1 z + \sum_{i=0}^{s} (F_{2,i} u^{(i)} + F_{3,i} y^{(i)}),$$

where (i) denotes the i-th derivative. Integer $s > 0$ is called index of the observer (O2).

(O3) $\dot{\hat{x}} = L_1\hat{x} + L_2 y + L_3 u + L_4 \dot{y}.$

(O4) $E\dot{\hat{x}} = L_1\hat{x} + L_2 y + L_3 u + L_4 \sum_{i=1}^{q} w_i,$

$$w_i^{(i)} = F_i(y - C\hat{x}) + M_i w, \quad i = 1, 2 \ldots q.$$

In the literature, observers of the form (O1) are called Luenberger observers [9, 11, 12]. Observers of the form (O2) are called generalized observers [13]. Observers of the form (O3) and (O4) are called proportional–derivative (PD) observers [14, 15], and proportional–integral (PI) observers [16, 17], respectively. The literature reflects that Luenberger observers (O1) are the most adopted among all other observers. It is because of the fact that Luenberger observers are explicit in nature and consequently simple to carry out for implementation. However, for descriptor systems with unknown inputs, designing of generalized observers is still an untouched area of research.

Darouach *et al.* [18] have utilized the ideas of generalized Sylvester equation and generalized inverse to design a Luenberger observer for system (1) under assumptions (H1) and (H2) as follows. Proportional Multiple-Integral observer of the form (O4) has also been designed using the assumptions (H1) and (H2) [17].

(H1) $\operatorname{rank}\begin{bmatrix} \bar{E} & \bar{A} & \bar{F} & 0 \\ 0 & \bar{E} & 0 & \bar{F} \\ 0 & \bar{C} & \bar{G} & 0 \\ 0 & 0 & 0 & \bar{G} \end{bmatrix} = \operatorname{rank}\begin{bmatrix} \bar{E} & \bar{F} \\ 0 & \bar{G} \end{bmatrix} + n + q,$

(H2) $\operatorname{rank}\begin{bmatrix} \bar{A} - \lambda\bar{E} & \bar{F} \\ \bar{C} & \bar{G} \end{bmatrix} = n + q \;\forall\, \lambda \in \bar{\mathbb{C}}^{+},$

where $\mathbb{C}$ stands for the set of complex numbers. $\bar{\mathbb{C}}^{+} = \{s | s \in \mathbb{C},\ Re(s) \geqslant 0\}$ is the closed right half complex plane.

Luenberger PD or PI observers for system (1) have been designed under the least restrictive assumptions (H1) and (H2). As a matter of fact, (H1) and (H2) are the generalizations of the impulse observability and detectability properties of linear descriptor systems without unknown inputs to (1), respectively [19]. We can obtain these conditions by substituting $\bar{F}$ and $\bar{G}$ as zero matrices in (H1) and (H2). Hou and Müller [13] have proved that the detectability is a necessary condition and therefore, can not be relaxed further for Luenberger or generalized observer design. It has been observed that many practical models do not fulfill (H1). To loosen up the condition (H1), observers of the form (O2) are introduced, which can be designed under milder conditions. In generalized observers, the minimum possible index is preferred to minimize the noise which may be occurred due appearance of derivatives terms.

In this paper, we replace the condition (H1) with the following proposed condition (H0) for designing index one generalized observer which can be understood as the main contribution of the paper.

$$\textbf{(H0)}\ \operatorname{rank}\begin{bmatrix}\bar{E} & \bar{A} & \bar{F} & 0 & 0\\ 0 & \bar{E} & 0 & \bar{F} & \bar{A}\\ 0 & \bar{C} & \bar{G} & 0 & 0\\ 0 & 0 & 0 & \bar{G} & \bar{C}\\ 0 & 0 & 0 & 0 & \bar{E}\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\bar{E} & \bar{F} & \bar{A}\\ 0 & \bar{G} & \bar{C}\\ 0 & 0 & \bar{E}\end{bmatrix} + n + q.$$

This work is devoted to design index one generalized observers of the form (O2) under the conditions (H0) and (H2). The proof of (H1)⇒(H0) is given in Section 2, which shows that (H0) is milder than (H1). On the other hand, an example that satisfies only the condition (H0) but not (H1), is given in Section 4. It is also notable that the condition (H0) is being proposed for the first time and has never been used in the literature for any other control theory purposes. In order to prove our results, we use orthogonal transformations to ensure the numerical stability of the proposed methods.

The notational convention is as follows. A matrix is called to be full column (row) rank provided that its rank is equal to the number of its columns (rows). 0 and I are the zero and identity matrices of compatible dimensions, respectively. Sometimes, to be specific, we use I_n to represent the identity matrix of dimension n. $\mathbb{R}^{m\times n}$ represents the $m \times n$ real matrix set. A^T stands for the transpose of a matrix A.

The rest of the paper is structured as follows: In Section 2, it is proved that the condition (H1) implies (H0). Section 3 describes the designing of the index one generalized observer for system (1). In Section 4, one real-life model is given to show the requirement and efficiency of the proposed technique. At the end, Section 5 concludes the paper.

2. System Decompositions and Preliminaries

In this section, we exhibit system decompositions and establish some fundamental theorems, which are used for observer design in the next section. Firstly, we state the following proposition which is used to develop subsequent results.

Proposition 1 [20] *Let* $X \in \mathbb{R}^{m_1\times r_1}$, $S \in \mathbb{R}^{m_1\times r_2}$, *and* $Y \in \mathbb{R}^{m_2\times r_2}$. *Then*

$$\operatorname{rank}\begin{bmatrix}X & S\\ 0 & Y\end{bmatrix} = \operatorname{rank} X + \operatorname{rank} Y,$$

if at least one of the following conditions holds.

(i) X is full row rank,

(ii) Y is full column rank.

Let rank $\bar{E} = n_0$ then there exists an orthogonal matrix P such that $P\bar{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}$, where E is a full row rank matrix. Other system matrices are also decomposed accordingly as $P\bar{A} = \begin{bmatrix} A \\ A_1 \end{bmatrix}$, $P\bar{B} = \begin{bmatrix} B \\ B_1 \end{bmatrix}$, $P\bar{F} = \begin{bmatrix} F \\ F_1 \end{bmatrix}$. Applying the above decompositions, (1) is a restricted system equivalent to the following system

$$E\dot{x} = Ax + Bu + Fv, \tag{2a}$$

$$y = Cx + Gv, \tag{2b}$$

where $E \in \mathbb{R}^{n_0 \times n}$, $A \in \mathbb{R}^{n_0 \times n}$, $B \in \mathbb{R}^{n_0 \times k}$, $F \in \mathbb{R}^{n_0 \times q}$, $C = \begin{bmatrix} A_1 \\ \bar{C} \end{bmatrix} \in \mathbb{R}^{t \times n}$, $G = \begin{bmatrix} F_1 \\ \bar{G} \end{bmatrix} \in \mathbb{R}^{t \times q}$, and $y = \begin{bmatrix} -B_1 u \\ \bar{y} \end{bmatrix} \in \mathbb{R}^t$ with $t = p + m - n_0$.

Let rank $G := q_1 \leqslant q$, taking the singular value decomposition (SVD) of G, there exist orthogonal matrices U and V such that $UGV = \begin{bmatrix} \Sigma_{q_1} & 0 \\ 0 & 0 \end{bmatrix}$, where Σ_{q_1} is a nonsingular diagonal matrix of rank q_1. System (2) can now be reformulated as follows.

$$E\dot{x} = \Phi x + Bu + F_{11}\Sigma_{q_1}^{-1} y_1 + F_{12} v_2\,, \tag{3a}$$

$$y_2 = C_{12} x, \tag{3b}$$

along with

$$y_1 = C_{11} x + \Sigma_{q_1} v_1, \tag{3c}$$

where $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Uy$, $\begin{bmatrix} C_{11} \\ C_{12} \end{bmatrix} = UC$, $v = V \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $FV = \begin{bmatrix} F_{11} & F_{12} \end{bmatrix}$, and $\Phi = A - F_{11}\Sigma_{q_1}^{-1} C_{11}$.

Similar transformations have been carried out in [9, 18]. For system (1) and (3), we have the following lemmas.

Lemma 1 *Descriptor system (1) satisfies (H1) if and only if descriptor system (3) satisfies the following condition.*

(H1.1) rank $\begin{bmatrix} E & F_{12} \\ C_{12} & 0 \end{bmatrix} = n + q - q_1$ *(full column rank).*

Lemma 2 *Descriptor system (1) satisfies (H2) if and only if descriptor system (3) satisfies the following condition.*

(H2.1) rank $\begin{bmatrix} \Phi - \lambda E & F_{12} \\ C_{12} & 0 \end{bmatrix} = n + q - q_1$ *(full column rank)* $\forall\, \lambda \in \bar{\mathbb{C}}^{+}$.

It is easy to prove Lemma 1 and Lemma 2 by applying orthogonal transformations P, U, and V in assumptions (H1) and (H2).

Now, let rank $\begin{bmatrix} E \\ C_{12} \end{bmatrix} = n_1$, then there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that $\begin{bmatrix} E \\ C_{12} \end{bmatrix} Q = \begin{bmatrix} E_1 & 0 \\ C_{12a} & 0 \end{bmatrix}$, where $\begin{bmatrix} E_1 \\ C_{12a} \end{bmatrix}$ is full column rank.

Let $x = Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\Phi Q = \begin{bmatrix} \Phi_1 & \Phi_2 \end{bmatrix}$. Then, system (3) can be written as follows.

$$E_1 \dot{x}_1 = \Phi_1 x_1 + \Phi_2 x_2 + Bu + F_{11} \Sigma_{q_1}^{-1} y_1 + F_{12} v_2 \,, \tag{4a}$$

$$y_2 = C_{12a} x_1 \,, \tag{4b}$$

We have the following theorems for systems (1) and (4). Proposition 1 is used throughout to establish the proofs.

Theorem 1 *Descriptor system (1) satisfies (H0) if and only if system (4) satisfies the following condition.*

(H0.1) rank $\begin{bmatrix} E_1 & \Phi_2 & F_{12} \\ C_{12a} & 0 & 0 \end{bmatrix} = n + q - q_1$ *(full column rank)*

Proof.

$$\operatorname{rank} \begin{bmatrix} \bar{E} & \bar{A} & \bar{F} & 0 & 0 \\ 0 & \bar{E} & 0 & \bar{F} & \bar{A} \\ 0 & \bar{C} & \bar{G} & 0 & 0 \\ 0 & 0 & 0 & \bar{G} & \bar{C} \\ 0 & 0 & 0 & 0 & \bar{E} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} P & 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & P \end{bmatrix} \begin{bmatrix} \bar{E} & \bar{A} & \bar{F} & 0 & 0 \\ 0 & \bar{E} & 0 & \bar{F} & \bar{A} \\ 0 & \bar{C} & \bar{G} & 0 & 0 \\ 0 & 0 & 0 & \bar{G} & \bar{C} \\ 0 & 0 & 0 & 0 & \bar{E} \end{bmatrix}$$

$$= \operatorname{rank} \begin{bmatrix} E & A & F & 0 & 0 \\ 0 & A_1 & F_1 & 0 & 0 \\ 0 & E & 0 & F & A \\ 0 & 0 & 0 & F_1 & A_1 \\ 0 & \bar{C} & \bar{G} & 0 & 0 \\ 0 & 0 & 0 & \bar{G} & \bar{C} \\ 0 & 0 & 0 & 0 & E \end{bmatrix} = \operatorname{rank} \begin{bmatrix} E & 0 & F & A \\ C & G & 0 & 0 \\ 0 & 0 & G & C \\ 0 & 0 & 0 & E \end{bmatrix} + \operatorname{rank} E$$

$$= \operatorname{rank} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & U & 0 & 0 \\ 0 & 0 & U & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} E & 0 & F & A \\ C & G & 0 & 0 \\ 0 & 0 & G & C \\ 0 & 0 & 0 & E \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ 0 & 0 & V & 0 \\ 0 & 0 & 0 & I \end{bmatrix} + \operatorname{rank} E$$

$$
= \text{rank}\begin{bmatrix} E & 0 & 0 & F_{11} & F_{12} & A \\ C_{11} & \Sigma_{q_1} & 0 & 0 & 0 & 0 \\ C_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Sigma_{q_1} & 0 & C_{11} \\ 0 & 0 & 0 & 0 & 0 & C_{12} \\ 0 & 0 & 0 & 0 & 0 & E \end{bmatrix} + \text{rank}\, E
$$

$$
= \text{rank}\begin{bmatrix} E & F_{11} & F_{12} & A \\ C_{12} & 0 & 0 & 0 \\ 0 & \Sigma_{q_1} & 0 & C_{11} \\ 0 & 0 & 0 & C_{12} \\ 0 & 0 & 0 & E \end{bmatrix} + q_1 + \text{rank}\, E
$$

$$
= \text{rank}\begin{bmatrix} E & F_{11} & F_{12} & A \\ C_{12} & 0 & 0 & 0 \\ 0 & \Sigma_{q_1} & 0 & C_{11} \\ 0 & 0 & 0 & C_{12} \\ 0 & 0 & 0 & E \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & -\Sigma_{q_1}^{-1} C_{11} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} + q_1 + \text{rank}\, E
$$

$$
= \text{rank}\begin{bmatrix} E & F_{12} & \Phi \\ C_{12} & 0 & 0 \\ 0 & 0 & C_{12} \\ 0 & 0 & E \end{bmatrix} \begin{bmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q \end{bmatrix} + 2q_1 + \text{rank}\, E
$$

$$
= \text{rank}\begin{bmatrix} E_1 & F_{12} & \Phi_1 & \Phi_2 \\ C_{12a} & 0 & 0 & 0 \\ 0 & 0 & C_{12a} & 0 \\ 0 & 0 & E_1 & 0 \end{bmatrix} + 2q_1 + \text{rank}\, E
$$

$$
= \text{rank}\begin{bmatrix} E_1 & F_{12} & \Phi_2 \\ C_{12a} & 0 & 0 \end{bmatrix} + \text{rank}\begin{bmatrix} E_1 \\ C_{12a} \end{bmatrix} + 2q_1 + \text{rank}\, E. \tag{5}
$$

Likewise, we can write

$$
\text{rank}\begin{bmatrix} \bar{E} & \bar{F} & \bar{A} \\ 0 & \bar{G} & \bar{C} \\ 0 & 0 & \bar{E} \end{bmatrix} = \text{rank}\begin{bmatrix} E_1 \\ C_{12a} \end{bmatrix} + q_1 + \text{rank}\, E. \tag{6}
$$

Equations (5) and (6) clearly reflect that the condition (H0) is equivalent to the condition (H0.1). □

Theorem 2 *Descriptor system (1) satisfies (H2) if and only if descriptor system (4) satisfies the following condition.*

(H2.2) $\text{rank}\begin{bmatrix} \Phi_1 - \lambda E_1 & \Phi_2 & F_{12} \\ C_{12a} & 0 & 0 \end{bmatrix} = n + q - q_1$ *(full column rank)* $\forall\, \lambda \in \bar{\mathbb{C}}^{+}$.

Proof. Since

$$\operatorname{rank}\begin{bmatrix}\Phi-\lambda E & F_{12}\\ C_{12} & 0\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\Phi-\lambda E & F_{12}\\ C_{12} & 0\end{bmatrix}\begin{bmatrix}Q & 0\\ 0 & I\end{bmatrix}$$
$$= \operatorname{rank}\begin{bmatrix}\Phi_1-\lambda E_1 & \Phi_2 & F_{12}\\ C_{12a} & 0 & 0\end{bmatrix}. \tag{7}$$

Hence, from Lemma 2 and equation (7), Theorem 2 is proved. □

The following corollary is immediate from any of Theorems 1 or 2.

Corollary 1 *The matrix* $\begin{bmatrix}\Phi_2 & F_{12}\end{bmatrix} \in \mathbb{R}^{n_0\times(n-n_1+q-q_1)}$ *is full column rank if the descriptor system (1) satisfies any of the assumptions (H0) or (H2).*

Theorem 3 *The following implication always holds. (H1) ⇒ (H0).*

Proof. It is obvious from (H1.1) in Lemma 1, that if the system fulfills the condition (H1), then rank $\begin{bmatrix}E\\ C_{12}\end{bmatrix} = n$ (full column rank) and hence, $\Phi_2 = \phi$ (empty). In that case, system (4) fulfills (H0.1), which is equivalent to the condition (H0) on system (1). □

3. Observer design

In this section, we will describe the methods to design index one generalized observers for system (1) under the assumptions (H0) and (H2). The design methods are based on work [9], where 'Luenberger' observer is designed for system (1) under the assumptions (H1) and (H2).

If we take vector x_2 as an unknown input, the combined unknown input in the dynamic equation (4a) would be $\begin{bmatrix}x_2\\ v_2\end{bmatrix}$. It is clear from the assumptions (H0.1) and (H2.1) and {Theorem 1 & 2, [9]} that considering the vectors $\begin{bmatrix}x_2\\ v_2\end{bmatrix}$ as unknown inputs, Luenberger observers can be designed for system (4) to estimate the states x_1 using the methods given in [9]. After estimating x_1, states x_2 and unknown inputs v_2 can be estimated by the equation (4a) itself, since from Corollary 1 matrix $\begin{bmatrix}\Phi_2 & F_{12}\end{bmatrix}$ is full column rank. Based on Sections 3 and 4 of the paper [9], the main result of the paper can be given in the form of the following theorem.

Theorem 4 *Let (H0) and (H2) be satisfied by the coefficient matrices of system (1). Then, there exists an index one generalized observer of the form (O2) for system (1).*

Now, assuming (H0) and (H2), we briefly present two algorithms for designing index one generalized observers (O2) for system (1). The reference [9] has proved that all the steps in Algorithm 1 and Algorithm 2 below are executable under assumptions (H0) and (H2). Prior to applying these algorithms, one must transform (1) into (4).

3.1. Observer design approach I

In this subsection, a generalized observer of the following form is designed:

$$\dot{z} = Nz + TBu + TF_{11}\Sigma_{q_1}^{-1}y_1 + Ly_2\,, \tag{8a}$$

$$\hat{x}_1 = z + My_2\,, \tag{8b}$$

$$\hat{x}_2 = \Sigma_1 E_1 \dot{\hat{x}}_1 - \Sigma_1 \Phi_1 \hat{x}_1 - \Sigma_1 Bu - \Sigma_1 F_{11}\Sigma_{q_1}^{-1}y_1\,, \tag{8c}$$

$$\hat{x}(t) = Q\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}, \tag{8d}$$

where $z \in \mathbb{R}^{n_1}$ and $\begin{bmatrix}\Sigma_1\\ \Sigma_2\end{bmatrix}$ is any left inverse of $\begin{bmatrix}\Phi_2 & F_{12}\end{bmatrix}$. Matrix Σ_2 is utilized in Remark 3 to estimate actual unknown inputs. The remaining problem is to compute N, T, L, and M of appropriate dimensions such that $\hat{x}(t) \to x(t)$ as $t \to \infty$ for arbitrary initial conditions $x(0)$ and $z(0)$. Algorithm 1 summarizes the steps to design these matrices.

Algorithm 1 Computational steps for construction of observer (8) for system (1)

Step 1. Compute a full row rank and left null matrix $T_0 \in \mathbb{R}^{(n_0+n_1-n-q+q_1)\times n_0}$ for matrix $\begin{bmatrix}\Phi_2 & F_{12}\end{bmatrix}$.

Step 2. Compute a full column rank R for matrix pair (T_0E_1, C_{12a}) such that $\operatorname{rank}\begin{bmatrix}I_{n_1} - RT_0E_1\\ C_{12a}\end{bmatrix} = \operatorname{rank}(C_{12a})$.

Step 3. Solve matrix equation $MC_{12a} = I_{n_1} - TE_1$ for matrix M, where $T = RT_0$.

Step 4. Find a matrix K such that $N = T\Phi_1 - KC_{12a}$ is a stable matrix (using pole placement or LMI approach).

Step 5. Calculate $L = K + NM$.

We may simplify system (8) further. Take $Q = \begin{bmatrix}Q_1 & Q_2\end{bmatrix}$, $Q_1 \in \mathbb{R}^{n\times n_1}$ and $Q_2 \in \mathbb{R}^{n\times(n-n_1)}$, then (8d) can be written as

$$\hat{x}(t) = Q_1\hat{x}_1(t) + Q_2\hat{x}_2(t). \tag{9}$$

Substituting values of $\hat{x}_1$ and $\hat{x}_2$ from (8b) and (8c), respectively, we obtain

$$\hat{x}(t) = Q_1 z + Q_1 M y_2 + Q_2 \Sigma_1 E_1 \dot{\hat{x}}_1 - Q_2 \Sigma_1 \Phi_1 \hat{x}_1 - Q_2 \Sigma_1 B u - Q_2 \Sigma_1 F_{11} \Sigma_{q_1}^{-1} y_1 . \tag{10}$$

Now once again, substituting the value of $\hat{x}_1$ and $\dot{\hat{x}}_1$ from (8b) in (10),

$$\begin{aligned}\hat{x}(t) &= (Q_1 - Q_2\Sigma_1\Phi_1)\, z + (Q_1 M - Q_2\Sigma_1\Phi_1 M)\, y_2 + Q_2\Sigma_1 E_1 \dot{z} \\ &\quad + W\dot{y}_2 - Q_2\Sigma_1 B u - Q_2\Sigma_1 F_{11}\Sigma_{q_1}^{-1} y_1 ,\end{aligned} \tag{11}$$

where $W = Q_2\Sigma_1 E_1 M$. Taking $\dot{z}$ from (8a) in (11) and simplifying it,

$$\begin{aligned}\hat{x}(t) &= Sz + \mathscr{U} y_2 + (Q_2\Sigma_1 E_1 T - Q_2\Sigma_1)\, Bu \\ &\quad + (Q_2\Sigma_1 E_1 T - Q_2\Sigma_1)\, F_{11}\Sigma_{q_1}^{-1} y_1 + W\dot{y}_2 .\end{aligned} \tag{12}$$

Combining (12) with (8a), the observer (8) can finally be simplified in the form of (O2) as

$$\dot{z} = Nz + TBu + TF_{11}\Sigma_{q_1}^{-1} y_1 + L y_2 , \tag{13a}$$

$$\hat{x}(t) = Sz + V\left(Bu + F_{11}\Sigma_{q_1}^{-1} y_1\right) + \mathscr{U} y_2 + W\dot{y}_2 , \tag{13b}$$

where,

$Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$, $Q_1 \in \mathbb{R}^{n\times n_1}$ and $Q_2 \in \mathbb{R}^{n\times(n-n_1)}$,
$S = Q_1 + Q_2\Sigma_1 E_1 N - Q_2\Sigma_1\Phi_1$, $\quad \mathscr{U} = Q_1 M + Q_2\Sigma_1 E_1 L - Q_2\Sigma_1\Phi_1 M$,
$V = Q_2\Sigma_1 E_1 T - Q_2\Sigma_1$, and

$$W = Q_2\Sigma_1 E_1 M. \tag{14}$$

3.2. Observer design approach II

In this subsection, we propose another generalized observer as follows

$$\dot{z} = \bar{N} z + T_0 Bu + T_0 F_{11}\Sigma_{q_1}^{-1} y_1 + \bar{L} y_2 , \tag{15a}$$

$$\hat{x}_1 = Rz + M y_2 , \tag{15b}$$

$$\hat{x}_2 = \Sigma_1 E_1 \dot{\hat{x}}_1 - \Sigma_1\Phi_1\hat{x}_1 - \Sigma_1 Bu - \Sigma_1 F_{11}\Sigma_{q_1}^{-1} y_1 , \tag{15c}$$

$$\hat{x}(t) = Q\begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}, \tag{15d}$$

where $z \in \mathbb{R}^{n_0-(q-q_1)-(n-n_1)}$. T_0, R, and M are the same from the previous subsection. The remaining matrices $\bar{N}$ and $\bar{L}$ would be calculated in such a way that $\hat{x}(t) \to x(t)$ as $t \to \infty\ \forall\, x(0), z(0)$. The observer design method is summarized in the following Algorithm 2.

Algorithm 2 Computational steps for construction of observer (15) for descriptor system (1)

Step 1. Repeat Steps 1–3 of Algorithm 1.

Step 2. Find matrix $\bar{K}$ such that $\bar{N} = T_0\Phi_1 R - \bar{K}C_{12a}R$ is a stable matrix.

Step 3. Calculate $\bar{L} = T_0\Phi_1 M + \bar{K} - \bar{K}C_{12a}M$.

Again, observer (15) can be formulated in the form of (O2) as

$$\dot{z} = \bar{N}z + T_0Bu + T_0F_{11}\Sigma_{q_1}^{-1}y_1 + \bar{L}y_2\,, \tag{16a}$$

$$\hat{x}(t) = \bar{S}z + V\left(Bu + F_{11}\Sigma_{q_1}^{-1}y_1\right) + \bar{U}y_2 + W\dot{y}_2\,, \tag{16b}$$

where, $\bar{S} = Q_1R + Q_2\Sigma_1E_1R\bar{N} - Q_2\Sigma_1\Phi_1R$ and $\bar{U} = Q_1M + Q_2\Sigma_1E_1R\bar{L} - Q_2\Sigma_1\Phi_1M$. Remaining matrices are the same as discussed in Subsection 3.1.

Derivation of (16b) can be done as given in Subsection 3.1, Observer design approach I.

Remark 1 *Approach I and II both provide reduced-ordered observers. To be specific, order of the observer given by approach I is n_1, which is* $\operatorname{rank}\begin{bmatrix}\bar{E} & \bar{A} & \bar{F}\\ 0 & \bar{E} & 0\\ 0 & \bar{C} & \bar{G}\end{bmatrix} - \operatorname{rank}\begin{bmatrix}\bar{E} & \bar{F}\\ 0 & \bar{G}\end{bmatrix}$ *and by approach II is $n_0-(q-q_1)-(n-n_1)$, which is* $\operatorname{rank}\begin{bmatrix}\bar{E} & \bar{A} & \bar{F}\\ 0 & \bar{E} & 0\\ 0 & \bar{C} & \bar{G}\end{bmatrix} - \operatorname{rank}\begin{bmatrix}\bar{A} & \bar{F}\\ \bar{E} & 0\\ \bar{C} & \bar{G}\end{bmatrix}$.

Remark 2 *From the equations (13b) and (16b), it is evident that if the matrix W is zero or empty, then the generalized observers discussed in this section result in Luenberger observers. Because, in this case, they do not contain derivatives of the output or input. Equation (14) reflects that if (H1) is satisfied, then W turns out to be empty because Q_2 is empty and observer design methods delivered in this article coincide with design techniques of Luenberger observers presented in [9]. However, there may be other cases where W is obtained as a zero matrix and system (1) satisfies only (H0) but not (H1).*

Remark 3 *After estimating the states, unknown input vector* $\hat{v} = V\begin{bmatrix}\hat{v}_1\\ \hat{v}_2\end{bmatrix}$ *may be estimated as follows.*

$$\hat{v}_1 = \Sigma_{q_1}^{-1} y_1 - \Sigma_{q_1}^{-1} C_{11}\hat{x}, \tag{17}$$

$$\hat{v}_2 = \Sigma_2 E_1 \dot{\hat{x}}_1 - \Sigma_2 \Phi_1 \hat{x}_1 - \Sigma_2 \left(Bu + F_{11}\Sigma_{q_1}^{-1} y_1\right). \tag{18}$$

4. Example

In this section, we apply our theory on a mathematical model of a physical system.

Example 1 Systems from constrained mechanics are generally modeled as [1,21]

$$\dot{\mathrm{x}}_1(t) = \mathrm{x}_2(t), \tag{19a}$$

$$\dot{\mathrm{x}}_2(t) = \mathscr{F}\mathrm{x}_1(t) + \mathscr{D}\mathrm{x}_2(t) + \mathscr{H}^T\mathrm{x}_3(t) + \mathscr{G}\mathrm{v}_1(t), \tag{19b}$$

$$0 = \mathscr{H}\mathrm{x}_1(t) + \mathrm{v}_2(t). \tag{19c}$$

State variables $\mathrm{x}_1(t)$ and $\mathrm{x}_2(t)$ represent the position vector and the velocity vector, respectively. Equation (19c) is a physical constraint that produces the force $\mathscr{H}^T\mathrm{x}_3(t)$. The input is being applied through a force $\mathscr{G}\mathrm{v}_1(t)$ in equation (19b) and affects the algebraic constraint (19c) through $\mathrm{v}_2(t)$. Moreover, matrix $\mathscr{H}$ is assumed to have full row rank. The physical system (19) can be formulated in the form (1), if we take

$$\bar{E} = \begin{bmatrix} I & 0 & 0\\ 0 & I & 0\\ 0 & 0 & 0\end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & I & 0\\ \mathscr{F} & \mathscr{D} & \mathscr{H}^T\\ \mathscr{H} & 0 & 0\end{bmatrix}, \quad \bar{F} = \begin{bmatrix} 0 & 0\\ \mathscr{G} & 0\\ 0 & I\end{bmatrix}, \quad \bar{B} = 0. \tag{20}$$

$x(t) = \begin{bmatrix}\mathrm{x}_1(t)\\ \mathrm{x}_2(t)\\ \mathrm{x}_3(t)\end{bmatrix}$, and $v(t) = \begin{bmatrix}\mathrm{v}_1(t)\\ \mathrm{v}_2(t)\end{bmatrix}$. For computational purpose, we consider the matrices $\mathscr{F}$, $\mathscr{D}$, $\mathscr{H}$, $\mathscr{G}$, and $v(t)$ as given below

$$\mathscr{F} = \begin{bmatrix} -2 & 1\\ 1 & -2\end{bmatrix}, \quad \mathscr{D} = \begin{bmatrix} 0.25 & 0\\ 0 & 0.25\end{bmatrix}, \quad \mathscr{H} = \begin{bmatrix}1 & -1\end{bmatrix}, \quad \mathscr{G} = \begin{bmatrix}1\\ 1\end{bmatrix} \quad \text{and} \quad v(t) = \begin{bmatrix}\sin(t)\\ \sin(t)\end{bmatrix}.$$

If we take $\bar{C} = \begin{bmatrix} -1 & 0 & 1 & 3 & 0\\ 4 & 1 & 0 & 1 & 0\\ 1 & 0 & 1 & 0 & 0\end{bmatrix}$, $\bar{G} = \begin{bmatrix} 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$, the system matrices (20) do not satisfy (H1). Thus, it is clear that the designing of Luenberger or PI observer is not possible for this system by the techniques available in [9, 17, 18]. However, as the system matrices fulfill the assumption (H0), the approaches explained in the previous section can be applied. Here we execute only Algorithm 2 as it provides a lesser order observer compared to Algorithm 1. Applying Algorithm 2, observer (15) matrices are obtained as follows

$$C_{11} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \end{bmatrix}, \; C_{12} = \begin{bmatrix} -1 & 0 & 1 & 3 & 0 \\ 4 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \; F_{11} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; F_{12} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix},$$

$$\Phi = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2 & 1 & 0.2500 & 0 & 1 \\ 1 & -2 & 0 & 0.25 & -1 \end{bmatrix}, \; E_1 = \begin{bmatrix} -0.9649 & 0.1506 & -0.0705 & -0.2032 \\ -0.2214 & -0.0337 & 0.2253 & 0.9482 \\ -0.0244 & -0.3160 & -0.9264 & 0.2032 \\ -0.1390 & -0.9361 & 0.2932 & -0.1355 \end{bmatrix},$$

$$\Phi_1 = \begin{bmatrix} -0.0244 & -0.3160 & -0.9264 & 0.2032 \\ -0.1390 & -0.9361 & 0.2932 & -0.1355 \\ 1.7023 & -0.4140 & 0.1347 & 1.4054 \\ -0.5569 & -0.0159 & -0.4478 & -2.1334 \end{bmatrix}, \; \Phi_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix},$$

$$C_{12a} = \begin{bmatrix} 0.5235 & -3.2750 & 0.0238 & 0 \\ -4.2201 & -0.3675 & 0.2367 & 0 \\ -0.9894 & -0.1654 & -0.9969 & 0 \end{bmatrix}, \; T_0 = \begin{bmatrix} 0.7071 & -0.7071 & 0 & 0 \\ -0.7071 & -0.7071 & 0 & 0 \end{bmatrix},$$

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.5301 & -0.8192 \\ -0.8658 & -0.5602 \end{bmatrix}, \; M = \begin{bmatrix} 0.0275 & -0.2214 & -0.0519 \\ -0.3008 & -0.0337 & -0.0152 \\ 0.0908 & 0.0209 & -1.0172 \\ -0.0251 & -0.0601 & 0.2283 \end{bmatrix},$$

$$\Sigma_1 = \begin{bmatrix} 0 & 0 & 0.5 & -0.5 \end{bmatrix}, \; \bar{N} = \begin{bmatrix} -0.2682 & -0.0400 \\ 0.3379 & -0.4313 \end{bmatrix}, \text{ and}$$

$$\bar{L} = \begin{bmatrix} -0.1630 & -0.2180 & 0.8702 \\ -0.2137 & -0.0660 & -0.4934 \end{bmatrix}.$$

The simulation results are displayed in Figure 1. All the computations have been executed in MATLAB 2021a. System (15a) is solved using ode45 solver. The vector $\hat{x}_1$ is differentiated by grad command using the same time discretization as used in ode45 solver.

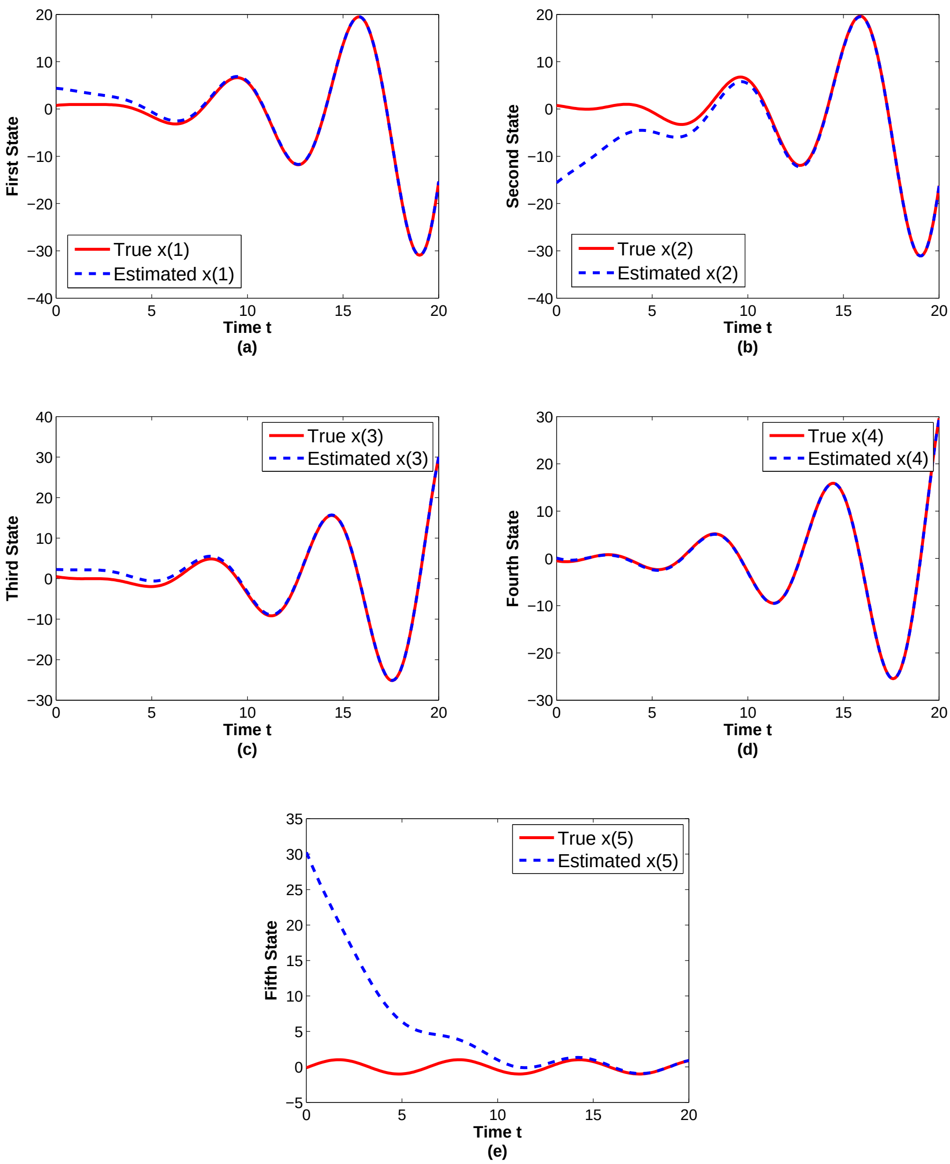

Figure 1: Original states and estimated states by Approach II

5. Conclusion

This paper presents novel sufficient conditions for the designing of generalized observers of index one for rectangular descriptor systems with the presence of unknown inputs. These conditions are presented in the form of algebraic constraints on the rank of coefficient matrices of the given descriptor system. Special cases are discussed where generalized observers coincide with Luenberger observers. Two different algorithms are proposed to summarize the design procedure, and the order of the observers is given in terms of system matrices. Simulation results are presented to validate our findings. Work on sufficient conditions for k-th indexed generalized observer is under progress.

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