

10.24425/acs.2023.145118

Archives of Control Sciences
Volume 33(LIX), 2023
No. 1, pages 155–178

On the constrained and unconstrained controllability of semilinear Hilfer fractional systems

Beata SIKORA

In the paper finite-dimensional semilinear dynamical control systems described by fractional-order state equations with the Hilfer fractional derivative are discussed. The formula for a solution of the considered systems is presented and derived using the Laplace transform. Bounded nonlinear function f depending on a state and controls is used. New sufficient conditions for controllability without constraints are formulated and proved using Rothe's fixed point theorem and the generalized Darbo fixed point theorem. Moreover, the stability property is used to formulate constrained controllability criteria. An illustrative example is presented to give the reader an idea of the theoretical results obtained. A transient process in an electrical circuit described by a system of Hilfer type fractional differential equations is proposed as a possible application of the study.

Key words: Hilfer fractional derivative, fractional systems, semilinear control systems, nonautonomous systems, Rothe's fixed point theorem, generalized Darbo fixed point theorem

1. Introduction

In recent decades, fractional differential calculus has attracted the attention of many scientists due to its increasingly widespread application. Control systems modeled by fractional differential equations appear in many problems in physics, mechanics, biology, and chemistry. Discussions of fractional differential equations and their practical applications can be found, among others, in the monographs: [5, 22, 27, 30, 34, 36, 37, 46].

It turns out that although fractional order systems do not have a semigroup property, it is still possible to establish relations between a fractional-order differential equation and a fractional flow of the corresponding dynamical system. One of the most important issues in control theory is controllability, i.e., the possibility of steering a control system from an initial state to a final state by means of admissible controls.

Copyright © 2023. The Author(s). This is an open-access article distributed under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives License (CC BY-NC-ND 4.0 <https://creativecommons.org/licenses/by-nc-nd/4.0/>), which permits use, distribution, and reproduction in any medium, provided that the article is properly cited, the use is non-commercial, and no modifications or adaptations are made

B. Sikora (e-mail: Beata.Sikora@polsl.pl) is with Department of Applied Mathematics, Silesian University of Technology, Kaszubska 23, 44-100 Gliwice, Poland.

Received 26.12.2022.

In recent years, numerous papers and monographs have appeared on the controllability of fractional order control systems. Various fractional operators have been used. The controllability of semilinear and nonlinear control systems with the Caputo derivative is studied in [3, 9, 28, 35, 39], among others, while the Riemann-Liouville derivative is considered in [7, 11, 12]. Fractional differential equations with the Hilfer derivative have been studied by many authors, see for example [8, 10, 13, 15, 24, 45, 47]. However, only a few papers have been published on the controllability of differential systems involving the Hilfer fractional derivative [21, 25, 31, 41].

The Hilfer fractional differential operator is increasingly used in modelling diffusion processes, financial crises, viscoelastic problems, and other problems. Some of the models are supported by experiments proving agreement with the behaviour of real processes, for example, in the search for relaxation spectra in glass-forming materials [14]. The aim of this work is to formulate and prove new criteria for the controllability of semilinear systems using the Hilfer derivative. For this purpose, Rothe's fixed point theorem was used. Fixed point theorems give as answer to the question whether a solution exists, i.e., a system can be steered to a final state $x(t_1)$ (the state $x(t_1)$ can be reached). Rothe's fixed point theorem has already been used in [48] for integer order nonlinear differential equation with integral boundary conditions, in [26] for semilinear system of ordinary differential equations, and in [39] for the Caputo fractional-order semilinear systems with delays in the control. The Darbo fixed point theorem has been used for studying controllability of the integer-order nonlinear differential systems in [6] and for the Caputo fractional-order nonlinear implicit systems with delays in [29].

The paper is organized as follows. Section 2 gives some introductory definitions, formulas, lemmas, and notations. Section 3 presents the mathematical model of the Hilfer fractional systems considered and the formula for a solution of the systems. The formula is derived using the Laplace transform. Some definitions of the constrained and unconstrained controllability of the systems are formulated. Constraints are established for the control values. The main results of the paper, contained in Section 4, are the criteria for unconstrained controllability of the Hilfer fractional system and constrained controllability when the set of admissible control values U is a convex and compact set containing 0 in its interior. All the theorems are proved. The theoretical results are illustrated with numerical and practical examples in Section 5. Finally, concluding remarks are made in Section 6.

2. Preliminaries

In this section we give some definitions, formulas, lemmas, and notations used throughout the paper. Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ be a function of the variable $t \in \mathbb{R}_+$

integrable over the closed interval $\langle 0, t \rangle$, $t > 0$. Moreover, let $D = \frac{d}{dt}$ be a first order differential operator and let Γ denote the gamma function.

Definition 1 [30] *The Riemann-Liouville fractional integral (left sided) of order $\alpha \in \mathbb{R}_+$ for the function f is called the integral operator I^α defined by the formula*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau) d\tau.$$

Definition 2 [13] *The Hilfer fractional derivative (left sided) of order α and type β for the function f is called the integro-differential operator $D^{\alpha,\beta}$ defined by the formula*

$$D^{\alpha,\beta} f(t) = I^{(1-\alpha)\beta} DI^{(1-\alpha)(1-\beta)} f(t),$$

where $n < \alpha < n + 1$, $0 \leq \beta \leq 1$, $n \in \mathbb{N}$, provided that the right side expression exists.

The two-parameter family of the Hilfer fractional derivatives is a generalization of both the Caputo derivative and the Riemann-Liouville derivative (definitions of both the derivatives can be found in [30]; applications to control theory can be found in [18, 20, 33, 38, 40, 44], among others). The Hilfer derivative allows us to interpolate between the fractional derivatives mentioned above. It is worth noting that for $0 < \alpha < 1$ and $\beta = 0$ the Hilfer fractional derivative is identical to the classical Riemann-Liouville fractional derivative D^α , i.e. $D^{\alpha,0} f(t) = D^\alpha f(t) = DI^{1-\alpha} f(t)$. For $0 < \alpha < 1$ and $\beta = 1$, on the other hand, the Hilfer fractional derivative reduces to the Caputo fractional derivative, i.e., $D^{\alpha,1} f(t) = {}^C D^\alpha f(t) = I^{1-\alpha} D f(t)$. Since the Hilfer fractional derivative is defined on the basis of the definite integral, it is a nonlocal operator. It has a “memory property”, which means that the current state depends not only on time, but also on previous states.

The Laplace transform of Hilfer’s derivative is of the form [13, 43]

$$\mathcal{L}[D^{\alpha,\beta} f(t)] = s^\alpha \mathcal{L}[f(t)] - s^{(1-\alpha)\beta} I^{(1-\alpha)(1-\beta)} f(0+),$$

where the initial-value condition $I^{(1-\alpha)(1-\beta)} f(0+)$ is the Riemann-Liouville integral of order $(1-\alpha)(1-\beta)$ evaluated for $t \rightarrow 0^+$. The fractional integral $I^{(1-\alpha)(1-\beta)} f(0+)$ remains constant for all t , although the function f varies.

Definition 3 [30], [19] *The two-parameter Mittag-Leffler function is defined by the formula*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}$$

for $\alpha, \beta \in \mathbb{R}_+$, $z \in \mathbb{C}$.

Let A be a bounded linear operator from X into X . Definition 3 can be extended to linear operators, hence

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}.$$

The following lemmas are also needed to formulate some controllability results. The symbol \exists stands for the existential quantifier, $\dim(Z)$ denotes the dimension of the space Z , and $\ker(R^*)$ is the kernel of (R^*) , where R^* is the adjoint operator of R .

Lemma 1 [17] *Let W and Z be Hilbert spaces, $G \in L(W, Z)$ and $R^* \in L(Z, W)$, and $\dim(Z) < +\infty$. The following conditions are equivalent:*

- (1) $\text{Rang}(R) = Z$,
- (2) $\ker(R^*) = \{0\}$,
- (3) $\exists \gamma > 0 \langle RR^*x, x \rangle > \gamma \|x\|^2, x \neq 0$,
- (4) $\exists (RR^*)^{-1} \in L(Z)$.

Lemma 2 [26] *Let (X, Σ, μ_1) be a measure space with $\mu_1(X) < +\infty$ and $1 \leq q < p < +\infty$. Then $L^p(\mu_1) \subset L^q(\mu_1)$ and*

$$\forall f \in L^p(\mu_1) \quad \|f\|_{L^q} \leq \mu_1(X)^{\frac{p-q}{pq}} \|f\|_{L^p}.$$

Definition 4 [2] *Let $(E, \|\cdot\|)$ be a Banach space, \mathcal{M}_E be a family of all nonempty bounded subset of E , and \mathcal{N}_E its subfamily of all relatively compact sets. A mapping $\mu: \mathcal{M}_E \rightarrow [0, +\infty)$ is called a measure of noncompactness in E if it satisfies the following conditions:*

- (1) $\ker(\mu) = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\ker(\mu) \subset \mathcal{N}_E$,
- (2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$,
- (3) $\mu(\overline{X}) = \mu(X)$,
- (4) $\mu(\text{Conv}X) = \mu(X)$,
- (5) $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda\mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$,
- (6) If $\{X_n\}$ is a nested sequence of closed sets from \mathcal{M}_E such that $\lim_{n \rightarrow +\infty} \mu(X_n) = 0$,

then the intersection set $X_\infty = \bigcap_{n=a}^{+\infty} X_n$ is nonempty.

The family $\ker(\mu)$ is called the kernel of the measure of noncompactness μ and \bar{X} , $\text{Conv}X$ stand for the closure and the close convex hull of X , respectively.

Let $C_n[0, t_1]$ be the space of continuous \mathbb{R}^n valued functions on $[0, t_1]$ with the norm

$$\|x\| = \max_{1 \leq i \leq n} |x_i(t)|, \quad t \in [0, t_1].$$

The function [6]

$$\omega(x, h) = \max_{1 \leq i \leq n} \omega(x_i, h),$$

where

$$\omega(x_i, h) = \sup\{|x_i(t) - x_i(s)| : |t - s| \leq h, t, s \in [0, t_1]\}$$

is called the modulus of continuity of a function $x \in C_n[0, t_1]$. Analogously, the function

$$\omega(S, h) = \sup\{\omega(x, h) : x \in S\},$$

is called the modulus of continuity of a bounded set $S \subset C_n[0, t_1]$. Moreover, all the functions $x \in S$ are equicontinuous if and only if

$$\omega_0(S) = \lim_{h \rightarrow 0^+} \omega(S, h) = 0.$$

If the space $(E, \|\cdot\|)$ is the Cartesian product of two spaces $(E_1, \|\cdot\|_{E_1})$ and $(E_2, \|\cdot\|_{E_2})$, and $\|\cdot\| = \max\{\|\cdot\|_{E_1}, \|\cdot\|_{E_2}\}$, then for any $S \subset E$ the measure of noncompactness μ is defined as

$$\mu(S) = \max\{\mu(S_1), \mu(S_2)\},$$

where S_1, S_2 means the natural projections of S into E_1, E_2 , respectively.

Other notations used in this paper are: $L^2([0, \infty), \mathbb{R}^m)$ is the Hilbert space of square integrable functions with values in \mathbb{R}^m , $L^2_{loc}([0, \infty), \mathbb{R}^m)$ is the linear space of locally square integrable functions with values in \mathbb{R}^m .

3. System description

This paper deals with semilinear fractional control systems described by the following Cauchy problem

$$D^{\alpha, \beta} x(t) = A x(t) + B(t)u(t) + f(t, x(t), u(t)) \quad (1)$$

$$I^{(1-\alpha)(1-\beta)} x(0+) = x_0 \quad (2)$$

for $t \geq 0$ and $0 < \alpha < 1, 0 \leq \beta \leq 1$, where

- $D^{\alpha, \beta}$ is the Hilfer fractional differential operator,

- $x(t) \in \mathbb{R}^n$ is a pseudo-state vector,
- $x_0 \in \mathbb{R}^n$ is a vector of initial conditions,
- $u \in L^2_{loc}([0, \infty), \mathbb{R}^m)$ is a control,
- A is a $n \times n$ matrix with real elements,
- $B(t)$ is a $n \times m$ matrix with continuous elements, satisfying the conditions $|b_{ij}(t)| \leq N, i = 1, 2, \dots, n, j = 1, 2, \dots, m, N \in \mathbb{R}_+$, for any $t \in [0, t_1]$,
- f is a given continuous nonlinear function $f: [0, t] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$.

The following norms are considered: $\|x\| = \max_{1 \leq i \leq n} |x_i|$, $\|B\| = \max_{1 \leq i \leq n} \sum_{j=1}^m |b_{ij}|$,

and $\|B\|_\infty = \sup_{0 \leq t \leq t_1} \|B\|$.

Theorem 1 For the given initial conditions $x_0 \in \mathbb{R}^n$ and a control $u \in L^2_{loc}([0, \infty), \mathbb{R}^m)$, there is a unique solution $x(t) \in \mathbb{R}^n$ of the semilinear Hilfer fractional-order system (1)–(2), for any $t \geq 0$, which has the following form

$$x(t) = x_0 t^{(\alpha-1)(1-\beta)} E_{\alpha, \alpha+(1-\alpha)\beta}(At^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(A(t-\tau)^\alpha) [B(t)u(\tau) + f(\tau, x(\tau), u(\tau))] d\tau. \quad (3)$$

Proof. To prove the theorem, we apply the Laplace transformation to the fractional equation (1), taking into account the initial condition (2). We have

$$s^\alpha \mathcal{L}[x(t)] - s^{(1-\alpha)\beta} I^{(1-\alpha)(1-\beta)} f(0+) = A \mathcal{L}[x(t)] + \mathcal{L}[B(t)u(t) + f(t, x(t), u(t))],$$

for any fixed $t \geq 0$. Thus

$$\begin{aligned} s^\alpha \mathcal{L}[x(t)] - A \mathcal{L}[x(t)] &= s^{(1-\alpha)\beta} x_0 + \mathcal{L}[B(t)u(t) + f(t, x(t), u(t))], \\ \mathcal{L}[x(t)] &= (s^\alpha I - A)^{-1} s^{(1-\alpha)\beta} x_0 + (s^\alpha I - A)^{-1} \mathcal{L}[B(t)u(t) + f(t, x(t), u(t))] \\ &= x_0 \mathcal{L}[t^{(\alpha-1)(1-\beta)} E_{\alpha, \alpha+(1-\alpha)\beta}(At^\alpha)] \\ &\quad + \mathcal{L}[t^{\alpha-1} E_{\alpha, \alpha}(At^\alpha)] \mathcal{L}[B(t)u(t) + f(t, x(t), u(t))]. \end{aligned}$$

According to the convolution theorem for the Laplace transformation we get

$$\begin{aligned} \mathcal{L}[x(t)] &= x_0 \mathcal{L}[t^{(\alpha-1)(1-\beta)} E_{\alpha, \alpha+(1-\alpha)\beta}(At^\alpha)] \\ &\quad + \mathcal{L}[t^{\alpha-1} E_{\alpha, \alpha}(At^\alpha)] * (B(t)u(t) + f(t, x(t), u(t))), \end{aligned}$$

And finally, with the help of the inverse Laplace transformation and the definition of the convolution, we obtain the solution of (1)–(2)

$$x(t) = x_0 t^{(\alpha-1)(1-\beta)} E_{\alpha, \alpha+(1-\alpha)\beta}(At^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(A(t-\tau)^\alpha) [B(t)u(\tau) + f(\tau, x(\tau), u(\tau))] d\tau.$$

Due to the uniqueness of the Laplace transform, the resulting solution (3) is unique. \square

A set of solutions for a given differential control system is also called a set of reachable states or an attainable set. As in the case of integer-order dynamical systems, we can define the attainable set for the fractional system (1)–(2).

Definition 5 *The attainable set from the initial state x_0 on $[0, t]$ for the Hilfer fractional system (1)–(2) is the set*

$$K(t) = \left\{ x(t) \in \mathbb{R}^n : x(t) = x_0 t^{(\alpha-1)(1-\beta)} E_{\alpha, \alpha+(1-\alpha)\beta}(At^\alpha) + \int_0^t (t-\tau)^{\alpha-1} E_{\alpha, \alpha}(A(t-\tau)^\alpha) [B(t)u(\tau) + f(\tau, x(\tau), u(\tau))] d\tau : u(t) \in \mathbb{R}^m \text{ for } t \in [0, t] \right\}. \quad (4)$$

for any fixed $t \geq 0$.

Next, we formulate several types definitions of controllability for the fractional system (1)–(2) on $[0, t_1]$. Let $S \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ be arbitrary nonempty sets.

Definition 6 *The Hilfer fractional system (1)–(2) is called controllable on $[0, t_1]$ from the initial state x_0 into S , if for each vector $\tilde{x} \in S$, there exist a control $\tilde{u} \in L^2([0, t_1], \mathbb{R}^m)$ such that $x(t_1) = \tilde{x}$.*

For $S = \mathbb{R}^n$ the fractional system (1)–(2) is called controllable on $[0, t_1]$ from the initial state x_0 . For $S = \{0\}$, the system (1)–(2) is called null controllable on $[0, t_1]$ from the initial state x_0 .

The following definitions of local and global controllability for the semilinear system (1)–(2) on $[0, t_1]$ are a consequence of corresponding definitions for integer-order systems presented in [23], among others.

Definition 7 *The semilinear Hilfer fractional system (1)–(2) is called locally controllable on $[0, t_1]$ if the attainable set $K(t_1)$ contains a certain neighborhood of zero in the space \mathbb{R}^n .*

Definition 8 *The semilinear fractional system (1)–(2) is called globally controllable on $[0, t_1]$ if it is controllable on $[0, t_1]$ for every initial complete state x_0 .*

Remark 1 *The system (1)–(2) is globally controllable on $[0, t_1]$ if $K(t_1) = \mathbb{R}^n$.*

The above definitions concern the controllability of semilinear systems without constraints. If we consider control values from the set $U \subset \mathbb{R}^m$, we speak of null, local or global U -controllability, respectively.

In the further course of the work we will write controllable instead of globally controllable for short.

4. Controllability criteria – main results

In this section we discuss controllability issues of system (1)–(2). We prove that under the assumed conditions, if the corresponding linear Hilfer fractional system

$$D^{\alpha, \beta} x(t) = Ax(t) + B(t)u(t), \quad (5)$$

$$I^{(1-\alpha)(1-\beta)} x(0+) = x_0 \quad (6)$$

is controllable on $[0, t_1]$, then the semilinear system (1)–(2) is also controllable on $[0, t_1]$. Moreover we give a control \tilde{u} that steers the semilinear system (1)–(2) from an initial state x_0 to a final state $x_1(t_1) = x(t_1, x_0, \tilde{u})$.

To formulate and prove a sufficient condition for the controllability of the system (6)–(7) without constraints on control values, Rothe's fixed point theorem is used.

Theorem 2 *(Rothe's fixed point theorem) [16, 42] Let E be a Banach space and $V \subset E$ be a closed convex subset such that zero of E is contained in the interior of V . Let $g: V \rightarrow E$ be a continuous mapping with $g(V)$ relatively compact (closure is compact) in E and $g(\partial V) \subset V$, where ∂V denotes the boundary of V . Then there is a point $x^* \in V$ such that $g(x^*) = x^*$.*

Let us denote, after [27],

$$\Phi(t) = t^{\alpha-1} E_{\alpha, \alpha}(At^\alpha). \quad (7)$$

The Mittag-Leffler function is of exponential order. Moreover, for $0 < \alpha < 1$, $\lim_{t \rightarrow +\infty} t^{\alpha-1} = 0$. Therefore, the inequality

$$\exists_{M>0} \exists_{\varrho \geq 0} \|\Phi(t)\| \leq Me^{\varrho t} \quad (8)$$

is valid for $t > 0$.

Let us define the operators $G, G_f: L^2([0, t_1], \mathbb{R}^m) \rightarrow \mathbb{R}^n$ for the semilinear Hilfer fractional system (1)–(2) as

$$G(u) = \int_0^{t_1} \Phi(t_1 - \tau) B(\tau) u(\tau) d\tau, \quad (9)$$

$$G_f(u) = \int_0^{t_1} \Phi(t_1 - \tau) B(\tau) u(\tau) d\tau + \int_0^{t_1} \Phi(t_1 - \tau) f(\tau, \tilde{x}(\tau), u(\tau)) d\tau \quad (10)$$

for $t_1 > 0$, where \tilde{x} is the unique solution of the fractional differential equation (1)–(2).

The controllability operator $G_f(u)$ can be expressed as a sum

$$G_f(u) = G(u) + H(u),$$

where $H: L^2([0, t_1], \mathbb{R}^m) \rightarrow \mathbb{R}^n$ is the nonlinear operator given by the formula

$$H(u) = \int_0^{t_1} \Phi(t_1 - \tau) f(\tau, \tilde{x}(\tau), u(\tau)) d\tau. \quad (11)$$

The adjoint operator $G^*: \mathbb{R}^n \rightarrow L^2([0, t_1], \mathbb{R}^m)$ of the operator G is

$$G^*(x) = B^*(\tau) \Phi^*(t_1 - \tau) x.$$

Remark 2 Starting from the classical Gramian matrix, the linear system (6)–(7) is controllable on $[0, t_1]$ if the controllability matrix

$$W(t_1) = \int_0^{t_1} \Phi(t - \tau) B(\tau) B^*(\tau) \Phi^*(t - \tau) d\tau \quad (12)$$

is nonsingular. This is equivalent to $\text{Rang}(G) = \mathbb{R}^n$. Moreover, by Remark 1, the semilinear system (1)–(2) is controllable on $[0, t_1]$ if and only if $\text{Rang}(G) = \mathbb{R}^n$ and $\text{Rang}(G_f) = \mathbb{R}^n$.

Theorem 3 Let the nonlinear function f satisfy the following condition

$$\|f(t, x(t), u(t))\| \leq a\|x(t)\| + b\|u(t)\|^\xi + c, \quad (13)$$

where a, b, c, ξ are real constants and $\frac{1}{2} \leq \xi < 1$. If the linear Hilfer fractional system (6)–(7) is controllable on $[0, t_1]$ and the inequality

$$\frac{1}{\gamma\sqrt{2}} n^2 N^2 M^3 a \sqrt{t_1} e^{aMt_1} \left(\frac{e^{2\varrho t_1} - 1}{2\varrho} \right)^{\frac{3}{2}} < 1, \quad (14)$$

holds for $\gamma > 0, N > 0, M > 0, \varrho \geq 0$, the semilinear Hilfer fractional system (1)–(2) is also controllable on $[0, t_1]$. A control steering the system (1)–(2) from the initial state x_0 to a final state $\tilde{x} = x(t_1)$ at time $t_1 > 0$ is given by the following formula

$$u(t) = B^*(\tau) \Phi^*(t_1 - t) (GG^*)^{-1} \left(\tilde{x} - x_0 t^{(\alpha-1)(1-\beta)} E_{\alpha, \alpha+(1-\alpha)\beta}(At^\alpha) - H(u) \right), \quad t \in [0, t_1]. \quad (15)$$

Proof. Without loss of generality, let us assume the initial condition $x_0 = 0$ and $c = 0$. For each $x \in \mathbb{R}^n$ fixed we define an operator $\Psi: L^2([0, t_1], \mathbb{R}^m) \rightarrow L^2([0, t_1], \mathbb{R}^m)$ by the formula

$$\Psi(u) = G^*(GG^*)^{-1}(x - H(u)).$$

It follows from Lemma 1(4) that $(GG^*)^{-1}$ exists, so Ψ is properly defined. Moreover, it follows from (3)

$$\|(GG^*)^{-1}x\| \leq \gamma^{-1}\|x\|.$$

Next we prove that the operator Ψ has a fixed point u that depends on x .

The nonlinear function f is continuous, therefore the operator H is continuous and, by (13), H is a compact operator.

Applying the Hölder inequality and condition (13), for any $u \in L^2([0, t_1], \mathbb{R}^m)$

$$\begin{aligned} \|H(u)\| &\leq \int_0^{t_1} M e^{\varrho(t_1-\tau)} f(\tau, \tilde{x}(\tau), u(\tau)) d\tau \\ &\leq \left(\int_0^{t_1} M^2 e^{2\varrho(t_1-\tau)} d\tau \right)^{\frac{1}{2}} \left(\int_0^{t_1} \|f(\tau, \tilde{x}(\tau), u(\tau))\|^2 d\tau \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^{t_1} M^2 e^{2\varrho(t_1-\tau)} d\tau \right)^{\frac{1}{2}} \left(\int_0^{t_1} (a^2 \|x(\tau)\|^2 + b^2 \|u(\tau)\|^{2\xi}) d\tau \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
 &\leq a \left(\int_0^{t_1} M^2 e^{2\rho(t_1-\tau)} d\tau \right)^{\frac{1}{2}} \left(\int_0^{t_1} \|x(\tau)\|^2 d\tau \right)^{\frac{1}{2}} \\
 &\quad + b \left(\int_0^{t_1} M^2 e^{2\rho(t_1-\tau)} d\tau \right)^{\frac{1}{2}} \left(\int_0^{t_1} \|u(\tau)\|^{2\xi} d\tau \right)^{\frac{1}{2}} \\
 &\leq a \left(\int_0^{t_1} M^2 e^{2\rho(t_1-\tau)} d\tau \right)^{\frac{1}{2}} \left(\int_0^{t_1} \left(\int_0^{t_1} \|B\|_\infty M e^{\rho(t_1-s)} \|u(s)\| ds \right. \right. \\
 &\quad \left. \left. + \int_0^{t_1} b M e^{\rho(t_1-s)} \|u(s)\|^\xi ds \right)^2 e^{2aMt_1} d\tau \right)^{\frac{1}{2}} \\
 &\quad + b \left(\int_0^{t_1} M^2 e^{2\rho(t_1-\tau)} d\tau \right)^{\frac{1}{2}} \left[\left(\int_0^{t_1} \|u(\tau)\|^{2\xi} d\tau \right)^{\frac{1}{2\xi}} \right]^\xi \\
 &\leq a\sqrt{t_1} \left(\int_0^{t_1} M^2 e^{2\rho(t_1-\tau)} d\tau \right)^{\frac{1}{2}} \left(\int_0^{t_1} \|B\|_\infty M e^{\rho(t_1-\tau)} \|u(\tau)\| d\tau \right. \\
 &\quad \left. + \int_0^{t_1} b M e^{\rho(t_1-\tau)} \|u(\tau)\|^\xi d\tau \right) e^{aMt_1} + b \left(\int_0^{t_1} M^2 e^{2\rho(t_1-\tau)} d\tau \right)^{\frac{1}{2}} \|u\|_{L^{2\xi}}^\xi \\
 &= a\sqrt{t_1} \left(\int_0^{t_1} M^2 e^{2\rho(t_1-\tau)} d\tau \right) \|B\|_\infty e^{aMt_1} \|u\|_{L^2} \\
 &\quad + ab\sqrt{t_1} e^{aMt_1} \left(\left(\int_0^{t_1} M^2 e^{2\rho(t_1-\tau)} d\tau \right) + b \left(\int_0^{t_1} M^2 e^{2\rho(t_1-\tau)} d\tau \right)^{\frac{1}{2}} \right) \|u\|_{L^{2\xi}}^\xi,
 \end{aligned}$$

where $L^{2\xi} = L^{2\xi}([0, t_1], \mathbb{R}^m)$ and $\int_0^{t_1} M^2 e^{2\rho(t_1-\tau)} d\tau = \frac{M^2}{2\rho} (e^{2\rho t_1} - 1)$. Moreover,

since $\frac{1}{2} \leq \xi < 1$, we obtain $1 \leq 2\xi < 2$.

Applying Lemma 2 and next taking into account the assumption $|b_{ij}(t)| \leq N$, we have

$$\|H(u)\| \leq M^2 a \sqrt{t_1} \|B\|_\infty e^{aMt_1} \frac{e^{2\varrho t_1} - 1}{2\varrho} \|u\| \leq M^2 a \sqrt{t_1} n N e^{aMt_1} \frac{e^{2\varrho t_1} - 1}{2\varrho} \|u\|.$$

It follows

$$\lim_{\|u\| \rightarrow +\infty} \frac{\|H(u)\|}{\|u\|} \leq M^2 a \sqrt{t_1} n N e^{aMt_1} \frac{e^{2\varrho t_1} - 1}{2\varrho},$$

where $\|u\| = \|u\|_{L^2}$.

Therefore,

$$\lim_{\|u\| \rightarrow +\infty} \frac{\|\Psi(u)\|}{\|u\|} \leq \|G^*(GG^*)^{-1}\| n N M^2 a \sqrt{t_1} e^{aMt_1} \frac{e^{2\varrho t_1} - 1}{2\varrho},$$

which implies

$$\lim_{\|u\| \rightarrow +\infty} \frac{\|\Psi(u)\|}{\|u\|} \leq \frac{1}{\gamma\sqrt{2}} n^2 N^2 M^3 a \sqrt{t_1} e^{aMt_1} \left(\frac{e^{2\varrho t_1} - 1}{2\varrho} \right)^{\frac{3}{2}}.$$

Setting $r = \frac{1}{\gamma\sqrt{2}} n^2 N^2 M^3 a \sqrt{t_1} e^{aMt_1} \left(\frac{e^{2\varrho t_1} - 1}{2\varrho} \right)^{\frac{3}{2}}$ and finding that, given the assumptions, $r < 1$, we conclude that for a fixed ε , $r < \varepsilon < 1$, there exists a sufficiently large $r_0 > 0$ such that

$$\|\Psi(u)\| \leq \varepsilon \|u\| = \varepsilon r_0.$$

Let $B(0, r_0)$ be the sphere with center zero and radius $r_0 > 0$, then $\Psi(\partial B(0, r_0)) \subset B(0, r_0)$. The operator Ψ is compact and maps the sphere $\partial B(0, r_0)$ into the interior of the ball $B(0, r_0)$. Therefore Rothe's fixed point theorem can be applied. It follows that there is a fixed point $u \in B(0, r_0) \subset L^2([0, t_1], \mathbb{R}^m)$ such that

$$u = G^*(GG^*)^{-1}(x - H(u)).$$

From $G(u) = x - H(u)$, taking $x = \tilde{x} - x_0 t^{(\alpha-1)(1-\beta)} E_{\alpha, \alpha+(1-\alpha)\beta}(At^\alpha)$, we obtain

$$\begin{aligned} \tilde{x} &= x_0 t^{(\alpha-1)(1-\beta)} E_{\alpha, \alpha+(1-\alpha)\beta}(At^\alpha) \\ &+ \int_0^{t_1} \Phi(t_1 - \tau) [B(\tau)u(\tau) d\tau + f(\tau, x(\tau), u(\tau))] d\tau. \end{aligned} \quad (16)$$

Thus, \tilde{x} is the solution of the system (1)–(2), and it is easy to verify that $\tilde{x} = x(t_1)$. Therefore the system is controllable on $[0, t_1]$.

From the above, we get the admissible control steering the system (1)–(2) from the initial state x_0 to a final state $\tilde{x} = x(t_1)$ at any time $t_1 > 0$, described as

$$u(t) = B^*(t)\Phi^*(t_1 - t)(GG^*)^{-1} \left(\tilde{x} - x_0 t^{(\alpha-1)(1-\beta)} E_{\alpha, \alpha+(1-\alpha)\beta}(At^\alpha) - H(u) \right),$$

$$t \in [0, t_1]. \quad \square$$

In order to formulate and prove the next criterion, the generalized Darbo fixed point theorem is applied.

Theorem 4 (The generalized Darbo fixed point theorem) [2] Let V be a nonempty, bounded, closed and convex subset of a Banach space E and let $F: V \rightarrow V$ be a continuous operator. Assume that there exists a constant $k \in [0, 1)$ such that

$$\mu(FX) \leq k\mu(X)$$

for any nonempty subset X of V , where μ is a measure of noncompactness defined in E . Then F has a fixed point in the set V .

Firstly, we define two operators

$$F_{(x,u)}^1(t) = x_0 t^{(\alpha-1)(1-\beta)} E_{\alpha, \alpha+(1-\alpha)\beta}(At^\alpha) \quad (17)$$

and

$$F_{(x,u)}^2(t) = B^*(\tau)\Phi^*(t_1 - t)W^{-1}(t_1) \left(\tilde{x} - x_0 t^{(\alpha-1)(1-\beta)} E_{\alpha, \alpha+(1-\alpha)\beta}(At^\alpha) - H(u) \right) + \int_0^{t_1} \Phi(t_1 - \tau) [B(\tau)F_{(x,u)}^1(\tau) d\tau + f(F_{(x,u)}^1(\tau), x(\tau), u(\tau))] d\tau, \quad (18)$$

where $W(t_1)$ and $H(u)$ are given by (12) and (11), respectively.

Next, we define the nonlinear transformation $F_{(x,u)}$ as

$$F_{(x,u)}(t) = [F_{(x,u)}^1(t), F_{(x,u)}^2(t)],$$

which is continuous in the Banach space $C_n^{\alpha, \beta}[0, t_1] \times C_m[0, t_1]$.

Theorem 5 Let the nonlinear function f satisfy the conditions:

$$\|f(t, x, u)\| \leq k_1, \quad (19)$$

$$\|f(t, x, u) - f(t, \bar{x}, u)\| \leq k_2 \|x - \bar{x}\| \quad (20)$$

for $x, \bar{x} \in \mathbb{R}^n$, $k_1, k_2 \in \mathbb{R}_+$, $0 \leq k_2 < 1$. If the linear Hilfer fractional system (6)–(7) is controllable on $[0, t_1]$, then the semilinear Hilfer fractional system (1)–(2) is controllable on $[0, t_1]$.

Proof. By the assumption, $|b_{ij}(t)| \leq N$ on $[0, t_1]$. Consider a set

$$\Theta = \{(x, u) : \|x\| \leq c_1, \|u\| \leq c_2\},$$

where $c_1, c_2 \in \mathbb{R}_+$ are defined as follows

$$c_1 = [(1 + N_1)|x_0| + N_1|x_1| \|\Phi(t_1)\| + k_1 t_1 (1 + N_1) \|\Phi(t_1)\|] \|\Phi(t_1)\|,$$

$$c_2 = N_2 [|x_0| + |x_1| \|\Phi(t_1)\| + k_1 t_1 \|\Phi(t_1)\|]$$

for

$$N_1 = \|W^{-1}(t_1)\| n^2 N^2 t_1 \|\Phi(t_1)\|,$$

$$N_2 = \|W^{-1}(t_1)\| n N \|\Phi(t_1)\|,$$

where $\Phi(t_1)$ is given by (7), $W(t_1)$ is defined by (12). The set Θ is bounded, closed, and convex in $C_n^{\alpha, \beta}[0, t_1] \times C_m[0, t_1]$. Moreover, the nonlinear transformation F maps Θ into Θ . It follows that for each pair $(x, y) \in \Theta$ the inequality

$$\omega(F_{(x,u)}^2, h) \leq \omega(B^* \Phi^*, h) q,$$

holds for

$$q = \sup\{W^{-1}(t_1)(\tilde{x} - x_0 t^{(\alpha-1)(1-\beta)}) E_{\alpha, \alpha+(1-\alpha)\beta}(At^\alpha) - H(u) : (x, u) \in \Theta\}.$$

Functions $F_{(x,u)}^2$ have a uniformly bounded modulus of continuity, because B and Φ do not depend on the choice of the points (x, u) in Θ , and hence they are equicontinuous. Moreover, functions $F_{(x,u)}^1$ are also equicontinuous as they have uniformly bounded Hilfer derivatives. The moduli of continuity of the functions $D^{\alpha, \beta} F_{(x,u)}^1$ is as follows

$$\begin{aligned} |D^{\alpha, \beta} F_{(x,u)}^1(t) - D^{\alpha, \beta} F_{(x,u)}^1(s)| &\leq |AF_{(x,u)}^1(t) - AF_{(x,u)}^1(s)| \\ &\quad + |B(t)F_{(x,u)}^1(t) - B(t)F_{(x,u)}^1(s)| + |f(t, x(t), F_{(x,u)}^1(t)) - f(s, x(s), F_{(x,u)}^1(s))| \\ &\leq |AF_{(x,u)}^1(t) - AF_{(x,u)}^1(s)| + |B(t)F_{(x,u)}^1(t) - B(t)F_{(x,u)}^1(s)| \\ &\quad + |f(t, x(t), F_{(x,u)}^1(t)) - f(t, x(s), F_{(x,u)}^1(t))| \\ &\quad + |f(t, x(s), F_{(x,u)}^1(t)) - f(s, x(s), F_{(x,u)}^1(s))| \\ &\leq \eta_1(|t - s|) + k_2|x(t) - x(s)| + \eta_2(|t - s|), \end{aligned}$$

because first two modules of the right side of the above inequality have the upper estimation $\eta_1(|t - s|)$, where η_1 is a nonnegative function such that $\lim_{h \rightarrow 0^+} \eta_1(h) = 0$, third module has the upper bound $k_2|x(t) - x(s)|$ due to the assumption (20), and

$\eta_2(|t-s|)$ (nonnegative function such that $\lim_{h \rightarrow 0^+} \eta_2(h) = 0$) is the upper estimation of the fourth module. Finally,

$$|D^{\alpha,\beta} F^1_{(x,u)}(t) - D^{\alpha,\beta} F^1_{(x,u)}(s)| \leq \eta(|t-s|) + k_2|x(t) - x(s)|,$$

where $\eta = \eta_1 + \eta_2$. This implies

$$\omega(D^{\alpha,\beta} F^1_{(x,u)}, h) \leq k_2\omega(D^{\alpha,\beta} x, h) + \eta(h).$$

Hence for any set $S \subset \Theta$: $\omega_0(F^2_{(x,u)} S) = 0$ and $\omega_0(D^{\alpha,\beta} F^1_{(x,u)} S) \leq k_2\omega_0(D^{\alpha,\beta} S_1)$, where S_1 is the natural projection of the set S on the space $C_n^{\alpha,\beta}[0, t_1]$. Defining the measure of noncompactness as $\mu(S) = \frac{1}{2}\omega_0(S)$, we have

$$\mu(F_{(x,u)} S) \leq k_2\mu(S).$$

Applying the Darbo fixed point theorem, we conclude that the function $F_{(x,u)}$ has a fixed point. It follows that there exist functions $\tilde{x} \in C_n^{\alpha,\beta}[0, t_1]$ and $\tilde{u} \in C_m[0, t_1]$ such that $\tilde{x}(t) = F^1_{(\tilde{x}, \tilde{u})}(t)$ and $\tilde{u}(t) = F^2_{(\tilde{x}, \tilde{u})}(t)$. Hence the control steering the system (1)–(2) from the initial state x_0 to a final state $\tilde{x}(t) = x(t_1)$ at any time $t_1 > 0$ is defined as

$$\tilde{u}(t) = B^*(\tau)\Phi^*(t_1 - t)W^{-1}(t_1) \left(\tilde{x} - x_0 t^{(\alpha-1)(1-\beta)} E_{\alpha, \alpha+(1-\alpha)\beta}(At^\alpha) - H(u) \right).$$

□

Theorem 6 Let $U \subset \mathbb{R}^m$ be a convex and compact set containing 0 in its interior, and let every $u(t) \in U$ be an admissible control. If the linear Hilfer fractional system (6)–(7) is controllable on $[0, t_1]$, the inequality (13) holds and $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$, $1 \leq i \leq n$, where λ_i are the eigenvalues of the matrix A , then the semilinear Hilfer fractional system (1)–(2) is null U -controllable on $[0, t_1]$.

Proof. We assume that inequality (13) is satisfied and the linear Hilfer fractional system (6)–(7) with unconstrained control is controllable on $[0, t_1]$. Applying Theorem 3, we conclude that the semilinear Hilfer fractional system (1)–(2) is controllable on $[0, t_1]$, while the control is unconstrained. Suppose further that $U \subset \mathbb{R}^m$ is a convex and compact set containing 0 in its interior and $u(t) \in U$. If the eigenvalues of the matrix A satisfy the condition $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$, $1 \leq i \leq n$, then the Hilfer fractional system (1)–(2) is asymptotically stable (see: [32]). Because of the asymptotic stability assumption, $x = 0$ is the solution of the system (1)–(2)

for the admissible control $u(t) = 0$. Using the null control $u(t) = 0$, the solution $x(t)$ of (1)–(2) satisfies the conditions

$$\lim_{t \rightarrow +\infty} x(t) = 0 \quad \text{and} \quad x(t_1) \in N(0),$$

for some $0 < t_1 < +\infty$, where $N(0)$ is a sufficiently small neighborhood of $0 \in \mathbb{R}^n$. From the assumption of convexity and compactness of the set $U \subset \mathbb{R}^m$, we conclude that any state $x(t_1)$ can be steered to $0 \in \mathbb{R}^n$ in finite time. Therefore the semilinear Hilfer fractional system (1)–(2) is null U -controllable on $[0, t_1]$. \square

Theorem 7 *Let $U \subset \mathbb{R}^m$ be a convex and compact set containing 0 in its interior, and let every $u(t) \in U$ be an admissible control. If the linear Hilfer fractional system (6)–(7) is controllable on $[0, t_1]$, inequalities (19)–(20) hold and $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$, $1 \leq i \leq n$, where λ_i are the eigenvalues of the matrix A , then the semilinear Hilfer fractional system (1)–(2) is null U -controllable on $[0, t_1]$.*

Proof. Assuming that inequalities (19)–(20) are satisfied and the linear Hilfer fractional system (6)–(7) with unconstrained control is controllable on $[0, t_1]$, we apply Theorem 5 and conclude that the semilinear Hilfer fractional system (1)–(2) is controllable on $[0, t_1]$ with unconstrained controls. If, additionally, $u(t) \in U$, $U \subset \mathbb{R}^m$ which is a convex and compact set containing 0 in its interior and the eigenvalues of the matrix A satisfy the condition $|\arg(\lambda_i)| > \frac{\alpha\pi}{2}$, $1 \leq i \leq n$, then the Hilfer fractional system (1)–(2) is asymptotically stable. The justification is just the same as in the proof of Theorem 6. \square

5. Illustrative examples

In this section, a numerical example is presented to illustrate the theoretical results obtained. Moreover, an example of a practical problem related to electric circuits is given, modeled by a semilinear Hilfer fractional differential equation, which is considered in this paper.

Example 1 (Numerical example)

Consider the following semilinear Hilfer fractional control system

$$D^{\frac{1}{3}, \frac{1}{2}} x(t) = Ax(t) + B(t)u(t) + f(t, x(t)), \quad (21)$$

$$I^{(1-\alpha)(1-\beta)} x(0+) = 0, \quad (22)$$

where $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & t \\ 1 & 0 \end{bmatrix}$, and $f(t, x(t), u(t)) = \begin{bmatrix} \sin x(t) \\ \cos u(t) \end{bmatrix}$, for $t \in [0, 1]$.

For $\alpha = \frac{1}{3}$ and $n = 2$ the matrix $\Phi(t)$ takes the form

$$\begin{aligned} \Phi(t) &= \sum_{k=0}^1 \frac{A^k t^{(k+1)\frac{1}{3}-1}}{\Gamma((k+1)\frac{1}{3})} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \frac{t^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)} \\ &= \begin{bmatrix} \frac{t^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} + \frac{t^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)} & 0 \\ 0 & \frac{t^{-\frac{2}{3}}}{\Gamma\left(\frac{1}{3}\right)} + \frac{2t^{-\frac{1}{3}}}{\Gamma\left(\frac{2}{3}\right)} \end{bmatrix}. \end{aligned}$$

Hence the Grammian matrix for the system (21)

$$\begin{aligned} W(1) &= \int_0^1 \Phi(t-\tau)B(t)B(t)^*\Phi^*(t-\tau)d\tau \\ &= \begin{bmatrix} \frac{3}{5\left(\Gamma\left(\frac{1}{3}\right)\right)^2} + \frac{1}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)} + \frac{3}{7\left(\Gamma\left(\frac{2}{3}\right)\right)^2} & 0 \\ 0 & \frac{-3}{\left(\Gamma\left(\frac{1}{3}\right)\right)^2} + \frac{12}{\left(\Gamma\left(\frac{2}{3}\right)\right)^2} \end{bmatrix} \end{aligned}$$

is nonsingular. Therefore, by Remark 2, the linear system corresponding to (21)-(22) is controllable on $[0, 1]$.

Moreover, f satisfies condition (13). That is

$$\|f(t, x(t))\| = \|\sin x(t) + \cos u(t)\| \leq \|\sin x(t)\| + \|\cos u(t)\| \leq |x(t)| + |u(t)|^{\frac{1}{2}}.$$

Since $a = 1$, $n = 2$ and $N = 1$, for $t \in [0, 1]$ we have

$$\begin{aligned} \frac{1}{\gamma\sqrt{2}}n^2N^2M^3a\sqrt{t_1}e^{aMt_1}\left(\frac{e^{2\varrho t_1}-1}{2\varrho}\right)^{\frac{3}{2}} &= \frac{4}{\gamma\sqrt{2}}M^3e^M\left(\frac{e^{2\varrho}-1}{2\varrho}\right)^{\frac{3}{2}} \\ &= \frac{2\sqrt{2}}{\gamma}M^3e^M\left(\frac{e^{2\varrho}-1}{2\varrho}\right)^{\frac{3}{2}}. \end{aligned}$$

For $\varrho \rightarrow 0$, $\left(\frac{e^{2\varrho} - 1}{2\varrho}\right)^{\frac{3}{2}} \rightarrow 1$, so γ and M can be found such that $\|\Phi(t_1)\| \leq M$ and $2\sqrt{2}M^3 e^M < \gamma$. Then condition (14) holds, and using Theorem 3, we conclude that the semilinear Hilfer fractional-order system (21)–(22) is controllable on $[0, 1]$.

Example 2 (Practical example)

Consider a transient process in the electrical circuit, that is, a process of transition from one operating mode to another, differing with parameters, shown in Fig. 1.

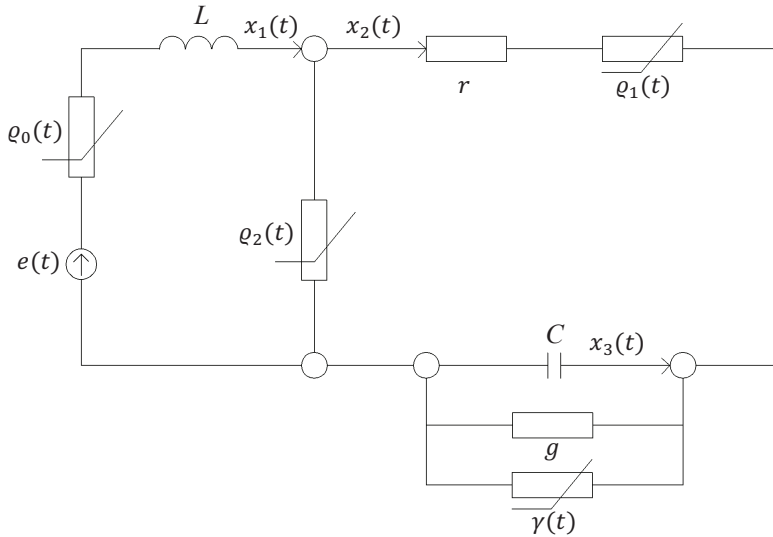


Figure 1: Electrical circuit

Applying the Hilfer fractional derivative, the following system of fractional differential equations describes the process.

$$\begin{cases} D^{\alpha,\beta} x_1(t) = -\frac{1}{L} x_2(t) - \frac{r}{L} x_3(t) + \frac{1}{L} e(t) - \frac{1}{L} \varrho_0(x_1(t)) - \frac{1}{L} \varrho_1(x_3(t)), \\ D^{\alpha,\beta} x_2(t) = -\frac{g}{C} x_2(t) + \frac{1}{C} x_3(t) - \frac{1}{C} \gamma(x_2(t)), \\ 0 = -x_2(t) - r x_3(t) + \varrho_2(x_1(t) - x_3(t)) - r_1(x_3(t)), \end{cases}$$

where $0 < \alpha < 1$, $0 \leq \beta \leq 1$, L is a given inductance, C is a given capacitance, I is the current intensity, U is the current voltage, $x_1(t) = I_L$, $x_2(t) = U_C$, $x_3(t) =$

I are coordinates of a state vector, $e(t) \in L^2_{\text{loc}}([0, \infty), \mathbb{R}^m)$ is a continuous voltage source (control), $L, C, r, g \in \mathbb{R}_+$ are given constants. Moreover, nonlinear resistances ϱ_0, ϱ_1 and ϱ_2 , and a nonlinear conductance γ are given continuous functions with continuous its Hilfer $D^{\alpha, \beta}$ derivative.

6. Concluding remarks

In the paper, the controllability of semilinear Hilfer fractional control systems is discussed. It was assumed that the nonlinear function f is smooth enough and satisfies either condition (13) or conditions (19)–(20). Definitions of controllability from a given initial state, local and global controllability on the interval $[0, t_1]$, and null controllability with constrained controls for the systems were formulated. The formula for solution of the Hilfer fractional-order initial value problem (1)–(2) was given and proved. The main result of the paper is new criteria for the controllability of the semilinear Hilfer fractional systems, which have been established and proved. The first criterion (Theorem 3) is based on Rothe's fixed point theorem and concerns the fractional differential system with unconstrained control. The second criterion (Theorem 5) is the constrained controllability criterion formulated on the basis of the Darbo fixed point theorem. The third and fourth criteria are the sufficient conditions for null controllability with constrained controls (Theorem 6 and Theorem 7). The numerical example was presented to show how the controllability of the systems under consideration can be checked using the criterion established (Example 1). The practical example was included to provide a possible application of the study (Example 2).

The Rothe's and generalized Darbo fixed point theorems are useful tools for studying the controllability of semilinear and nonlinear fractional order dynamical systems. The presented results can be generalized to the case of the function f dependent on a delayed argument.

References

- [1] S. ABBAS, M. BENCHOHRA, N. HAMIDI, and J.J. NIETO: Hilfer and Hadamard fractional differential equations in Fréchet spaces. *TWMS Journal of Pure and Applied Mathematics*, **10**(1), (2019), 102–116.
- [2] A. AGHAJANI, J. BANAŚ, and N. SABZALI: Some generalizations of Darbo fixed point theorem and applications. *Bulletin of the Belgian Mathematical Society – Simon Stevin*, **20**(2), (2013), 345–358. DOI: [10.36045/bbms/1369316549](https://doi.org/10.36045/bbms/1369316549).

- [3] K. BALACHANDRAN, J.Y. PARK, and J.J. TRUJILLO: Controllability of nonlinear fractional dynamical systems. *Nonlinear Analysis*, **75**(4), (2012), 1919–1926. DOI: [10.1016/j.na.2011.09.042](https://doi.org/10.1016/j.na.2011.09.042).
- [4] J. BANAŚ: On measures of noncompactness in Banach space. *Commentationes Mathematicae Universitatis Carolinae*, **21**(1), (1980), 131–143.
- [5] R. CAPONETTO, G. DONGOLA, L. FORTUNA, and I. PETRÁŠ : Fractional Order Systems: Modeling and Control Applications, In: *World Scientific Series on Nonlinear Science*, **72**, 2010. DOI: [10.1142/7709](https://doi.org/10.1142/7709).
- [6] C. DACKA: On the controllability of a class of nonlinear systems. *IEEE Transactions on Automatic Control*, **25**(2), (1980), 263–266. DOI: [10.1109/TAC.1980.1102287](https://doi.org/10.1109/TAC.1980.1102287).
- [7] A. DEBBOUCHE and D.F.M. TORRES: Approximate controllability of fractional nonlocal delay semilinear systems in Hilbert spaces. *International Journal of Control*, **86**(9), (2013), 1577–1585. DOI: [10.1080/00207179.2013.791927](https://doi.org/10.1080/00207179.2013.791927).
- [8] K.M. FURATI, M.D. KASSIM and N.E. TATAR: Existence and uniqueness for a problem involving Hilfer fractional derivative. *Computers & Mathematics with Applications*, **64**(6), (2012), 1616–1626. DOI: [10.1016/j.camwa.2012.01.009](https://doi.org/10.1016/j.camwa.2012.01.009).
- [9] F. GE, Y. CHEN, C. KOU, and I. PODLUBNY: On the regional controllability of the sub-diffusion process with Caputo fractional derivative. *Fractional Calculus and Applied Analysis*, **19** (2016), 1262–1281. DOI: [10.1515/fca-2016-0065](https://doi.org/10.1515/fca-2016-0065).
- [10] E. GEROLYMATOU, I. VARDOLAKIS, and R. HILFER: Modelling infiltration by means of a nonlinear fractional diffusion model. *Journal of Physics D: Applied Physics*, **39** (2006), 4104–4110. DOI: [10.1088/0022-3727/39/18/022](https://doi.org/10.1088/0022-3727/39/18/022).
- [11] A. HAQ: Partial-approximate controllability of semi-linear systems involving two Riemann-Liouville fractional derivatives. *Chaos, Solitons & Fractals*, **157** (2022), 111923. DOI: [10.1016/j.chaos.2022.111923](https://doi.org/10.1016/j.chaos.2022.111923).
- [12] A. HAQ and N. SUKAVANAM: Existence and partial approximate controllability of nonlinear Riemann-Liouville fractional systems of higher order. *Chaos, Solitons & Fractals*, **165** (2022), 112783. DOI: [10.1016/j.chaos.2022.112783](https://doi.org/10.1016/j.chaos.2022.112783).
- [13] R. HILFER: *Applications of Fractional Calculus in Physics*. World Scientific, Singapore, 2000.

- [14] R. HILFER: Experimental evidence for fractional time evolution in glass forming materials. *Chemical Physics*, **284**(1-2), (2002), 399–408. DOI: [10.1016/S0301-0104\(02\)00670-5](https://doi.org/10.1016/S0301-0104(02)00670-5).
- [15] R. HILFER, Y. LUCHKO, and Ž. TOMOVSKI: Operational method for solution of the fractional differential equations with the generalized Riemann-Liouville fractional derivatives. *Fractional Calculus & Applied Analysis*, **12**(3), (2009), 299–318.
- [16] G. ISAC: On Rothe’s fixed point theorem in a general topological vector space. *An. St. Univ. Ovidius Constanta*, **12**, (2004), 127–134 .
- [17] E. ITURRIAGA and H. LEIVA: A characterization of semilinear surjective operators and applications to control problems. *Applied Mathematics*, **1**(4), (2010), 265–273. DOI: [10.4236/am.2010.14033](https://doi.org/10.4236/am.2010.14033).
- [18] T. KACZOREK: Realization problem for fractional continuous-time systems. *Archives of Control Sciences*, **18**(1), (2008), 43–58.
- [19] T. KACZOREK: *Selected Problems of Fractional Systems Theory*. Lecture Notes in Control and Information Science, **411**, 2011.
- [20] T. KACZOREK: A new method for computation of positive realizations of fractional linear continuous-time systems. *Archives of Control Sciences*, **28**(4), (2018), 511–525. DOI: [10.24425/acs.2018.125481](https://doi.org/10.24425/acs.2018.125481).
- [21] K. KAVITHA, V. VIJAYAKUMAR, R. UDHAYAKUMAR, N. SAKTHIVEL, and K.S. NISAR: A note on approximate controllability of the Hilfer fractional neutral differential inclusions with infinite delay. *Mathematical Methods in the Applied Science*, John Wiley & Sons, 2020, pp. 1–20. DOI: [10.1002/mma.7040](https://doi.org/10.1002/mma.7040).
- [22] A.A. KILBAS, H.M. SRIVASTAVA, and J.J. TRUJILLO: *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, **204**, 2006.
- [23] J. KLAMKA: *Controllability of Dynamical Systems*. Kluwer Academic Publishers, 1991.
- [24] K.D. KUCHE, A. FERNANDEZ, and H.M. FAHAD: On tempered Hilfer fractional derivatives with respect to functions and the associated fractional differential equations. *Chaos, Solitons & Fractals*, **163**, (2022). DOI: [10.1016/j.chaos.2022.112547](https://doi.org/10.1016/j.chaos.2022.112547).

- [25] V. KUMAR, M. MALIK, and D. BALEANU: Results on Hilfer fractional switched dynamical system with non-instantaneous impulses. *Pramana – Journal of Physics*, **96**, 172 (2002). DOI: [10.1007/s12043-022-02411-1](https://doi.org/10.1007/s12043-022-02411-1).
- [26] H. LEIVA: Rothe’s fixed point theorem and controllability of semilinear nonautonomous systems. *Systems and Control Letters*, **67**(1), (2014), 14–18. DOI: [10.1016/j.sysconle.2014.01.008](https://doi.org/10.1016/j.sysconle.2014.01.008).
- [27] A. MONJE, Y. CHEN, B.M. VIAGRE, D. XUE and V. FELIU: *Fractional-order Systems and Controls. Fundamentals and Applications*. Springer-Verlag, 2010.
- [28] R.J. NIRMALA, K. BALACHANDRAN, L. RODRIGUEZ-GERMA, and J.J. TRUJILLO: Controllability of Nonlinear Fractional Delay Dynamical Systems. *Reports on Mathematical Physics*, **77**(1), (2016), 87–104. DOI: [10.1016/S0034-4877\(16\)30007-6](https://doi.org/10.1016/S0034-4877(16)30007-6).
- [29] R.J. NIRMALA and K. BALACHANDRAN: The Controllability of Nonlinear Implicit Fractional Delay Dynamical Systems. *International Journal of Applied Mathematics and Computer Science*, **27**(3), (2017), 501–513. DOI: [10.1515/amcs-2017-0035](https://doi.org/10.1515/amcs-2017-0035).
- [30] I. PODLUBNY: Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. In: *Mathematics in Science and Engineering*, Academic Press, 1999.
- [31] B. RADHAKRISHNAN and T. SATHYA: Controllability of Hilfer fractional Langevin dynamical system with impulse in an abstract weighted space. *Journal of Optimization Theory and Applications*, **195**(1), (2022), 265–281. DOI: [10.1007/s10957-022-02081-4](https://doi.org/10.1007/s10957-022-02081-4).
- [32] H. REZAZADEH, H. HOSSEIN AMINIKHAH, and A.H. REFAHI SHEIKHANI: Stability analysis of Hilfer fractional differential systems. *Mathematical Communications*, **21** (2016), 45–64.
- [33] M. RÓŻAŃSKI, B. SIKORA, A. SMUDA, and R. WITUŁA: On theoretical and practical aspects of Duhamel’s integral. *Archives of Control Sciences*, **31**(4), (2021), 815–847. DOI: [10.24425/acs.2021.139732](https://doi.org/10.24425/acs.2021.139732).
- [34] J. SABATIER, O.P. AGRAWAL, and J.A. TENREIRO MACHADO: Advances in Fractional Calculus. In: *Theoretical Developments and Applications in Physics and Engineering*. Springer-Verlag, 2007.

- [35] R. SAKTHIVEL, Y. REN, and N.I. MAHMUDOV: On the approximate controllability of semilinear fractional differential systems. *Computers and Mathematics with Applications*, **62**(3), (2011), 1451–1459. DOI: [10.1016/j.camwa.2011.04.040](https://doi.org/10.1016/j.camwa.2011.04.040).
- [36] S.G. SAMKO, A.A. KILBAS, and O.I. MARICHEV: *Fractional integrals and derivatives: theory and applications*. Gordon and Breach Science Publishers, 1993.
- [37] H. SHENG, Y.Q. CHEN, and T.S. QIU: *Fractional Processes and Fractional-Order Signal Processing*. Springer, New York, 2012.
- [38] B. SIKORA and J. KLAMKA: Cone-type constrained relative controllability of semilinear fractional systems with delays. *Kybernetika*, **53**(2), (2017), 370–381. DOI: [10.14736/kyb-2017-2-0370](https://doi.org/10.14736/kyb-2017-2-0370).
- [39] B. SIKORA: On application of Rothe’s fixed point theorem to study the controllability of fractional semilinear systems with delays. *Kybernetika*, **55**(4), (2019), 675–698. DOI: [10.14736/kyb-2019-4-0675](https://doi.org/10.14736/kyb-2019-4-0675).
- [40] B. SIKORA and N. MATLOK: On controllability of fractional positive continuous-time linear systems with delay. *Archives of Control Sciences*, **31**(1), (2021), 29–51. DOI: [10.24425/acs.2021.136879](https://doi.org/10.24425/acs.2021.136879).
- [41] V. SINGH: Controllability of Hilfer fractional differential systems with non-dense domain. *Numerical Functional Analysis and Optimization*, **40**(13), (2019), 1572–1592. DOI: [10.1080/01630563.2019.1615947](https://doi.org/10.1080/01630563.2019.1615947).
- [42] J.D.R. SMART: *Fixed Points Theorems*. Cambridge University Press, 1974.
- [43] H.M. SRIVASTAVA and Ž. TOMOVSKI: Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel. *Applied Mathematics and Computation*, **211**(1), (2009), 198–210. DOI: [10.1016/j.amc.2009.01.055](https://doi.org/10.1016/j.amc.2009.01.055).
- [44] J. TOKARZEWSKI: Zeros and the output-zeroing problem in linear fractional-order systems. *Archives of Control Sciences*, **18**(4), (2008), 437–451.
- [45] V. VIJAYAKUMAR and R. UDHAYAKUMAR: A new exploration on existence of Sobolev-type Hilfer fractional neutral integro-differential equations with infinite delay. *Numerical Methods for Partial Differential Equations*, (2020), 1–17. DOI: [10.1002/num.22550](https://doi.org/10.1002/num.22550).
- [46] B.J. WEST, M. BOLOGNA, and P. GRIGOLINI: *Physics of Fractal Operators*. Springer, New York, 2003.

-
- [47] A. WONGCHAROEN, K. SOTIRIS, S.K. NTOUYAS, and J. TARIBOON: Boundary value problems for Hilfer fractional differential inclusions with nonlocal integral boundary conditions. *Mathematics*, **8**(11), (2020), 1905. DOI: [10.3390/math8111905](https://doi.org/10.3390/math8111905).
- [48] G. ZHANG, T. ZHANG, and T. ZHANG: Fixed point theorems of Rothe and Altman types about convex-power condensing operator and application. *Applied Mathematics and Computation*, **214**, (2009), 618–623. DOI: [10.1016/j.amc.2009.04.033](https://doi.org/10.1016/j.amc.2009.04.033).