10.24425/acs.2023.148882

Archives of Control Sciences
Volume 33(LXIX), 2023
No. 4*, pages 801–827*

Equivalent diagrams of fractional order elements

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This paper presents equivalent impedance and operator admittance systems for fractional order elements. Presented models of fractional order elements of the type: $s^{\alpha}L_{\alpha}$ and $1/s^{\alpha}C_{\alpha}$, $(0 < \alpha < 1)$ were obtained using the Laplace transform based on the expansion of the factor sign to an infinite fraction with varying degrees of accuracy – the continued fraction expansion method (CFE). Then circuit synthesis methods were applied. As a result, equivalent circuit diagrams of fractional order elements were obtained. The obtained equivalent schemes consist both of classical RLC elements, as well as active elements built based on operational amplifiers. Numerical experiments were conducted for the constructed models, presenting responses to selected input signals.

Key words: fractional order derivative, Laplace transform for fractional order systems, CFE method, circuit synthesis, numerical experiments

1. Introduction

Two trends can be noted in the literature on elements described by fractional order equations:

1. Modeling physical phenomena with fractional order elements.
2. Creating equivalent diagrams for fractional order elements.

Regarding the first group of issues, the area of fractional order differential calculus covers electrical engineering in the broadest sense and concerns,

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The article was realized within the framework of the internship of Sebastian Różowicz, Ph.D., realized at the Rzeszow University of Technology from 1 March 2021, to 30 June 2021, Internship topic: Application of mathematical methods to systems analysis in electrical engineering.

Received 26.04.2023. Revised 18.10.2023.

among other things, the description and modeling of: supercapacitors [1–3], electrolytes [4, 5], lossy inductive elements [6, 7] long line [8, 9], relaxation phenomena of organic dielectric materials [10], viscoelastic phenomena [11–14], diffusion phenomena [15–17], heating process and heat conduction [18].

The second group includes the creation of equivalent diagrams in the form of ladder systems of passive elements or active elements [1, 19–21]. Implementations of such elements (time- or frequency-related) are generally divided into realizations by approximations of L_β, C_α elements through ladder structures and synthesis of *RL* or *RC* structures [19], and realizations of fractional-order elements using electronic active circuits such as a gyrator or generalized impedance converter GIC [20, 21]. There have also been other studies on new ways to realize secondary fractional-order elements, as using a field-effect transistor. These generalized models have come to be known as quasi-conductance, pseudo-conductance, or quasi-inductive or pseudo-inductive elements. Even though that the mathematical tools used in their analysis are more theoretically involved and more complicated than in the classical implementation, it turns out that traditional circuit models are replaced by models derived from differential-integral calculus of fractional order. The solution of a fractional order differential equation describing such systems is possible using the Laplace transform method. Then, an algebraic equation is transformed using simple mathematical rules to obtain a solution in the operator s domain. The final solution of the differential equation is obtained by applying the inverse Laplace transform. Determining the inverse Laplace transform is done by decomposing the function into simple fractions. In practical applications, the non-infinite-order systems of continuous-time systems can be successfully approximated by higher-order systems that maintain a constant phase in the selected frequency band. This can be successfully accomplished with the Continued Fraction Expansion (CFE) method [22].

In previous publications, equivalent diagrams of elements for fractional order derivatives applicable only to a specific α value (e.g. $\alpha = 0.5$) were presented [23]. This paper presents a wide range of equivalent impedance and operator admittance diagrams of fractional-order elements: $s^\alpha L_\alpha i 1/s^\alpha C_\alpha$, (for any α values in the range $0 < \alpha < 1$), obtained by the expansion of the factor s^α (applying the Laplace transform) in to a repeating decimal with different degrees of accuracy (CFE method), and passive or active circuit synthesis methods.

The Introduction of the paper discusses the problem. Section 2 is devoted to a synthetic presentation of the mathematical methods used in the analysis carried out – the CFE method for determining the Laplace transform for fractional order systems is discussed. The essential parts of the work are contained in Section 3, which presents the synthesis of fractional order elements for different orders of approximation, including for negative impedance and using an operational amplifier. The results of the theoretical analysis of the developed equivalent diagrams of fractional order elements are also included. Section 4 contains the results of numerical experiments for selected orders of approximation of fractional order systems, showing the responses to input signals because it suggests that the responseses are like impulse, unit step function and sinusoidal excitation.

2. CFE Method – (Continued fraction expansion)

Determination of the inverse Laplace transform in the form of a function represented by a quotient of polynomials of the variable s with integer powers, for single poles is carried out by decomposing the function into simple fractions. Therefore, one should approximate the factor s^α with a multinomial, in which the s appears in integer powers. And this is where the CFE method has come into play, making such the approximation possible.

CFE is derived from the expansion in to a repeating decimal of the expression $(1+x)^\alpha$ for $0 \leqslant \alpha \leqslant 1$:

$$(1+x)^\alpha \cong \cfrac{1}{1-\cfrac{\alpha x}{1+\cfrac{(1+\alpha)x}{2+\cfrac{(1-\alpha)x}{3+\cfrac{(2+\alpha)x}{2+\cfrac{2(1-\alpha)x}{5}}}}}} \tag{1}$$

substitute $x = s - 1$ and, taking the subsequent expressions for consideration, one obtains approximations to the required order.

According to this method, factor s^α can be presented as the quotient of polynomials of the variable s and α order derivative – these variables occur here in integer powers [24, 25].

$$s^\alpha \cong \frac{N(s,\alpha)}{D(s,\alpha)} = \frac{\sum\limits_{k=0}^{A} P_{Ak}(\alpha)s^{A-k}}{\sum\limits_{k=0}^{A} Q_{Ak}(\alpha)s^{A-k}}, \tag{2}$$

where: A – order of approximation, $P_{Ak}(\alpha)$, $Q_{Ak}(\alpha)$ – α polynomials of order A.

These approximations according to [24] are respectively:

– 1st order approximation:

$$s^\alpha \cong \frac{(1+\alpha)s + (1-\alpha)}{(1-\alpha)s + (1+\alpha)}; \tag{3}$$

– 2nd order approximation:

$$s^{\alpha} \cong \frac{\left(\alpha^2 + 3\alpha + 2\right) s^2 + \left(-2\alpha^2 + 8\right) s + \left(\alpha^2 - 3\alpha + 2\right)}{\left(\alpha^2 - 3\alpha + 2\right) s^2 + \left(-2\alpha^2 + 8\right) s + \left(\alpha^2 + 3\alpha + 2\right)}\,; \tag{4}$$

– 3rd order approximation:

$$s^{\alpha} \cong \frac{P_{30}s^3 + P_{31}s^2 + P_{32}s + P_{33}}{Q_{30}s^3 + Q_{31}s^2 + Q_{32}s + Q_{33}}\,, \tag{5}$$

where:

$$P_{30} = Q_{33} = \alpha^3 + 6\alpha^2 + 11\alpha + 6,$$
$$P_{31} = Q_{32} = -3\alpha^3 - 6\alpha^2 + 27\alpha + 54,$$
$$P_{32} = Q_{31} = 3\alpha^3 - 6\alpha^2 - 27\alpha + 54,$$
$$P_{33} = Q_{30} = -\alpha^3 + 6\alpha^2 - 11\alpha + 6,$$

– 4th order approximation:

$$s^{\alpha} \cong \frac{P_{40}s^4 + P_{41}s^3 + P_{42}s^2 + P_{43}s + P_{44}}{Q_{40}s^4 + Q_{41}s^3 + Q_{42}s^2 + Q_{43}s + Q_{44}}\,, \tag{6}$$

where:

$$P_{40} = Q_{44} = \alpha^4 + 10\alpha^3 + 35\alpha^2 + 50\alpha + 24,$$
$$P_{41} = Q_{43} = -4\alpha^4 - 20\alpha^3 + 40\alpha^2 + 320\alpha + 384,$$
$$P_{42} = Q_{42} = 6\alpha^4 - 150\alpha^2 + 864,$$
$$P_{43} = Q_{41} = -4\alpha^4 + 20\alpha^3 + 40\alpha^2 - 320\alpha + 384,$$
$$P_{44} = Q_{40} = \alpha^4 - 10\alpha^3 + 35\alpha^2 - 50\alpha + 24.$$

In the paper [24] it was shown that the approximation of order $A = 5$ gives an accuracy which is relatively best for polynomials of the fifth degree. Increasing the order of approximation in addition to increased number of components in the polynomials gives a negligible improvement of accuracy. It means that increasing the order of approximation, in addition to increased number of components in the polynomials, and thus larger number of poles and a more complex numerical implementation, gives a relatively negligible change in the solution, which is interpreted as a slight improvement of accuracy.

Assuming the order of approximation to be $A = 5$, these polynomials take the following form:

$$
\begin{aligned}
&P_{50}(\alpha) = Q_{55}(\alpha) = -\alpha^5 - 15\alpha^4 - 85\alpha^3 - 225\alpha^2 - 274\alpha - 120,\\
&P_{51}(\alpha) = Q_{54}(\alpha) = 5\alpha^5 + 45\alpha^4 + 5\alpha^3 - 1005\alpha^2 - 3250\alpha - 3000,\\
&P_{52}(\alpha) = Q_{53}(\alpha) = -10\alpha^5 - 30\alpha^4 + 410\alpha^3 + 1230\alpha^2 - 4000\alpha - 12000,\\
&P_{53}(\alpha) = Q_{52}(\alpha) = 10\alpha^5 - 30\alpha^4 - 410\alpha^3 + 1230\alpha^2 + 4000\alpha - 12000,\\
&P_{54}(\alpha) = Q_{51}(\alpha) = -5\alpha^5 + 45\alpha^4 - 5\alpha^3 - 1005\alpha^2 + 3250\alpha - 3000,\\
&P_{55}(\alpha) = Q_{50}(\alpha) = \alpha^5 - 15\alpha^4 + 85\alpha^3 - 225\alpha^2 + 274\alpha - 120.
\end{aligned}
\tag{7}
$$

It is worth noting that the polynomials appearing in the approximations contain integer powers of s, so known methods can be used to determine the inverse transform.

3. Synthesis of fractional order elements

Fractional order systems modeling is proving to be an indispensable tool for simulation, identification and control of some automation systems [26–32]. Cited works show that the main area of application of fractional order operators is control theory, however, their capabilities also suit other areas, such as the theory of electrical circuits. Attempts have already been made (among other works [33–35]), but they were few and do not exhaust the subject. A good example is the inductive and capacitive elements occurring in some electrical circuits, the behavior of which cannot be accurately described by the classical method. Only the use of fractional order operators to describe them for the parameter α $(0 \leqslant \alpha \leqslant 1)$ gives satisfactory results.

We will consider the elements L_α and C_α defined for $(0 < \alpha < 1)$ via the following equations:

$$u_L(t) = L_\alpha \frac{D^\alpha i_L(t)}{dt^\alpha}, \tag{8}$$

$$i_C(t) = C_\alpha \frac{D^\alpha u_C(t)}{dt^\alpha}. \tag{9}$$

By performing the Laplace transformation of the above equations with zero initial conditions, their operator impedances can be obtained in the form:

$$Z_{L_\alpha}(s) = s^\alpha L_\alpha, \tag{10}$$

$$Z_{C_\alpha}(s) = \frac{1}{s^\alpha C_\alpha} \tag{11}$$

or admittance by the formulas:

$$Y_{L_\alpha}(s) = \frac{1}{s^\alpha L_\alpha}, \tag{12}$$

$$Y_{C_\alpha}(s) = s^\alpha C_\alpha . \tag{13}$$

Comparing the dependencies, it is easy to see that the impedance operator of the element (10) and the admittance operator (13) have the same form – they differ only by a factor. The same is true for admittance (12) and impedance (11). Thus, given these relationships, it is sufficient to synthesize only the impedance operator of the elements L_α and C_α. In the following subsections, such synthesis is presented for approximations of orders 1 to 5.

3.1. Element synthesis for 1st order approximation

Given equations (2) and (10) for the 1st order approximation, the impedance synthesis of element L_α can be made according to the following relationship:

$$Z_{L_\alpha} = s^\alpha L_\alpha = L_\alpha \frac{(1+\alpha)s + (1-\alpha)}{(1-\alpha)s + 1 + \alpha} = L_\alpha \frac{(1+\alpha)}{(1-\alpha)} + \frac{-4L_\alpha \alpha}{(1-\alpha)^2 s + (1-\alpha^2)} . \tag{14}$$

Using the decomposition of straight fractions and switching to the form used in the synthesis of the two-port elements [36], we obtain:

$$Z_{L_\alpha} = R_0 + \frac{-k_i}{s + \sigma_i}, \tag{15}$$

where:

$$R_0 = L_\alpha \frac{(1+\alpha)}{(1-\alpha)}, \qquad k_i = \frac{4L_\alpha \alpha}{(1-\alpha)^2}, \qquad \sigma_i = \frac{(1-\alpha^2)}{(1-\alpha)^2} . \tag{16}$$

It should be noted that the second component of the formula (15) represents the impedance Z_1, consisting of a parallel combination of resistance R_1 and a capacitor with capacitance C_1, the form of which according to [12, 24] is as follows:

$$R_1 = \frac{k_i}{\sigma_i} = \frac{4L_\alpha \alpha}{(1-\alpha^2)}, \qquad C_1 = \frac{1}{k_i} = \frac{(1-\alpha)^2}{4L_\alpha \alpha} . \tag{17}$$

Negative impedance synthesis can be performed in many ways [14, 25] – the authors chose the simplest one, using an operational amplifier. The equivalent diagram of the 1st order element was developed as shown in Fig. 1.

In an similar way, the impedance of element C_α can be synthesized according to relations (2) and (11):

$$Z_{C_\alpha} = \frac{1}{s^\alpha C_\alpha} = \frac{1}{C_\alpha}\frac{(1-\alpha)s + 1 + \alpha}{(1+\alpha)s + 1 - \alpha} = \frac{1}{C_\alpha}\frac{(1-\alpha)}{(1+\alpha)} + \frac{1}{C_\alpha}\frac{\dfrac{4\alpha}{(\alpha+1)}}{(1+\alpha)s + (1-\alpha)} . \tag{18}$$

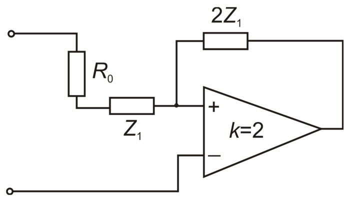

Figure 1: Equivalent diagram of the $Z_{L_\alpha}(s) = s^\alpha L_\alpha$ 1st order element

Using the decomposition of simple fractions and switching to the form used in the synthesis of two-port elements [11], we get:

$$Z_{C_\alpha} = R_0 + \frac{k_i}{s + \sigma_i}, , \tag{19}$$

where:

$$R_0 = \frac{1}{C_\alpha}\frac{(1-\alpha)}{(1+\alpha)}, \quad k_i = \frac{1}{C_\alpha}\frac{4\alpha}{(1+\alpha)^2}, \quad \sigma_i = \frac{(1-\alpha)}{(1+\alpha)}. \tag{20}$$

The second component of formula (15) represents impedance Z_1 consisting of a parallel combination of resistance R_1 and a capacitor with capacitance C_1, whose values according to [11] are:

$$R_1 = \frac{k_i}{\sigma_i} = \frac{4\alpha}{C_\alpha(1-\alpha^2)}, \quad C_1 = \frac{1}{k_i} = C_\alpha\frac{(1+\alpha)^2}{4\alpha}. \tag{21}$$

Thus, the results of synthesizing the impedance of element C_α of order 1 can be presented using passive elements connected as shown in Fig. 2.

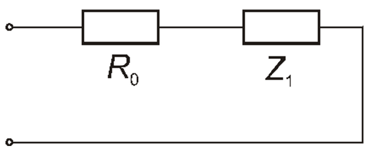

Figure 2: Equivalent diagram of the $Z_{L_\alpha}(s) = s^\alpha L_\alpha$ 1st order element

3.2. Element synthesis for 2nd order approximation

According to relations (3) and (10) for the 2nd order approximation, the impedance synthesis of element L_α can be represented in the following form:

$$Z_{L_\alpha} = s^\alpha L_\alpha = L_\alpha \frac{(\alpha^2 + 3\alpha + 2)\, s^2 + (-2\alpha^2 + 8)\, s + (\alpha^2 - 3\alpha + 2)}{(\alpha^2 - 3\alpha + 2)\, s^2 + (-2\alpha^2 + 8)\, s + (\alpha^2 + 3\alpha + 2)}. \tag{22}$$

Writing the impedance in the form of a continued fraction (Cauer's method [16]), we get the form:

$$Z_{L_\alpha} = s^\alpha L_\alpha = R_1 + \cfrac{1}{Y_2 + \cfrac{1}{R_3 + \cfrac{1}{Y_4 + \cfrac{1}{R_5}}}}, \tag{23}$$

where:

$$\begin{aligned} R_1 &= L_\alpha \frac{\left(\alpha^2 + 3\alpha + 2\right)}{\left(\alpha^2 - 3\alpha + 2\right)}, \qquad Y_2 = s\frac{(\alpha - 1)\left(\alpha^2 - 3\alpha + 2\right)}{12L_\alpha \alpha(\alpha + 2)} = sC_2, \\ R_3 &= \frac{-12L_\alpha \alpha(\alpha + 2)^2}{\alpha^4 + 4\alpha^3 - 15\alpha^2 - 4\alpha + 14}, \\ R_5 &= \frac{144L_\alpha \alpha(\alpha - 1)}{\left(\alpha^2 + 3\alpha + 2\right)\left(\alpha^3 + 2\alpha^2 - 22\alpha + 28\right)}, \\ Y_4 &= s\frac{\left(\alpha^3 + 2\alpha^2 - 22\alpha + 28\right)\left(-\alpha^3 - 5\alpha^2 + 10\alpha + 14\right)}{144L_\alpha \alpha(\alpha - 1)(\alpha + 2)} = sC_4 . \end{aligned} \tag{24}$$

This corresponds to the equivalent impedance diagram of the ladder structure shown in Fig. 3.

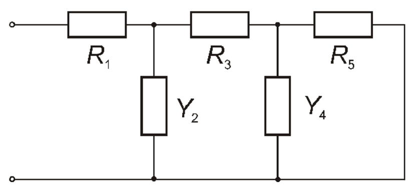

Figure 3: ELadder structure corresponding to the impedance diagram (23)

To assess whether a given component of the ladder structure is passive or active, determine the sign of the values of R_m $(m = 1,\ 3,\ 5)$ and C_n $(n = 2,\ 4)$ for the assumed $0 < \alpha < 1$. For this purpose, the graph of the variation of various parameters depending on the fractional order α was made.

It can be concluded from Figures 4 and 5, it can be concluded that for the assumed values $0 < \alpha < 1$, only the element R_1 is positive and the other elements are negative and they can be represented in the form of active systems as in Fig. 6.

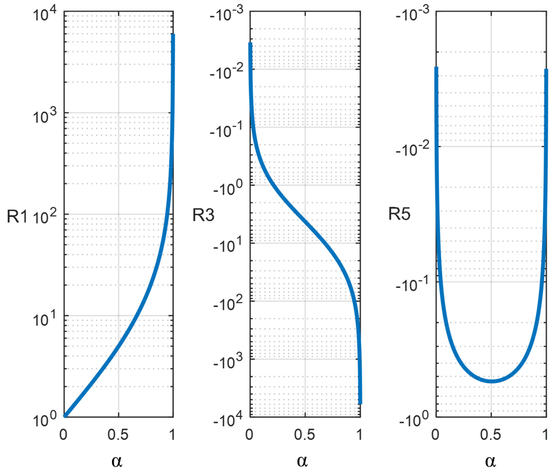

Figure 4: Values of elements R_m ($m = 1,\ 3,\ 5$) for impedance $Z_{L_\alpha}(s) = s^\alpha L_\alpha$

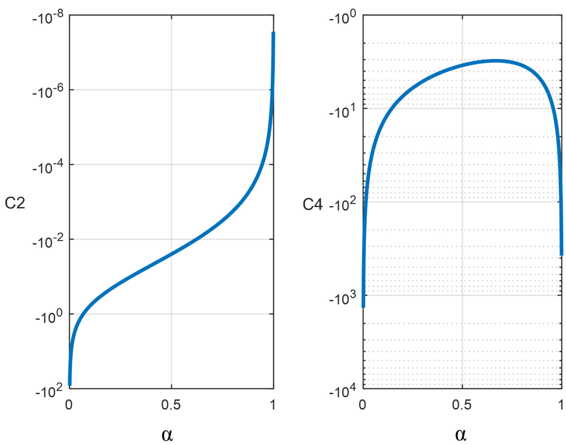

Figure 5: Values of elements C_n ($n = 2,\ 4$) for impedance $Z_{L_\alpha}(s) = s^\alpha L_\alpha$

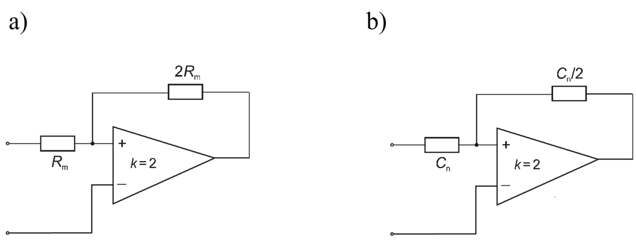

Figure 6: Equivalent diagram of active elements for: a) negative resistance ($m = 3,\ 5$), b) negative capacitance ($n = 2,\ 4$)

Similarly, the $Z_{C_\alpha}(s)$ impedance synthesis was performed, obtaining the following expression:

$$Z_{C_\alpha} = \frac{1}{s^\alpha C_\alpha} = \frac{1}{C_\alpha}\frac{\left(\alpha^2 - 3\alpha + 2\right)s^2 + \left(-2\alpha^2 + 8\right)s + \left(\alpha^2 + 3\alpha + 2\right)}{\left(\alpha^2 + 3\alpha + 2\right)s^2 + \left(-2\alpha^2 + 8\right)s + \left(\alpha^2 - 3\alpha + 2\right)}. \tag{25}$$

Similarly an equivalent diagram was obtained in the form of the ladder structure shown in Fig. 3, where the individual elements have the value:

$$\begin{aligned}
R_1 &= \frac{\left(\alpha^2 - 3\alpha + 2\right)}{C_\alpha\left(\alpha^2 + 3\alpha + 2\right)}, \\
Y_2 &= -sC_\alpha\frac{(\alpha + 1)\left(\alpha^2 + 3\alpha + 2\right)}{12\alpha(\alpha - 2)} = sC_2, \\
R_3 &= \frac{1}{C_\alpha}\frac{-12\alpha(\alpha - 2)^2}{(\alpha + 1)\left(-\alpha^3 + 5\alpha^2 + 10\alpha - 14\right)}, \\
R_5 &= \frac{144\alpha(\alpha + 1)}{C_\alpha\left(\alpha^2 - 3\alpha + 2\right)\left(-\alpha^3 + 2\alpha^2 + 22\alpha + 28\right)}, \\
Y_4 &= sC_\alpha\frac{\left(-\alpha^3 + 5\alpha^2 + 10\alpha - 14\right)\left(-\alpha^3 + 2\alpha^2 + 22\alpha + 28\right)}{144\alpha(\alpha + 1)(\alpha - 2)} = sC_4.
\end{aligned} \tag{26}$$

The values of the elements R_m and C_n for the impedance $Z_{C_\alpha}(s) = 1/s^\alpha C_\alpha$ depending on the parameter α are shown in Figures 7 and 8.

It follows from Fig. 7 and 8, it is found that for the assumed values of $0 < \alpha < 1$, all elements are positive, so they can be represented as passive elements as shown in Figure 3.

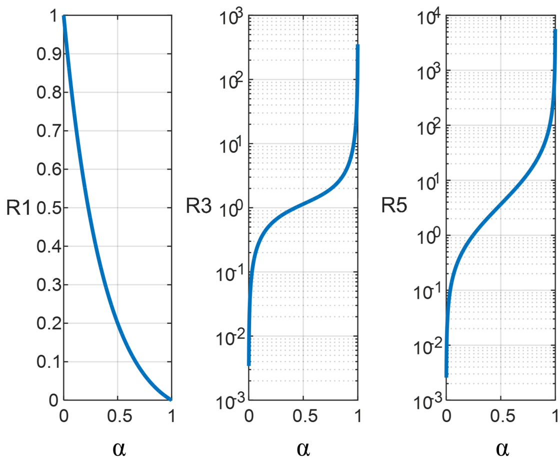

Figure 7: Values of elements R_m $(m = 1,\ 3,\ 5)$ for impedance $Z_{C_\alpha}(s)$

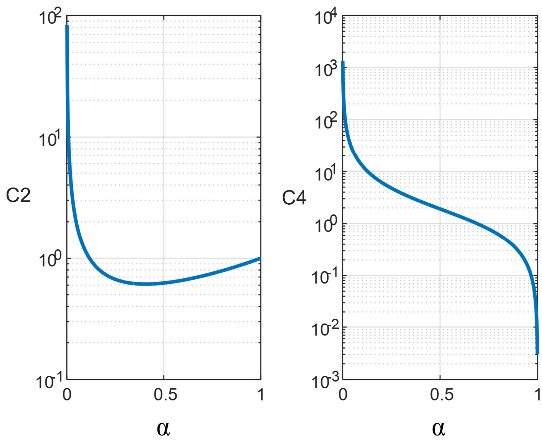

Figure 8: Values of elements R_m $(m = 1,\ 3,\ 5)$ for impedance $Z_{C_\alpha}(s)$

3.3. Element synthesis for 3rd order approximation

Considering the 3rd order approximation, according to relations (2) and (5), (6), the impedance operator of the element L_α has the form:

$$Z_{L_\alpha} = s^\alpha L_\alpha = L_\alpha \frac{N(s,\alpha)}{D(s,\alpha)}, \tag{27}$$

where:

$$\begin{aligned} N(s,\alpha) &= (\alpha^3 + 6\alpha^2 + 11\alpha + 6)s^3 + (-3\alpha^3 - 6\alpha^2 + 27\alpha + 54)s^2 \\ &\quad + (3\alpha^3 - 6\alpha^2 - 27\alpha + 54)s + (-\alpha^3 + 6\alpha^2 - 11\alpha + 6), \\ D(s,\alpha) &= (-\alpha^3 + 6\alpha^2 - 11\alpha + 6)s^3 + (3\alpha^3 - 6\alpha^2 - 27\alpha + 54)s^2 \\ &\quad + (-3\alpha^3 - 6\alpha^2 + 27\alpha + 54)s + \alpha^3 + 6\alpha^2 + 11\alpha + 6. \end{aligned}$$

This leads to the following form of the continued fraction:

$$Z_{L^\alpha} = s^\alpha L = R_1 + \cfrac{1}{Y_2 + \cfrac{1}{R_3 + \cfrac{1}{Y_4 + \cfrac{1}{R_5 + \cfrac{1}{Y_6 + \cfrac{1}{R_7}}}}}}, \tag{28}$$

where:

$$R_1 = -L_\alpha \frac{(\alpha^3 + 6\alpha^2 + 11\alpha + 6)}{(\alpha^3 - 6\alpha^2 + 11\alpha - 6)},$$

$$R_3 = \frac{24L_\alpha\alpha(\alpha+2)\left(\alpha^2 + 5\alpha + 6\right)}{(\alpha-1)\left(\alpha^4 + 8\alpha^3 - 52\alpha^2 + 28\alpha + 87\right)},$$

$$R_5 = \Big[-600L_\alpha\alpha\left(-2\alpha^3 + 2\alpha^2 + 23\alpha - 23\right)\left(-2\alpha^3 - 6\alpha^2 + 23\alpha + 69\right)\Big] \Big/ \Big[\left(\alpha^4 + 5\alpha^3 - 73\alpha^2 + 205\alpha - 174\right)\left(2\alpha^6 + 18\alpha^5 - 15\alpha^4 - 579\alpha^3 - 81\alpha^2 + 3153\alpha + 2686\right)\Big],$$

$$R_7 = \frac{24000\alpha\left(\alpha^2 - 3\alpha + 2\right)}{C_\alpha\left(\alpha^3 + 6\alpha^2 + 11\alpha + 6\right)\left(2\alpha^5 + 6\alpha^4 - 91\alpha^3 - 273\alpha^2 + 2657\alpha - 4029\right)},$$

$$Y_2 = s\frac{-(\alpha-1)\left(\alpha^3 - 6\alpha^2 - 11\alpha - 6\right)}{24L_\alpha\alpha\left(\alpha^2 + 5\alpha + 6\right)},$$

$$Y_4 = s\frac{-\left(\alpha^4 + 8\alpha^3 - 52\alpha^2 + 28\alpha + 87\right)\left(\alpha^4 + 5\alpha^3 - 73\alpha^2 + 205\alpha - 174\right)}{600L_\alpha\alpha(\alpha+2)\left(-2\alpha^3 + 2\alpha^2 + 23\alpha - 23\right)},$$

$$Y_6 = s\Big[\left(2\alpha^5 + 6\alpha^4 - 91\alpha^3 - 273\alpha^2 + 2657\alpha - 4029\right)\left(2\alpha^6 + 18\alpha^5 - 15\alpha^4 - 579\alpha^3 - 81\alpha^2 + 3153\alpha + 2686\right)\Big] \Big/ \Big[2400L_\alpha\alpha\left(\alpha^2 - 3\alpha + 2\right) \cdot \left(-2\alpha^3 - 6\alpha^2 + 23\alpha + 69\right)\Big],$$

The ladder structure showing the equivalent impedance diagram (28) is shown in Figure 9.

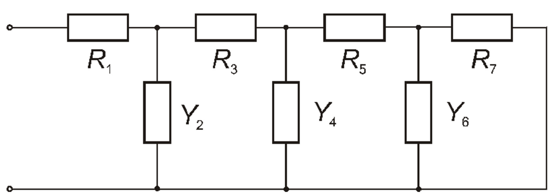

Figure 9: Ladder structure corresponding to the equivalent impedance diagram (28)

Then, as for the 2nd order approximation, the determination of the values signs of individual elements was carried out, using the graphs of various parameters variation depending on the fractional order α, shown in Figures 10 and 11.

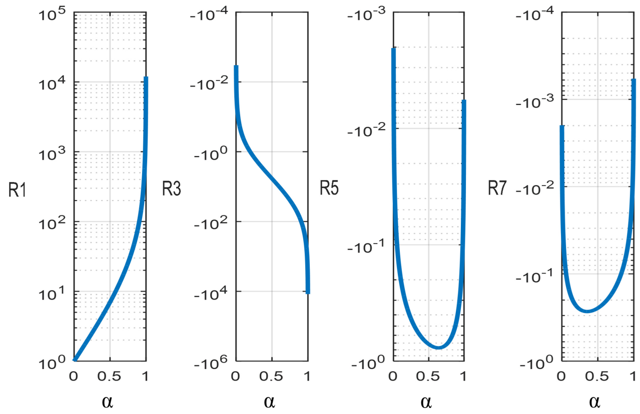

Figure 10: Values of elements R_m $(m = 1,\ 3,\ 5,\ 7)$
for impedance $Z_{L_\alpha}(s) = s^\alpha L_\alpha$

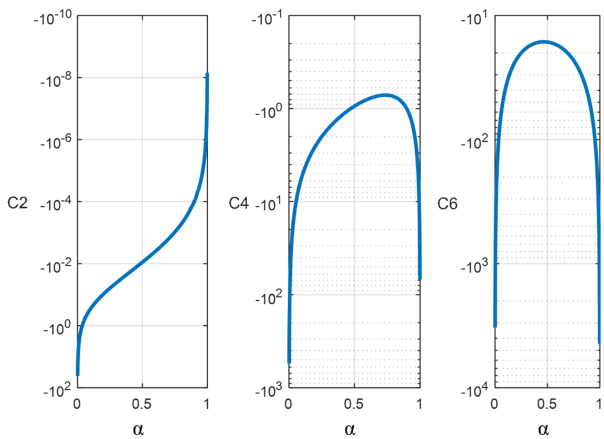

Figure 11: Values of elements C_n $(n = 2,\ 4,\ 6)$
for impedance $Z_{L_\alpha}(s) = s^\alpha L_\alpha$

It can be seen from Figures 10 and 11 that for the assumed values of $0 < \alpha < 1$, only the R_1 element is positive, while the other elements are negative. It is possible, therefore, to present them in the form of active components as shown in Fig. 6.

Performing the synthesis of impedance $Z_{C_\alpha}(s)$ analogously we obtain:

$$Z_{C_\alpha} = \frac{1}{s^\alpha C_\alpha} = \frac{N(s,\alpha)}{C_\alpha D(s,\alpha)}, \tag{29}$$

where:

$$N(s,\alpha) = (-\alpha^3 + 6\alpha^2 - 11\alpha + 6)s^3 + (3\alpha^3 - 6\alpha^2 - 27\alpha + 54)s^2 + (-3\alpha^3 - 6\alpha^2 + 27\alpha + 54)s + \alpha^3 + 6\alpha^2 + 11\alpha + 6\,,$$

$$D(s,\alpha) = (\alpha^3 + 6\alpha^2 + 11\alpha + 6)s^3 + (-3\alpha^3 - 6\alpha^2 + 27\alpha + 54)s^2 + (3\alpha^3 - 6\alpha^2 - 27\alpha + 54)s + (-\alpha^3 + 6\alpha^2 - 11\alpha + 6).$$

In this case, there is also an equivalent diagram in the form of the ladder structure shown in Fig. 9, where the individual elements have the value:

$$R_1 = -\frac{(\alpha^3 - 6\alpha^2 + 11\alpha - 6)}{C_\alpha\,(\alpha^3 + 6\alpha^2 + 11\alpha + 6)}\,,$$

$$R_3 = \frac{24\alpha(\alpha-2)\,(\alpha^2 - 5\alpha + 6)}{C_\alpha(\alpha+1)\,(-\alpha^4 + 8\alpha^3 + 52\alpha^2 + 28\alpha - 87)}\,,$$

$$R_5 = \left[600\alpha\left(-2\alpha^3 - 2\alpha^2 + 23\alpha + 23\right)\left(-2\alpha^3 + 6\alpha^2 + 23\alpha - 69\right)\right] \Big/ \left[C_\alpha\left(-\alpha^4 + 5\alpha^3 + 73\alpha^2 + 205\alpha + 174\right)\left(-2\alpha^6 + 18\alpha^5 + 15\alpha^4 - 579\alpha^3 + 81\alpha^2 + 3153\alpha - 2686\right)\right],$$

$$R_7 = \frac{-24000\alpha\,(\alpha^2 + 3\alpha + 2)}{C_\alpha\,(\alpha^3 - 6\alpha^2 + 11\alpha - 6)\,(2\alpha^5 - 6\alpha^4 - 91\alpha^3 + 273\alpha^2 + 2657\alpha + 4029)}\,,$$

$$Y_2 = s\frac{-(\alpha-1)\,(\alpha^3 - 6\alpha^2 - 11\alpha - 6)}{24L_\alpha\alpha\,(\alpha^2 + 5\alpha + 6)}\,,$$

$$Y_4 = sC_\alpha\frac{(-\alpha^4 + 8\alpha^3 + 52\alpha^2 + 28\alpha - 87)\,(-\alpha^4 + 5\alpha^3 + 73\alpha^2 + 205\alpha + 174)}{600\alpha(\alpha-2)\,(-2\alpha^3 - 2\alpha^2 + 23\alpha + 23)}\,,$$

$$Y_6 = sC_\alpha\left[\left(2\alpha^5 - 6\alpha^4 - 91\alpha^3 + 273\alpha^2 + 2657\alpha + 4029\right)\left(-2\alpha^6 + 18\alpha^5 + 15\alpha^4 - 579\alpha^3 + 81\alpha^2 + 3153\alpha - 2686\right)\right] \Big/ \left[2400\alpha\left(\alpha^2 + 3\alpha + 2\right) \cdot \left(-2\alpha^3 + 6\alpha^2 + 23\alpha - 69\right)\right].$$

Determinations of the sign of the value of individual elements can be read from the graphs shown in Figures 12 and 13.

It follows from the above results, that for the assumed values of $0 < \alpha < 1$, all elements are positive, so they can be presented in the form of passive elements in the structure shown in Fig. 9.

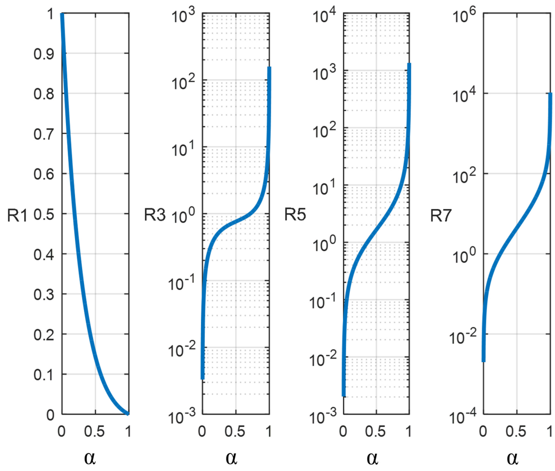

Figure 12: Values of elements R_m $(m = 1,\ 3,\ 5,\ 7)$ for impedance $Z_{C_\alpha}(s)$

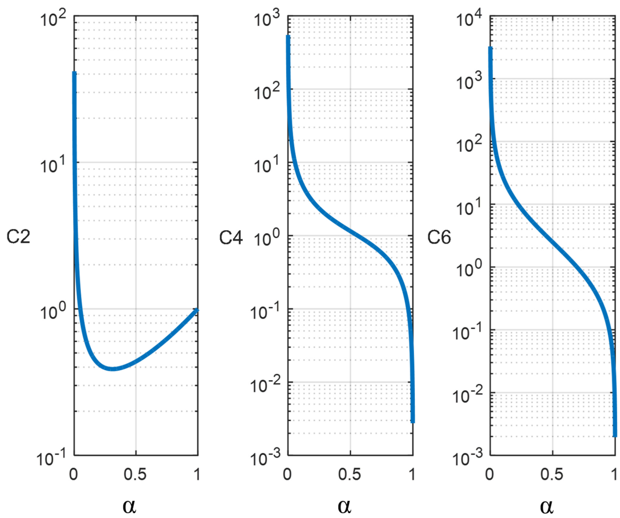

Figure 13: Values of elements C_n $(n = 2,\ 4,\ 6)$ for impedance $Z_{C_\alpha}(s)$

3.4. Element synthesis for 5th order approximation

According to relations (2) and (7), for the 5th order approximation, the synthesis of the impedance of the element L_α can be represented in the following form:

$$Z_{L_\alpha} = s^\alpha L_\alpha = L_\alpha \frac{N(s,\alpha)}{D(s,\alpha)}, \tag{30}$$

where:

$$
\begin{aligned}
N(s,\alpha) = &-s^5(\alpha^5 + 15\alpha^4 + 85\alpha^3 + 225\alpha^2 + 274\alpha + 120)\\
&- \; s^4(-5\alpha^5 - 45\alpha^4 - 5\alpha^3 + 1005\alpha^2 + 3250\alpha + 3000)\\
&- \; s^3(10\alpha^5 + 30\alpha^4 - 410\alpha^3 - 1230\alpha^2 + 4000\alpha + 12000)\\
&+ \; s^2(10\alpha^5 - 30\alpha^4 - 410\alpha^3 + 1230\alpha^2 + 4000\alpha - 12000)\\
&- \; s(5\alpha^5 - 45\alpha^4 + 5\alpha^3 + 1005\alpha^2 - 3250\alpha + 3000)\\
&+ \; (\alpha^5 - 15\alpha^4 + 85\alpha^3 - 225\alpha^2 + 274\alpha - 120),\\
D(s,\alpha) = &\; s^5(\alpha^5 - 15\alpha^4 + 85\alpha^3 - 225\alpha^2 + 274\alpha - 120)\\
&- \; s^4(5\alpha^5 - 45\alpha^4 + 5\alpha^3 + 1005\alpha^2 - 3250\alpha + 3000)\\
&+ \; s^3(10\alpha^5 - 30\alpha^4 - 410\alpha^3 + 1230\alpha^2 + 4000\alpha - 12000)\\
&- \; s^2(10\alpha^5 - 30\alpha^4 - 410\alpha^3 - 1230\alpha^2 + 4000\alpha + 12000)\\
&- \; s(-5\alpha^5 - 45\alpha^4 - 5\alpha^3 + 1005\alpha^2 + 3250\alpha + 3000)\\
&- \; \alpha^5 - 15\alpha^4 - 85\alpha^3 - 225\alpha^2 - 274\alpha - 120) \,.
\end{aligned}
$$

This leads to the following form of the continued fraction:

$$
Z_{L^\alpha} = s^\alpha L = R_1 + \cfrac{1}{Y_2 + \cfrac{1}{R_3 + \cfrac{1}{Y_4 + \cfrac{1}{R_5 + \cfrac{1}{Y_6 + \cfrac{1}{R_7 + \cfrac{1}{Y_8 + \cfrac{1}{R_9 + \cfrac{1}{Y_{101}}}}}}}}}}, \tag{31}
$$

where:

$$
Y_{101} = Y_{10} + \frac{1}{R_{11}}\,, \qquad R_1 = -\frac{L_\alpha X}{A}\,,
$$

$$
R_3 = \frac{60 L_\alpha \alpha(\alpha+2)B}{(\alpha-1)D}\,, \quad R_5 = \frac{-1680 L_\alpha \alpha E N}{FM}\,,
$$

$$R_7 = \frac{-94080L_\alpha \alpha HJ}{GK}, \qquad R_9 = \frac{205752960L_\alpha \alpha IP}{ST},$$
$$R_{11} = \frac{14814213120L_\alpha \alpha U}{WX}, \quad Y_2 = -\frac{s(\alpha-1)A}{60L_\alpha \alpha B},$$
$$Y_4 = \frac{-sDM}{1680L_\alpha \alpha(\alpha+2)E}, \quad Y_6 = \frac{sFG}{94080L_\alpha \alpha HN},$$
$$Y_8 = \frac{-sKS}{205752960L_\alpha \alpha JP}, \quad Y_{10} = \frac{sTW}{14814213120L_\alpha \alpha IU}$$

in which:

$$A = \alpha^5 - 15\alpha^4 + 85\alpha^3 - 225\alpha^2 + 274\alpha - 120,$$
$$B = \alpha^4 + 14\alpha^3 + 71\alpha^2 + 154\alpha + 120,$$
$$U = \alpha^4 - 10\alpha^3 + 35\alpha^2 - 50\alpha + 24,$$
$$X = \alpha^5 + 15\alpha^4 + 85\alpha^3 + 225\alpha^2 + 274\alpha + 120,$$
$$D = \alpha^6 + 17\alpha^5 - 347\alpha^4 + 1781\alpha^3 - 3014\alpha^2 - 718\alpha + 4440,$$
$$E = -11\alpha^5 - 88\alpha^4 + 28\alpha^3 + 1412\alpha^2 + 1639\alpha - 2980,$$
$$F = 11\alpha^9 + 242\alpha^8 + 253\alpha^7 - 32318\alpha^6 + 61941\alpha^5 + 775038\alpha^4 - 2065013\alpha^3 - 4304642\alpha^2 + 9234488\alpha + 10793360,$$
$$G = -11\alpha^9 - 132\alpha^8 + 1727\alpha^7 + 26268\alpha^6 - 373461\alpha^5 + 114212\alpha^4 + 4151273\alpha^3 - 34133148\alpha^2 + 79510072\alpha - 64760160,$$
$$H = -143\alpha^8 + 429\alpha^7 + 10844\alpha^6 - 33390\alpha^5 - 266601\alpha^4 + 866583\alpha^3 + 1851944\alpha^2 - 7288998\alpha + 4859332,$$
$$I = -26\alpha^7 - 130\alpha^6 + 3003\alpha^5 + 15015\alpha^4 - 109014\alpha^3 - 545070\alpha^2 + 1260557\alpha + 6302785,$$
$$J = 143\alpha^8 + 1287\alpha^7 - 8270\alpha^6 - 100170\alpha^5 + 66261\alpha^4 + 2599749\alpha^3 + 3347554\alpha^2 - 21866994\alpha - 48593320,$$
$$K = -143\alpha^{13} - 3432\alpha^{12} - 7881\alpha^{11} + 450066\alpha^{10} + 3280146\alpha^9 - 26747556\alpha^8 - 220536722\alpha^7 + 849643992\alpha^6 + 6033756453\alpha^5 - 12563224068\alpha^4 - 73449872997\alpha^3 + 46429080324\alpha^2 + 351444161944\alpha + 249121581456,$$
$$M = \alpha^4 + 23\alpha^3 - 208\alpha^2 + 538\alpha - 444,$$
$$N = -11\alpha^4 - 77\alpha^3 + 17\alpha^2 + 1043\alpha + 1788,$$

$$
\begin{aligned}
P &= -26\alpha^9 + 156\alpha^8 + 2717\alpha^7 - 17862\alpha^6 - 75981\alpha^5 + 636066\alpha^4 + 61403\alpha^3 \\
&\quad - 6909258\alpha^2 + 13866127\alpha - 7563342, \\
S &= -143\alpha^{12} - 1287\alpha^{11} + 29442\alpha^{10} + 290718\alpha^9 - 2988516\alpha^8 - 32711004\alpha^7 \\
&\quad + 313189594\alpha^6 + 798379461\alpha^5 - 14064211011\alpha^4 + 23835719331\alpha^3 \\
&\quad + 167166761124\alpha^2 - 724611496824\alpha + 830405271520, \\
T &= 26\alpha^{12} + 520\alpha^{11} - 625\alpha^{10} - 78800\alpha^9 - 393088\alpha^8 + 3651250\alpha^7 \\
&\quad + 33877750\alpha^6 - 53154250\alpha^5 - 908017682\alpha^4 - 688762570\alpha^3 \\
&\quad + 7623780475\alpha^2 + 17329217450\alpha + 9841626744, \\
W &= 26\alpha^9 + 130\alpha^8 - 5436\alpha^7 - 27175\alpha^6 + 455763\alpha^5 + 2278815\alpha^4 \\
&\quad - 21629165\alpha^3 - 108145825\alpha^2 + 1071353531\alpha - 2050338905.
\end{aligned}
$$

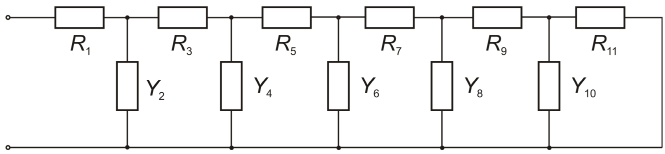

Figure 14: Ladder structure corresponding to the equivalent impedance diagram (31)

Due to the complexity of the values of the individual elements, their sign was determined using the graphs shown in Figures 15–18.

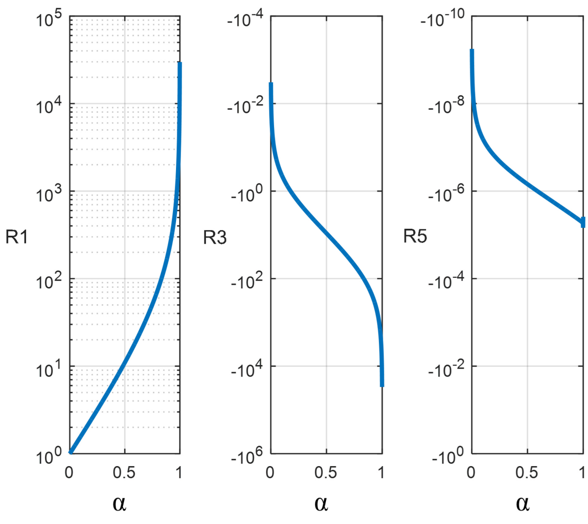

Figure 15: Values of elements R_m $(m = 1,\ 3,\ 5)$ for impedance $Z_{L_\alpha}(s)$

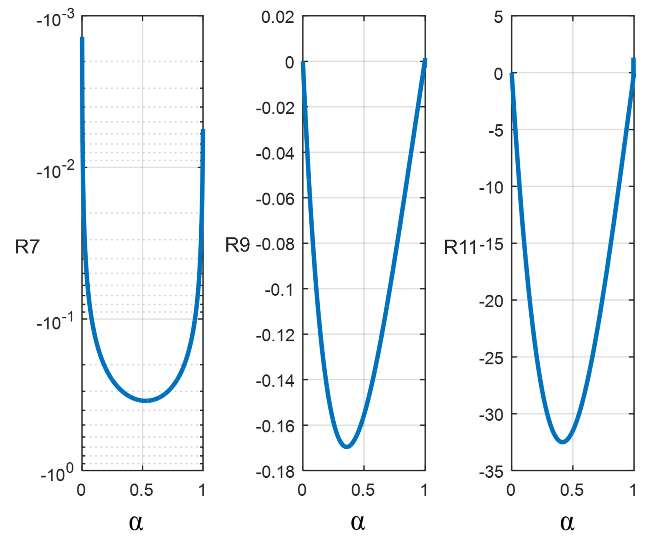

Figure 16: EValues of elements R_m $(m = 7,\ 9,\ 11)$
for impedance $Z_{L_\alpha}(s)$

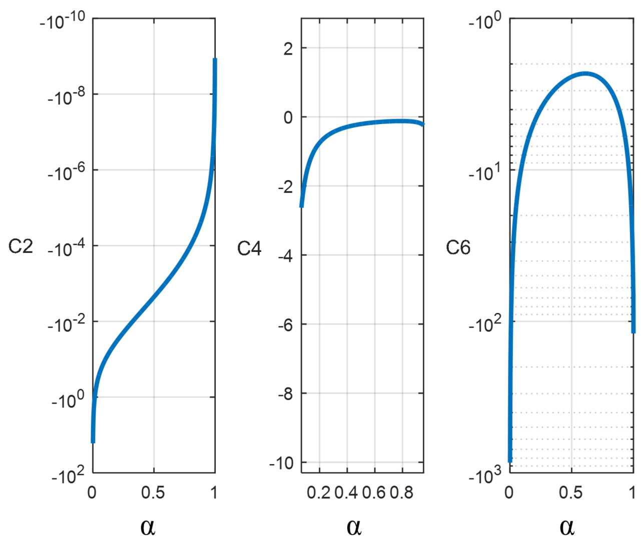

Figure 17: Values of elements C_m $(m = 2,\ 4,\ 6)$
for impedance $Z_{L_\alpha}(s)$

Diagrams shown in Figures 15–18 show that only R_1 (in the diagram from Figure 14) is positive and can be represented as a simple resistor, while the other R_n elements $(n = 3,\ 5,\ 7,\ 9,\ 11)$, in the range considered are negative and

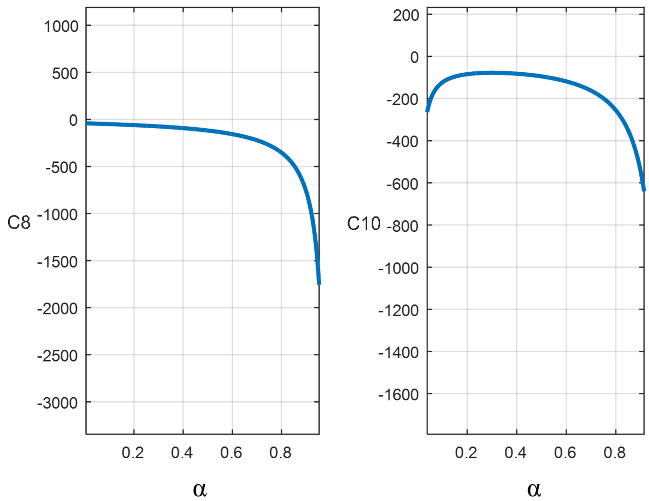

Figure 18: Values of elements C_m $(m = 8,\ 10)$ for impedance $Z_{L_\alpha}(s)$

therefore take the form of the active elements shown in Figure 6a. As for the elements C_m $(m = 2,\ 4,\ 6,\ 8,\ 10)$, they are all negative, so they will take the form of Fig. 6b. As can be seen, for 5th-order approximation, the equivalent scheme is complex and contains a large number of active elements.

4. Numerical experiments

The purpose of numerical experiments presented in this chapter is to determine the response of energy storage elements or electrical circuits to different types of excitation. Knowing the responses to specific signals, it is possible to obtain dynamics description of the analyzed circuit. Therefore, signals in the form of impulse function, unit step function and sinusoidal excitation were used for selected orders of approximation.

Fractional order systems realized for approximation of order 3 and 5 were selected to check the response to impulse function excitation. The results are presented in Figures 19 and 20.

The responses of the systems for the approximation of order 2 and 4 to unit step function excitation are presented in Figures 21 and 22

Simulations comparing the responses to harmonic excitation of the fractional order element for approximation of order 4 and 5 are presented in Figures 23 and 24.

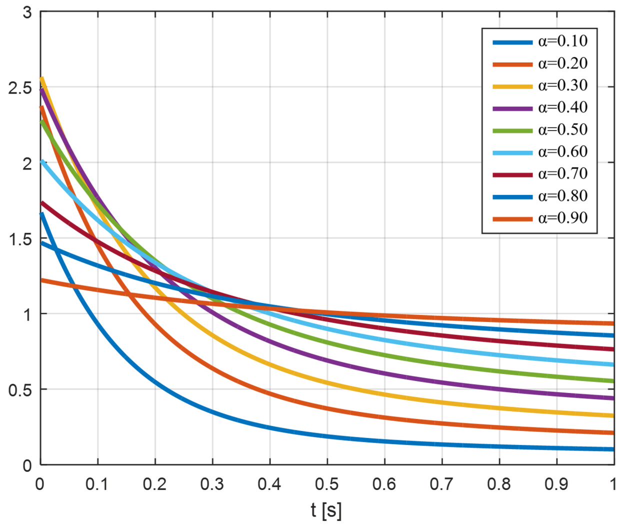

Figure 19: The response of a fractional order element to an impulse function excitation for the 3rd order approximation

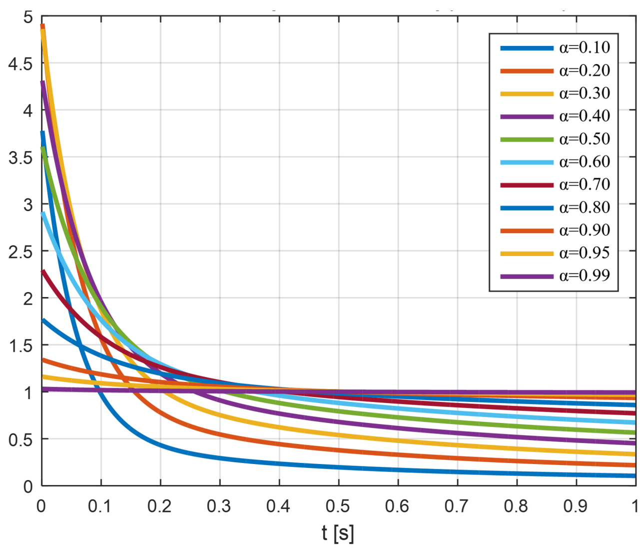

Figure 20: The response of a fractional order element to an impulse function excitation for the 5th order approximation

Analyzing the presented results (which are responses to individual excitations) in the form of time characteristics for different system approximations will allow, in practice, a direct assessment of the dynamic properties of the system and

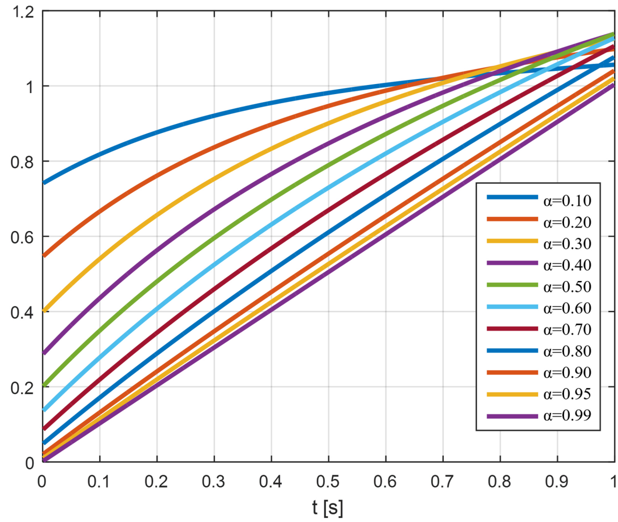

Figure 21: The response of a fractional order element a unit step function excitation for the 2nd order approximation

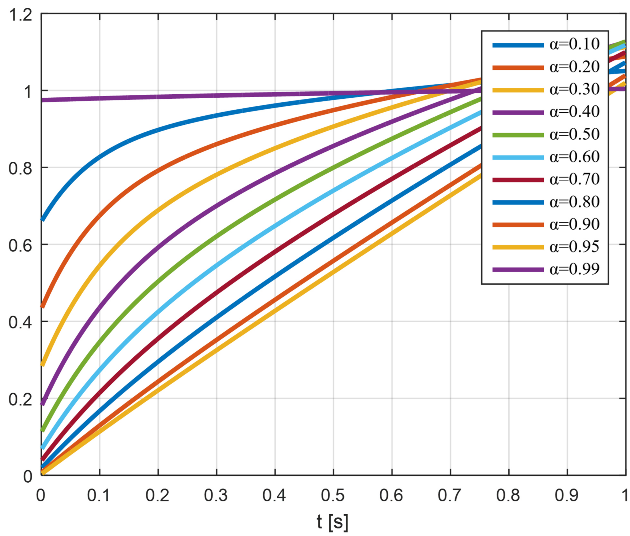

Figure 22: The response of a fractional order element a unit step function excitation for the 4th order approximation

the determination of the optimal fractional order of the modeled real system or circuit.

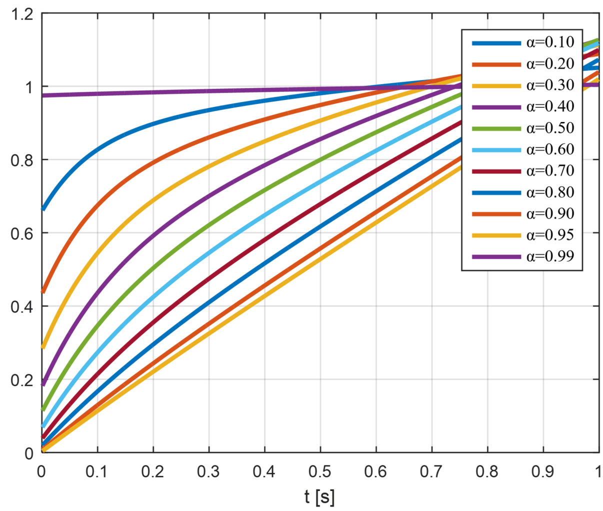

Figure 23: The response of a fractional order element to sinusoidal excitation for the 4th order approximation

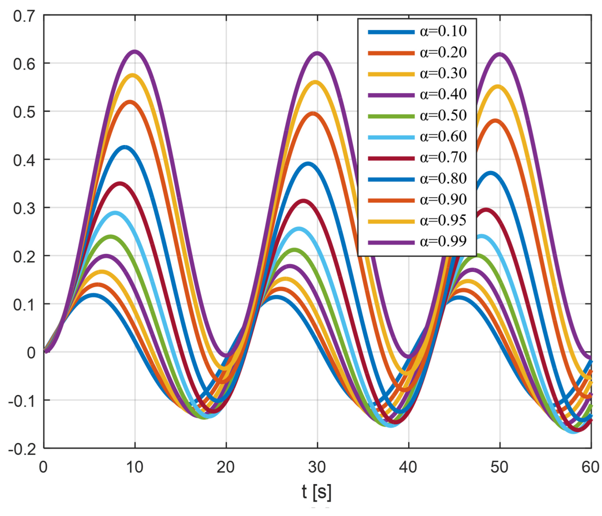

Figure 24: The response of a fractional order element to sinusoidal excitation for the 5th order approximation

5. Conclusion

The use of the CFE method to determine the inverse transform and thus solve the differential equation of the fractional order is an interesting are of research. Using approximations of higher and higher orders, a more and more accurate solution was obtained at the expense of expanding the calculation (polynomials of higher degrees and thus, more poles). Increasing the order of derivatives to 0.9, characteristics of fractional order elements lie closer and closer to the classical first-order derivative. Considering the position of the transforms poles, it can be seen that they lie in the left half-plane or on the imaginary axis of the complex plane, and that with decreasing order of the derivative they lie closer and closer to the zero point.

Currently, high power conversion efficiency is being pursued through the continued development of high-power semiconductor devices. Classical analysis of the operation of such systems does consider all the phenomena occurring there. The use of analysis based on fractional-order derivatives is quite complicated, but the effort put into mathematical considerations can be very profitable, as it allows correcting the errors of classical analysis and thus, designing systems with high energy efficiency. This is an issue of great importance, as such systems are increasingly important and have many practical applications.

In the previous publications, equivalent diagrams of fractional order elements applicable only to a specific α value (e.g. $\alpha = 0.5$) [23] were presented. This paper presents equivalent diagrams for arbitrary α values in the range 0–1.

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