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Review paper

Research on general theory and methodology in geodesy in Poland in 2019–2022

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Abstract: We present a summary of research carried out in 2019–2022 in Poland in the area of general theory and methodology in geodesy. The study contains a description of original contributions by authors affiliated with Polish scientific institutions. It forms part of the national report presented at the 28th General Assembly of the International Union of Geodesy and Geophysics (IUGG) taking place on 11-20 July 2023 in Berlin, Germany. The Polish authors developed their research in the following thematic areas: robust estimation and its applications, prediction problems, cartographic projections, datum transformation problems and geometric geodesy algorithms, optimization and design of geodetic networks, geodetic time series analysis, relativistic effects in GNSS (Global Navigation Satellite System) and precise orbit determination of GNSS satellites. Much has been done on the subject of estimating the reliability of existing algorithms, but also improving them or studying relativistic effects. These studies are a continuation of work carried out over the years, but also they point to new developments in both surveying and geodesy. We hope that the general theory and methodology will continue to be so enthusiastically developed by Polish authors because although it is not an official pillar of geodesy, it is widely applicable to all three pillars of geodesy.

Keywords: robust estimation, relativistic effects, prediction problems, datum transformation, geodetic time series

1. Introduction

This contribution is part of the Polish National Report on geodesy to the International Association of Geodesy (IAG) presented at the General Assembly of the International Union of Geodesy and Geophysics (IUGG) on the topic of general theory and methodology in geodesy. The research described below was carried out by authors affiliated with Polish scientific institutions in the period 2019–2022. The report is, so to speak, a continuation of the previous report published by Borkowski et al. (2019), which described the original achievements of Polish researchers in the period 2015-2018. Most of the studies published by Polish researchers were carried out within international collaboration and published in JCR-indexed journals. They included analyses conducted on robust estimation and its applications, prediction problems, cartographic projections, datum transformation problems, geometric geodesy algorithms, optimization and design of geodetic networks, geodetic time series analysis, relativistic effects in the Global Navigation Satellite System (GNSS) and precise orbit determination of GNSS satellites. The research reported focused on discussing the Msplit estimation and developing its modifications to achieve better stability of the method. Within the topic of prediction problems, the authors concentrated on improving the least squares collocation method and its comparison with kriging for different types of geodetic datasets. Much has also been done to improve cartographic projections and improve transformation between different geodetic coordinates. The attempts have been made to analyse geodetic networks, to assess their reliability and optimise them. An effort has also been made to estimate the defect of the existing networks. An issue that is also frequently addressed by Polish authors concerns geodetic time series analysis. In this field, the authors estimated the effectiveness of existing methods, but also improved existing approaches resulting in the increase of the applicability of time series. Finally, the attempts were made to explain relativistic effects using GNSS observations.

2. Robust estimation and its applications

Since its introduction by Wisniewski (2009, 2010) M_{split} estimation has been experiencing constant development. Recent advancements by Duchnowski and Wisniewski (2019, 2020) consider robustness of $M_{\text{split}\,(q)}$ estimation method from both theoretical and empirical perspectives. M_{split} estimation is a method that can be derived as a development (or generalization) of M-estimation theory, under an assumption that a functional model for observations may be broken up into q competitive ones. To study the issue of robustness of $M_{\text{split}\,(q)}$ estimation local and global breakdown points are introduced, denoted as (LBdP), and (GBdP), respectively. In general, a breakdown point informs how many blunders a method may handle until it fails to provide acceptable results. Since $M_{\text{split}\,(q)}$ estimation concerns q competitive models, LBdP measures robustness of adjacent pair of parameters θ_i, $\theta_{(i+1)}$ whilst GBdP measures the overall robustness of estimates coming from all competitive models. The authors state that the maximum value of LBdP is 50% but, what is striking, exceeding this value, $M_{\text{split}\,(q)}$ estimation does not break down. This is explained by swapping places by adjacent estimates and the point is called a reversal point. On the other hand, from general perspective $M_{\text{split}\,(q)}$ is not robust to outliers since GBdP is zero, but if the number of competitive functional models will be extended to q+1 then outlying observations will be included therein. This operation cleans the remaining functional models and this may be perceived as robustness of M_{split} estimation method. Empirical analysis shows equivalence between empirical LBdP and corresponding theoretical derivations. However, GBdP strongly relies on prescribed number of competitive models q. If the value matches reality M_{split} method is a good alternative to conventional M-estimation what has been proven empirically with the use of Huber's method. Conversely, when q is mismatched then the method may fail. Also, Wisniewski and Zienkiewicz (2021a) examined robustness of (squared) M_{split} estimation confirming high efficiency of the method in identifying gross errors and assigning them to competitive model with respect to a model that includes clean observations. The authors point out also very important feature of M_{split} estimation which is the independence of subjectively selected parameters controlling robustness (e.g., tuning constants). This particular feature should attract attention and contribute to further development of the family of M_{split} estimation methods.

Generalization of M_{split} estimation in the framework of errors-in-variables (EIV) model was introduced by Wisniewski (2022) under the name of Total M_{split} estimation. The EIV model allows for inclusion random errors in both observations and a design matrix what makes it more realistic. The procedure itself is similar to that one of conventional M_{split} estimation with the replacement of classical versions of competitive models with EIV ones. Such a replacement makes the optimization problem much more complex than it is in the conventional case. A Lagrange objective function, that takes into account EIV competitive models as side conditions, is used to derive the solution algorithm that is based on the Gauss–Newton method applied to linearized EIV models. TM_{split} estimation absorbs M_{split} estimation in a sense that if a design matrix $\mathbf{A}$ is not contaminated by random errors, then TM_{split} and M_{split} estimators coincide.

On the other hand, Wyszkowska and Duchnowski (2019) proposed a modification of M_{split} estimation where the least-squares based objective function is replaced with L1 norm optimization problem. This new variant is called the absolute M_{split} estimation (AMS). The authors found this variant less sensitive to outliers or inadequate assignment of an observation to one of the competitive functional models than the conventional squared M_{split} estimation (SMS). Also, the solution seems to be more stable with respect to an adopted initial guess that begins the iterative procedure.

Interesting and up-to-date application of M_{split} estimation was presented by Wyszkowska and Duchnowski (2022). The authors apply the method to processing of Terrestrial Laser Scanning (TLS) data. In fact, they modify the method to satisfy the needs of robustness against outliers since the basic variant of M_{split} estimation was not meant to be such a method in conventional understanding of M-estimation. This improvement relies on modification of original influence functions (also objective and weighting functions) of SMS and AMS estimations into Huber's and Tukey's like influence function and are named SMSH, SMST and AMSH, AMSR, respectively. Practical aspect of this contribution concentrates on processing heterogenous TLS data (including outliers) with newly developed methods and confronting results obtained with their classical counterparts. The study shows that SMS and AMS may fail whilst SMST and AMSR provide satisfactory results. Huber's like variants seem to be more sensitive to outliers. The examples reveal that the newly introduced robustified version of M_{split} estimation outperforms considered M-estimators which fail when a data set includes a relatively high number of blunders. In addition, M_{split} estimation may yield correct results even if the percentage of outliers exceeds 50%.

Zienkiewicz (2022) proposed a new version of the squared M_{split} estimation which allows the estimation of competitive parameters in a split functional model in constrained datums. The emphasis is put on robustness and efficiency of the method in geodetic network deformation analysis. The presented approach can be successfully used in the process of identifying reference datum.

In Wisniewski and Zienkiewicz (2021b) the theory of $M_{\text{split}\,(q)}$ estimation has been supplemented with the valuable missing element that is a derivation of covariance matrices for $M_{\text{split}\,(q)}$ estimators. This extends the possible use of the method since accuracy analysis is, in fact, a fundamental step in every adjustment procedure. Construction of appropriate covariance matrices is based on an empirical influence function and estimators of variance coefficients (scale factors). The paper presents two versions of covariance matrices for M_{split} estimators originating from a single general formula but derived with different assumptions and applying different variance coefficients. Despite the fact that they differ in assumptions they provide comparable values of standard deviations of M_{split} estimates.

3. Prediction problem

The Least Squares Collocation (LSC) method, introduced by Torben Krarup and Helmut Moritz, has been of interest to the surveying community since its development in the 1960s, and this interest has been sustained to the present. Ligas (2022) compared LSC with geostatistical method of kriging. The methods are compared under the same conditions, however, it is known that they were created to satisfy different needs thus they are equipped with specialized tools needed in particular field of application. The author demonstrates equivalence between the methods under the assumption of a second-order structure of a random function. It is shown that simple kriging (constant and known mean value) is equivalent to the least-squares prediction (interpolation) formula whilst filtered version of simple kriging is a counterpart of least-squares collocation with random errors. Universal kriging (unknown and spatially varying mean value – trend) as a filter and its special case ordinary kriging (unknown and constant mean) are equivalent to least-squares collocation with parameters. The paper clarifies also the issue of exact and filtered prediction. These two variants provide the same values on newly predicted (out-of-sample) points and filtered kriging, in this case, has a smaller prediction variance since the total signal is less variable than observed data. On the other hand, they provide different prediction results on observed data-points (in-sample). In this case, the exact model honors the data giving the same value of prediction as observed at a given point with zero prediction variance. In contrast, the filtered model provides the predicted value different from observed one (non-exact prediction) with non-zero prediction variance. Both versions provide exactly the same outcomes (in corresponding configurations of points) when there is no noise present in the data (or it is neglected).

The problem of uncorrelated noise variance in least-squares spatial prediction was studied numerically, with the application of local gravity data and EGM2008 model by Jarmolowski (2019). The meaning of noise variance level is explained in this paper, and the sources of the noise are carefully examined. The studies applied LSC and revealed its relationships with the spectral signal properties. The terrestrial Bouguer anomalies, have a large variance at higher signal frequencies, i.e. their power spectral density (PSD) decreases slowly when the spatial resolution increases. The same quantity was calculated from the EGM2008 model using various degrees of spherical harmonic expansion. The different degrees of harmonics were used to remove some spectral part of the signal from the data, which revealed the relation of the noise variance with medium and high-frequency signal parts. The observed statistical quantities proved that the noise level is related with signal spectral range and data spatial resolution. The paper provided a relevant proof that the noise is not solely dependent on the measurement error and explained geometrical meaning of the regularization requirement.

The use of techniques from the kriging family to the interpolation of ionospheric total electron content (TEC) data was the subject of research by a team from the University of Warmia and Mazury (UWM). The work is focused on the local TEC models but also on new global ionosphere map (GIM) developed at UWM. The detrending and parametrization of LSC, which is equivalent to Simple Kriging (SKR), Ordinary Kriging (OKR) and Universal Kriging (UKR) were studied with respect to ionospheric TEC determined at GNSS stations (Jarmolowski et al., 2021). The studies proved similar accuracy derived from different parametric modelling techniques, but a special attention should be put on the parametrization and detrending issues. It was found that local detrending with higher order polynomial surfaces applied to UKR deform kriging modeling results. This deformation is especially high in case of outliers occurrence together with a higher-order detrending in UKR.

Wisniewski and Kaminski (2020) introduced a method for estimating and predicting vertical deformations based on a total least squares collocation method (TLSC). It is a generalization of least squares collocation in which conventional solution was replaced with that of total least squares. Vertical deformation field is treated therein as a random field, i.e., a collection of random variables indexed by a set of planar coordinates. It is assumed that the model underlying observed deformations consists of deterministic and stochastic parts. Generality of LSC in this particular application manifests itself through the opportunity of prediction of displacements on points not being a part of a control network (named extended control points therein) in a single coherent numerical procedure. This extends monitoring of deformations to points that are not observed directly due to different reasons like covering, no access etc. TLSC solution is iterative and the convergence is usually reached after 8–10 iterations but this is dependent on the magnitude of noise present or assumed in data, on the covariance function model and its parameters adopted for computations. In addition, TLSC approach allows not only for the determination of the deterministic and random displacements on control points and extended control points, but also for the estimation of the value of random disturbance at these points.

4. Cartographic projections, datum transformation problems and geometric geodesy algorithms

The problem of triaxiality of reference ellipsoids approximating shapes of celestial bodies, including the Earth and Moon, has attracted attention of geodesists and cartographers for the last decades (Burša and Šima, 1980). This led to the development of algorithms concerning conversion between Cartesian and planetographic coordinates (Feltens, 2009), geodesic lines (Panou, 2013), fitting triaxial ellipsoids (Panou et al., 2020), and cartographic projections (Nyrtsov et al., 2013; Pedzich, 2017) to mention only a few contributions. The work by Pedzich (2019) inscribes in this stream. It presents a construction of a low distortion conformal projection of a triaxial ellipsoid that is based on Chebyshev's theorem on minimization of distortions in the class of conformal projections. The procedure of constructing such a projection consists of several steps which in brief may be listed out as: determination of conformal coordinates on a triaxial ellipsoid, finding formulas for a scale of linear distortion and convergence of meridians, determination of projection functions that link rectangular coordinates to conformal coordinates and determining the coefficients of the polynomials used to approximate the conformal projection that meets Chebyshev's criterion. This elaborate procedure may be examined step-by-step therein. It is worth mentioning that development of cartographic projections for triaxial ellipsoids is driven by outer space exploration since many of the celestial objects as a first approximation may be considered as bodies of such shapes (e.g., Mimas, Enceladus moons of Saturn, or Amalthea a moon of Jupiter, asteroids) therefore it is an up-to-date research field within cartography.

Conversion between Cartesian x, y, z and geodetic ϕ, λ, h coordinates belongs to classical problems of computational geodesy. Although it has already many solutions, new ones are constantly emerging. Kadaj (2020) has modified Getchell's method (Getchell, 1972) by applying new initial guess to the original iterative procedure of Getchell's reducing at the same time the number of necessary iterations. The author includes also Newton's iteration for the solution of the Getchell's nonlinear equation. Both modifications are equipped with a theoretical convergence analysis. The methods keep stability in the entire range of latitudes $(-90°, 90°)$ and a practical range of geodetic heights. The exception is the geocenter region where a great majority of conversion algorithms lose their stability or ability to converge. The implementations of the algorithms in DELPHI programming language are provided in the appendix of the paper what is particularly valuable.

In addition to the problem of coordinate conversion, a problem that is constantly attracting the attention of the academic community is the issue of coordinate transformations. Ligas and Prochniewicz (2021) gave a closed-form solution to the point-wise weighted rigid-body transformation (only rotations and translations involved, weights assigned to points not to separate coordinates) for asymmetric and symmetric cases. Asymmetric case assumes that either a source system or a target system is subject to random errors. On the other hand, symmetric case enables the inclusion of random errors in both systems simultaneously. A collection of developed estimation algorithms use Procrustean framework to find solutions for transformation parameters in all considered adjustment scenarios. The solutions are based on the use of polar decomposition and singular value decomposition of matrices resulting from the optimality criteria. Provided solutions are attractive since they require neither linearisation nor iteration. In addition, formulas are universal in the sense that they fit to 2D and 3D transformation problems without modifications. Theoretical considerations are supplemented with step-by-step algorithms and numerical examples. Ligas (2020) extends the solution given for a rigid-body transformation to the general similarity transformation (Helmert's transformation) also applying point-wise weighting scheme and solving the problem with Procrustean approach. For asymmetric cases, i.e., either a source system (Q) or a target system (P) is subject to random errors, the solutions may be presented in closed-form. The most general case, where both coordinate systems are considered erroneous simultaneously, does not have such a solution. It can be solved iteratively but avoiding linearization process, with an initial guess for parameters obtained from, e.g., a closed-form asymmetric case. Nevertheless, the general case have some special instances which may be presented in closed-form. Those special instances include: equal weight matrices for both coordinate systems $\mathbf{W}_Q = \mathbf{W}_P = \mathbf{W}$ (for arbitrary $\mathbf{W}$), $\mathbf{W}_Q = \mathbf{W}_P = k\mathbf{I}$ (a scalar multiple of an identity matrix), and $\mathbf{W}_Q = \frac{1}{k_Q}\mathbf{W}$, $\mathbf{W}_P = \frac{1}{k_P}\mathbf{W}$ (different scalar multiples of the same weight matrix). The approach presented in this work absorbs the solutions presented by Chang (2015, 2016). In case of the scale factor $s = 1$, the solutions reduce to the ones of a rigid-body transformation given by Ligas and Prochniewicz (2021). They are based on polar decomposition of a certain matrix resulting from optimality criterion (minimization of a Frobenius norm under the side conditions for a rotation matrix) and fit to both 2D and 3D transformation problems without modification.

The work by Kadaj (2021a) relates PL-ETRF89 and PL-ETRF2000 frames through transformational approach using two variants, i.e., theoretical and empirical ones. The first one is solely based on 7-parameter transformation model (Helmert transformation) fit to 330 points of the POLREF network. The second variant is based on the same 330 POLREF points with additional points of the adjusted first class triangulation network (app. 6500 points) that were used to generate an interpolation grid. Gridded differences between coordinates of two systems are the input data for a bilinear interpolation. Comparison of two approaches shows that the similarity transformation alone is not able to fully model the relation between the two reference frames. On the other hand, the empirical approach recommended for practical use reduces to some degree local deformations of the PL-ETRF89 (older) while converting coordinates of points to the PL-ETRF2000 (newer).

Another problem of geometric geodesy – geodesic line on a rotation ellipsoid – that has a rich history dating back to 18th century and strong Polish accent due to works by Thaddeus Vincenty (Tadeusz Szpila) (Vincenty, 1975) was again undertaken by Nowak and Nowak Da Costa (2022). They presented new and effective solutions for geodesic related problems. The authors use a cylindrical coordinate system and a newly introduced geometric construction, i.e., an equatorial geodesic triangle, to derive a novel set of formulas for the longitude, distance along the geodesic and the area of a geodesic polygon. The novelty of the contribution is multifaceted. It introduces an already mentioned equatorial geodesic triangle as a supportive tool for determining the area of a geodesic polygon. It uses Napier's rules to determine the difference between forward and reverse azimuths of a geodesic arc. It also applies a new parameterization of the geodetic inverse problem. The solution is equally well suited for oblate and prolate ellipsoids. The phenomenon of bifurcation having impact on the solution of the inverse geodetic problem is studied. In fact, the paper constitutes a significant contribution to studies on the behavior of a geodesic line on rotational ellipsoids.

5. Optimization and Design of Geodetic Networks

Analyses of control networks

Monitoring of deformation of the engineering structures is an important function of surveying and geodesy. Periodically measured geodetic control networks usually represents the area or object of investigation and the results of their analyses indicate the condition of the tested object. Fundamental role in those analyses plays the identification of stable potential reference points (PRPs). In this issue the robust S-transformation approach has already been used by Chen (1983) and Caspary and Borutta (1987). This approach is still being developed and modified along with the development of geodetic network analysis methods. In this terms, a new approach for the identification of stable potential reference points was proposed by Nowel (2019). The idea of the approach is inspired by the theory of squared $M_{\mathrm{split}(q)}$ estimation and lies in the assumption that between the control networks adjusted in two considered epochs can simultaneously exist many congruences (matchings) which differ by the transformation parameters; in the conventional robust S-transformation existence of one congruence is assumed. That is, one model is assumed to realize a congruence in a subgroup of stable PRPs, and other models may realize different congruences in unstable PRPs. Nowel (2019) proposed statistical hypothesis tests to determine the number of congruence models in a given case and whether the best model is correct. The presented approach was evaluated on the basis of four experiments described in detail.

The deformation congruence models were also investigated in the study of Nowel (2020). The author discussed two widely used approaches to the specification of deformation congruence models: the global congruence test (GCT) procedure and new concept involving combinatorial possibilities (two methods) pointing out their weaknesses. The GCT procedure is based on consecutive point-by-point specification and may suffer from so-called displacement smearing. The combinatorial methods generate another weakness, namely the problem of comparison of different-dimensional models. To address this weaknesses, Nowel (2020) discussed a new combinatorial procedure, denoted as CIDIA (combinatorial iterative detection–identification–adaptation (DIA) testing procedure). This new procedure relied on the appropriate use of combinatorics capabilities and generalized likelihood ratio tests performed in DIA steps, and overcame both of the aforementioned weaknesses. To demonstrate the proposed CIDIA procedure against existing methods, the author presented four experiments.

In the problem of analyses of geodetic control networks also an important role plays the network sensitivity analyses and the Minimal Detectable Displacement (MDD) as its measure. This problem was introduced to the scientific discussion by Baarda (1968) and Pelzer (1972), and became an element of network analysis (Niemeier 1982; Niemeier et al., 1982). This issue was later the subject of many scientific discussions. A new perspective on the problem was proposed by Proszynski and Lapinski (2021). The authors considered the possibility of using MDD as a source of supporting information in a priori analyses of control networks accuracy and in significance tests of displacements. Proszynski and Lapinski (2021) formulated a theoretical basis that would enable to investigate the problem and examine the possibility of modifying these procedures so as to use the supporting information contained in the MDD. The investigations were focused on a probabilistic aspect of combining detectability and confidence as well as significance and detectability by the superimposition of the corresponding ellipsoids and their joint analysis. Based on the analysis of MDD support, the authors proposed two options of modifying the confidence and significance thresholds related to single point displacements for practical use. As they emphasize, the task has not yet been the subject of research presented in the literature in the field of geodetic control networks so, the presented approach can be considered as a new proposal extending the application area of the MDD.

Reliability of geodetic networks analyses

An important element of the analysis of geodetic networks is the reliability of the network and problems related to the measures of this reliability. In that respect Proszynski and Kwasniak (2019) investigated the effect of observation correlations on the basic characteristics of reliability matrix as oblique projection operator. The authors developed the theoretical basis for the use of reliability matrices in designing the positioning systems. They formulate (connected with the effect of observation correlations) properties of an oblique projection operator and provide a more detailed knowledge on variability of the response-based reliability measures with respect to the increase in observation correlations. They also compare the behavior of the response-based reliability measures and of the so-called testing-based measures (i.e. generalized reliability numbers and MDBs) with the increase in observation correlations.

The issue of reliability of geodetic networks was also raised by Proszynski and Lapinski (2019). The authors studied reliability of non-distorting connection (NDC) of engineering survey network. They developed and analysed reliability measures for both, joint and two step adjustment models of NDC networks. The theory was illustrated by numerical examples based on two test networks. The authors showed that the reliability description based on indices obtained from the joint model should be supplemented with the indices from the two-step model. They also identified a relationship between the strength of a new network in terms of internal reliability and mutual controllability of the coordinates of connection points. The provided indices of external reliability include information related to individual connection points and the transformation parameters, so they are more meaningful for the design of non-distorting network connection.

Optimization of geodetic networks

Another current problem concerning the analysis of geodetic networks in engineering surveying is their optimization. An interesting approach presented by Mrowczynska and Sztubecki (2021) consists of the optimization of the measurement and control network structure based on minimizing the objective function defined in the form of information entropy and evolutionary algorithms. The authors accepted the minimum value of the parameter vector entropy as the objective function, whereas evolutionary algorithms were used as an optimization method. Theoretical considerations were supported by numerical analyses based on a test network in three versions: with linear only, angular only and mixed (linear and angular) observations. The proposed approach allows to establish a hierarchy of importance of the individual linear and angular observations as well as determination of the optimal observations number and their mutual distribution in geodetic networks before measurements. In connection to this it allows for shortening the measurement time without reducing the accuracy of the network, and for optimizing the geodetic network structure.

Configuration defect of geodetic networks

The classical problem of identifying and locating local configuration defects of geodetic networks was considered by Kadaj (2021b). The problem of incorrectly defined network structure due to missing data or errors in point numbering prevents the adjustment of such a network. In networks with a small number of points, this problem can be easily detectable and solved by simple, non-automatic analyses of the network structure. In large networks, automatic procedures for identifying and locating this kind of defect must be applied. The author described the use of Tikhonov's regularization method for this purpose. The approach was implemented by the author into the GEONET system and its effectiveness has been confirmed on the example of the adjustment of a large network consisting of over 6,000 network points.

6. Geodetic time series analysis

Geodetic time series analysis has been quite popular in Poland in 2019–2022. First of all, it concerns time series analysis of GNSS station displacements. The researchers used various GNSS permanent station networks, and analyzed both the deterministic as well as the stochastic part of the displacements. Klos et al. (2020) presented two alternative approaches to modelling the deterministic part of GNSS station displacement time series.

The authors compared the conventional approach, which takes into account the trend, the annual component and the semi-annual component, to an approach in which the seasonal curves are replaced with a high-resolution hydrological model assimilating GRACE. The effectiveness of using both approaches was tested by determining the nature of the stochastic part of the time series. The authors proved that the new approach to determining the deterministic part allows reducing the correlation present in the residuals, i.e. whitening them. This is due to the fact that frequencies higher than half a year are modelled by the hydrological model. This proves that some stations are sensitive to short-term changes in the terrestrial hydrosphere. These results have quite large implications especially for the use of GNSS permanent stations as hydrological sensors. Nistor et al. (2021) analysed solutions for a set of 200 EPN GNSS stations obtained in a second reprocessing campaign (Repro-2). The authors focused on determining the nature of the stochastic part of the series. They found that the displacement time series for most of the European stations have a flicker noise character and that the uncertainties of station velocity are significantly lower using the classical Maximum Likelihood Estimation (MLE; Langbein and Svarc, 2019) method than using the MIDAS (Median Interannual Difference Adjusted for Skewness) method (Blewitt et al., 2016). This is quite understandable, as the latter method is advertised by the authors as being robust to offsets and therefore does not require offsets to be removed before velocity determination. Ray et al. (2019) analyzed the displacement time series of 13 permanent GNSS stations located in the Himalayan region of Nepal. They used the Lomb-Scargle method and the nonlinear Levenberg-Marquardt algorithm to determine the index of the power-law noise based on the spectra of time series. They found that the character of the time series depends on the frequency band, i.e. the series has the character of white noise from the highest frequencies up to the frequency of 21 days, and the character of power-law noise for the rest of the frequencies.

Attempts have also been made to study the effectiveness of the algorithms used for geodetic time series analysis, and even to improve them. Najder (2020) analysed the effectiveness of the Finding Outliers and Discontinuities In Time Series (FODITS) program for detecting offsets in GNSS displacement time series. The program is implemented in the Bernese GNSS Software environment (Dach et al., 2015). The author concluded that although automatic detection of spike epochs is much faster than manual detection, one should take into account the errors generated by the algorithm. Maciuk et al. (2020) proposed a simplified presentation of results on the nature of the stochastic part of time series obtained by the Allan variance method. The authors applied the method for five-year-long time series of GPS satellite clock corrections and declared that the method allows qualitative and quantitative expression of the type of noise using the Allan variance without the need for integration steps.

Klos et al. (2019) showed how important the correct analysis of time series is in terms of application, and how the integration of parameters determined for different techniques can support the interpretability of the results. The authors performed analyses for the eastern Pacific area, which is characterized by frequent earthquakes. The largest earthquake, called Tohoku-Oki, affected the area in 2010. It was clearly recorded by GNSS permanent stations located in the area and is visible in the time series of position changes as a function of postseismic deformation. The earthquake was also recorded in observations of sea-level changes at tide-gauges, which show a clear offset in the series at the earthquake epoch. This offset will cause the trends of sea-level to be biased. The authors proposed that the amplitude of the offset should be determined from the time series of GNSS station displacements, whose sensitivity to earthquakes has already been proven, and this value should be implemented to correct tide-gauge observations. The above approach makes the changes in sea-level estimated from tide-gauge observations have better agreement with altimetric observations than before. This is of great importance when including tide-gauges from the Pacific area in studies of sea-level changes.

A definite novelty is the analysis of determining the sensitivity of GNSS stations to environmental effects. Klos et al. (2021) attempted to identify the sensitivity of a GNSS system to environmental crustal loading effects: non-tidal atmospheric, non-tidal oceanic and hydrospheric loading. The authors used time series of vertical displacement changes recorded for GNSS stations located in Asia. The study area was chosen because the atmospheric effect is highly significant there and reliably modelled in present-day environmental loading models. It was observed that the GNSS system is sensitive to changes in environmental loading at different time scales. The hydrospheric effect is well visible in the displacement time series for time scales from seasonal above, the oceanic effect is only visible for short periods, while the atmospheric effect is visible on all time scales, but with varying intensity.

In addition to time series analyses of GNSS station displacements, crustal displacements resulting from terrestrial hydrosphere loading were also analysed. Lenczuk et al. (2020) analyzed river catchments across Europe. Displacements were estimated based on hydrological models and observations from GRACE (Gravity Recovery and Climate Experiment; Tapley et al., 2004) gravity mission. They found that the hydrology-induced displacements of Earth's crust are pronounced for the eastern European area, with a clear trend indicating a long-term decrease in water resources and a pronounced amplitude of the annual signal indicating large variations in water resources through the year. They then used the Singular Spectrum Analysis (SSA) method to determine inter-annual signals. The differences between the displacements derived for GRACE and those predicted by two hydrological models were analysed. They found that the hydrological models underestimate the trend and inter-annual signals observed by GRACE for eastern Europe. The above analyses can support the interpretation of displacements recorded by other surveying techniques, or serve as a reference when testing their sensitivity to changes in the hydrosphere.

7. Relativistic effects in GNSS

The satellites of the Galileo navigation system are orbiting in general on circular orbits. Unfortunately, the first pair of the system accidentally has been launched into highly eccentric orbits. This misplacement of satellites was exploited by Sosnica et al. (2021) to investigate the impact of general relativistic effects on the satellites orbits. The authors used the post-Newtonian parameterization of general relativity and first-order Gaussian perturbations to derive formulas describing theoretical perturbations of Keplerian parameters of Earth-orbiting satellites. The study focused on three general relativity effects: Schwarzschild effect, Lense–Thirring effect and geodetic precession – De Sitter effect. Sosnica et al. (2021) found that the change of the semi-major axis Δa for circular orbits ($e = 0$) is independent of the orbital height and equals on average:

$$\Delta a = -4\frac{GM}{c^2} = -17.74 \text{ mm} \tag{1}$$

for all Earth orbiting satellites. G denotes the gravity constant, M is the mass of the Earth, c is the speed of the light. The mean changes of the semi-major axis is equal to the double value of the Schwarzschild radius for the Earth, i.e., the hypothetical size of the black hole of the mass equal to the Earth's mass. For eccentric orbits, the mean offset due to the Schwarzschild effect changes over time and equals to –7.8 and –28.3 mm in perigee and apogee, respectively for Galileo E14 and E18 (Sosnica et al., 2021). De Sitter and Lense–Thirring cause small offsets of the semi-major axis for a satellite at Galileo orbit heights of about +0.46 and −0.07 mm, respectively.

The authors also found that the geodetic precession depends on the elevation of the Sun above the orbital plane. The rate of the ascending node depends on the maximum possible elevation of the Sun above the orbital plane, which was not considered before in the literature (the constant rate due to the geodetic precession was considered so far). A nodal rate of 55.6 and 6.8 µas/day is caused by De Sitter and Lense–Thirring, respectively, on Galileo satellite in eccentric orbits. Thus, the total effect changes the position of the Galileo orbit node by 8.46 mm after one day and more than 3 m when accumulating data from 1 year. Thus, the effect is measurable using the current techniques of space geodesy.

The effect of general relativity on satellite orbits geometry has also been the subject of research provided by Sosnica et al. (2022). The authors used three years of GPS, GLONASS, and Galileo data to retrieve the general relativistic effects acting on the orbit geometry described as the semi-major axis and eccentricity. The mean observed offset of the semi-major axis is –17.41 mm, which gives a relative error versus the expected value of 0.36‰ when estimating the post-Newtonian parameters

8. Precise orbit determination of GNSS satellites

Satellite position errors are one of the most important factors affecting the accuracy of positioning with the use of the GNSS. Research to improve this accuracy focuses, among others, on evaluation of various factors crucial for the precise orbit determination like infrared radiation, albedo and direct solar radiation pressure. In that regard Bury et al. (2020) studied the impact of the aforementioned forces on GNSS satellites. The authors focused on evaluation of the magnitude and the characteristic periods of accelerations caused by the forces. The studies were conducted in relation to Empirical Orbit Model (ECOM), which in the Sun-satellite-Earth reference frame, decompose the accelerations acting on the GNSS satellite in three directions: D-from the satellite toward the Sun, Y-along the solar panel rotation axis and B-perpendicular to D and Y axes, completing the right-handed orthogonal frame. The authors pointed and assessed the perturbations absorbed by the extended ECOM2 and the consequences of neglecting higher order of ECOM coefficients. They reported that the largest periodic perturbations absorbed by the ECOM2 in B and D directions are at 371 and 146 mm respectively. However, Galileo in eccentric orbits – E14 and E18 – are subject to the different perturbing forces. As a result, the non-considered once-per-revolution accelerations in D and twice-per revolution acceleration in B cause the errors up to 154 and 37 mm, respectively. Therefore, using the proper a priori model for Galileo satellites is indispensable to achieve the highest quality of the geodetic products (Bury et al., 2021). The authors developed also the equation describing the relation between the periodic error of the position (A_m) and the periodic acceleration acting on satellites in direction Q:

$$A_m = \sqrt{Q^2_{cn_{\text{ref}}} + Q^2_{sn_{\text{ref}}}} \cdot \frac{1}{n^2_{\text{ref}}} \cdot \left(\frac{T}{2\pi}\right)^2, \tag{2}$$

where $n_{\text{ref}} = 1, 2, 3, \ldots$ denotes the perturbation of the once-, twice-, ternary-, quadruple-per-revolution, etc., T is the satellite revolution period and Q_c and Q_s denote the sine and cosine acceleration components.

On the other hand Kur et al. (2021) investigated the possible consequences of introducing Inter-Satellite Links (ISL) in the orbit determination process. The of three constellation types of orbital planes (GPS-like with 24 satellites on six orbital planes, GPS-real with real satellite positions and Galileo-like with 24 satellites on three orbital planes) and seven different ISL connectivity schemes (intra-plane closed, intra-plane open), nearest (inter-plane, dual one-way), nearest (general, dual one-way), nearest (general, one-way), sequential (dual one-way) and sequential (one-way)) were investigated. Simulated ISL measurements were included in the orbit and clock estimation process. The authors demonstrated strong contribution of ISLs to minimize of the orbit errors, in general, and help to minimize satellite clock estimation errors.

9. Conclusions

Although general theory and methodology is not an official pillar of geodesy, the field is widely developed because it allows for the improvement of algorithms used in all three pillars of geodesy. The results of the researches presented in this study are an overview of the achievements of Polish scientists in this field in the period of 2019–2022. On the one hand, the studies are continuation of work carried out for years, the summary of which can be found in the studies, e.g. Borkowski and Kosek (2015) and Borkowski et al. (2019). On the other hand, they indicate new directions of development in both geodesy and surveying. The most important achievements include:

- new approach for the identification of stable potential reference points inspired by the theory of squared $M_{\text{split}(q)}$ estimation proposed by Nowel (2019) and investigations of the deformation congruence models based on combinatorial methods (Nowel, 2020);
- new proposal of application of the use of Minimal Detectable Displacement in the network sensitivity analyses derived by Proszynski and Lapinski (2021);

- development of the theoretical basis for the use of reliability matrices in designing the positioning systems (Proszynski and Kwasniak, 2019) and reliability analysis for non-distorting connection (NDC) of engineering survey network (Proszynski and Lapinski, 2019);
- new approach of the optimization of the measurement and control network structure proposed by Mrowczynska and Sztubecki (2021). The approach is based on minimizing of the objective function defined in the form of information entropy and evolutionary algorithms;
- analyzes regarding of the classical problem of identifying and locating local configuration defects of geodetic networks (Kadaj, 2021b);
- investigations on the impact of general relativistic effects on the satellites orbits (Sosnica et al., 2021; 2022);
- research to improve the accuracy of satellites orbits based on the study of perturbing forces on GNSS satellites (Bury et al., 2020; 2021) as well as introducing Inter-Satellite Links in the orbit determination process (Kur et al., 2021);
- development of cartographic projections for triaxial ellipsoids and their applications for mapping of celestial bodies (Pedzich, 2019);
- research on closed-form solutions for a similarity transformation and its variants (e.g., a rigid-body transformation) within the Procrustean framework (Ligas, 2020; Ligas and Prochniewicz, 2021);
- derivation of novel set of formulas for the longitude, distance along the geodesic and the area of a geodesic polygon (Nowak and Nowak Da Costa, 2022);
- continuous development of M_{split} estimation in terms of robustness analysis, introduction of errors-in-variables (EIV) solution, further robustification against outliers, accuracy analysis, e.g., Duchnowski and Wisniewski (2019, 2020), Wisniewski and Zienkiewicz (2021a); Wisniewski (2022), Wyszkowska and Duchnowski (2022);
- demonstration of equivalence between the least squares collocation and geostatistical method of kriging for scalar random fields under the assumption of 2^{nd} order stationarity (Ligas, 2022);
- investigations on noise-related problems and detrending issues in least squares collocation and kriging, application of total least squares collocation to deformation analysis (Jarmolowski, 2019; Jarmolowski et al., 2021; Wisniewski and Kaminski, 2020);
- improved understanding of geophysical processes through better modelling of geodetic time series (Klos et al., 2019, 2020, 2021; Lenczuk et al., 2020; Maciuk et al., 2020; Najder, 2020; Nistor et al., 2021).

Author contributions

Conceptualization: A.K., M.L., M.T.; original draft preparation, editing and reviewing: A.K., M.L., M.T.

Data availability statement

No datasets were used in this research.

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