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A boundary value problem for a non-linear difference equation

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A boundary value problem for a non-linear difference equation of order three is considered. We show that this equation can be interpreted as the equation satisfied by the value function in a stochastic optimal control problem. We thus obtain an expression for the solution of the non-linear difference equation that can be used to find an explicit solution to this equation. An example is presented.

Key words: higher-order difference equations, optimal control, dynamic programming, first-passage time, homing problem

1. Introduction

We consider the following third-order non-linear difference equation:

$$\begin{aligned} 0 = {} & (1-c_2)F^2(n) + (c_2 d_1 - c_1 - d_1)F(n) + c_1 d_2 F(n+1) \\ & + (1-c_2) d_1 F(n+2) + c_1(1-d_2)F(n+3) \\ & + [c_2 F(n) + (1-c_2)F(n+2)]\,[d_2 F(n+1) + (1-d_2)F(n+3)] \\ & + c_1 d_1 \end{aligned} \tag{1}$$

for $n \in \{0, 1, \ldots, k-1\}$, where $k \in \{2, 3, \ldots\}$. The real constants c_i and d_i, for $i = 1, 2$, must satisfy the conditions

$$c_1,\ d_1 > 0 \quad \text{and} \quad c_2,\ d_2 \in (0,1). \tag{2}$$

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Moreover, the equation is subject to the boundary conditions

$$F(n) = 0 \quad \text{if} \;\; n < 0 \;\; \text{or} \;\; n \geqslant k. \tag{3}$$

Many authors have studied the problem of the existence of non-trivial solutions to boundary value problems for non-linear difference equations; see, for instance, Wang and Zhou [8] and the references therein. Other authors are looking for explicit solutions to such equations. Stević *et al.* [7], for example, found closed-form solutions of

$$x_n = \frac{x_{n-2}x_{n-k-2}}{x_{n-k}\left(a_n + b_n x_{n-2}x_{n-k-2}\right)} \tag{4}$$

for $n \in \{0, 1, \ldots\}$, $k \in \mathbb{N}$ and given initial values, where a_n and b_n are real numbers. See also the numerous references cited therein.

In this note, we will show that Eq. (1) can be associated with a stochastic optimal control problem. Using the equation satisfied by the value function in this problem, we can derive an explicit expression for the solution to Eqs. (1), (3).

In Section 2, we will obtain the expression for the solution of our boundary value problem. A particular problem will be solved explicitly in Section 3, and we will end this note with a few concluding remarks in Section 4.

2. Associated optimal control problem

Let $\{X_n, n = 0, 1, \ldots\}$ be a random walk starting from $X_0 = x \in C \subset \mathbb{Z}$. Thus, we can write that

$$X_{n+1} = X_n + \epsilon_n\,, \tag{5}$$

where ϵ_n is a random variable equal to 1 with probability $p \in (0, 1)$, and to -1 with probability $q := 1-p$. We consider the controlled process $\{X_n^u,\ n = 0, 1, \ldots\}$ defined by

$$X_{n+1}^u = X_n^u + u_n + \epsilon_n\,, \tag{6}$$

where the control variable u_n is equal to either 1 or 2.

Next, we define the *first-passage time*

$$\tau(x) = \inf\{n > 0 : X_n^u \notin C \mid X_0^u = x\}. \tag{7}$$

Lefebvre and Kounta [5] studied the problem of finding the value u_n^* of the control variable that minimizes the expected value $\mathbb{E}[J(x)]$, where $J(x)$ is the cost function

$$J(x) := \sum_{n=0}^{\tau(x)-1} (u_n^2 + \lambda), \tag{8}$$

in which λ is a positive constant.

The above problem is a particular *homing problem*; see Whittle [9, p. 289] and [10]. The author has published numerous papers on homing problems; see, for instance, [3] and [4]. Other papers on this topic are the ones by Kuhn [2], Makasu [6] and Kounta and Dawson [1].

To solve our stochastic optimal control problem, we can use *dynamic programming*. First, we define the *value function*

$$F(x) = \min_{u_n,\ n=0,\ldots,\tau(x)-1} \mathbb{E}[J(x)]. \tag{9}$$

That is, $F(x)$ denotes the smallest expected cost incurred when starting from x. We have

$$\begin{aligned}
F(x) &:= \min_{u_n,\ n=0,\ldots,\tau(x)-1} \mathbb{E}\left[\sum_{n=0}^{\tau(x)-1} (u_n^2+\lambda)\right] \\
&= \min_{u_n,\ n=0,\ldots,\tau(x)-1} \mathbb{E}\left[u_0^2+\lambda+\sum_{n=1}^{\tau(x)-1} (u_n^2+\lambda)\right] \\
&= \min_{u_n,\ n=0,\ldots,\tau(x)-1} \left\{u_0^2+\lambda+\mathbb{E}\left[\sum_{n=1}^{\tau(x)-1} (u_n^2+\lambda)\right]\right\} \\
&= \min_{u_0} \left\{u_0^2+\lambda+\min_{u_n,\ n=1,\ldots,\tau(x)-1} \mathbb{E}\left[\sum_{n=1}^{\tau(x)-1} (u_n^2+\lambda)\right]\right\} \\
&= \min_{u_0} \left\{u_0^2+\lambda+\mathbb{E}\left[F(X_1^u)\right]\right\},
\end{aligned} \tag{10}$$

where the last equality follows from *Bellman's principle of optimality*.

We can now state the following proposition.

Proposition 1 *The value function $F(x)$ satisfies the dynamic programming equation*

$$F(x) = \min_{u_0} \left\{u_0^2+\lambda+(1-p)F(x+u_0-1)+pF(x+u_0+1)\right\} \tag{11}$$

for $x \in C$. The boundary condition is

$$F(x) = 0 \quad \textit{if } x \notin C. \tag{12}$$

Since we assumed that $u_n \in \{1,2\}$, Eq. (11) becomes

$$\begin{aligned}
F(x) = \min\Big\{&1+\lambda+(1-p)F(x)+pF(x+2), \\
&4+\lambda+(1-p)F(x+1)+pF(x+3)\Big\}.
\end{aligned} \tag{13}$$

Let us denote the value function $F(x)$ by $F_i(x)$ if we take $u_0 = i$, and let $k_i := i^2 + \lambda$, for $i = 1, 2$. We then deduce from Eq. (11) that the function $F_1(x)$ satisfies the linear second-order difference equation

$$F_1(x) = k_1 + (1 - p) F_1(x) + p F_1(x + 2). \tag{14}$$

Similarly, we have

$$F_2(x) = k_2 + (1 - p) F_2(x + 1) + p F_2(x + 3). \tag{15}$$

Assume that the set C is given by $\{0, 1, \ldots, k - 1\}$, where $k \in \{2, 3, \ldots\}$. We will solve Eqs. (14) and (15) subject to the boundary conditions $F_i(k) = F_i(k + 1) = F_i(k + 2) = 0$, for $i = 1, 2$. Once we have obtained the solutions to both (14) and (15), we can compute the value function $F(x)$ explicitly.

Corollary 1 *If $C = \{0, 1, \ldots, k - 1\}$, then we can write that*

$$\begin{aligned} F(x) = \min \Big\{ & k_1 + (1 - p) \min\{F_1(x), F_2(x)\} \\ & + p \min\{F_1(x + 2), F_2(x + 2)\}, \\ & k_2 + (1 - p) \min\{F_1(x + 1), F_2(x + 1)\} \\ & + p \min\{F_1(x + 3), F_2(x + 3)\} \Big\}. \end{aligned} \tag{16}$$

This equation is valid for $x = 0, \ldots, k - 1$.

Now, we also have the following proposition.

Proposition 2 *If $C = \{0, 1, \ldots, k - 1\}$, then the function $F(x)$ satisfies the following non-linear third-order difference equation:*

$$\begin{aligned} 0 = {} & p F^2(x) - (k_1 + p k_2) F(x) + (1 - p) k_1 F(x + 1) \\ & + p k_2 F(x + 2) + p k_1 F(x + 3) - p F(x) [F(x + 2) + F(x + 3)] \\ & + p (1 - p) [-F(x) F(x + 1) + F(x) F(x + 3) + F(x + 1) F(x + 2)] \\ & + p^2 F(x + 2) F(x + 3) + k_1 k_2 \end{aligned} \tag{17}$$

for $x = 0, 1, \ldots, k - 1$, subject to the boundary conditions $F(x) = 0$ if $x = k, k + 1, k + 2$. Moreover, we set $F(x) = 0$ if $x < 0$.

Proof. Making use of the formula

$$\min\{a, b\} = \frac{1}{2} \{a + b - |a - b|\}, \tag{18}$$

we can write that

$$
\begin{aligned}
&2F(x) - (k_1+k_2) - (1-p)\,[F(x)+F(x+1)] - p\,[F(x+2) + F(x+3)] \\
&\quad = -\big|k_1-k_2 + (1-p)\,[F(x) - F(x+1)] + p\,[F(x+2) - F(x+3)]\big|. \qquad (19)
\end{aligned}
$$

Squaring both sides of the above equation and then simplifying, we obtain Eq. (17). □

Remark 1 *Because $u_n \in \{1, 2\}$ and $\epsilon_n \in \{-1, 1\}$, the value of the controlled process $\{X_n^u, n = 0, 1, \ldots\}$ cannot decrease. This is the reason why we set $F(x)$ equal to zero if $x < 0$ and we use the boundary conditions $F(k) = F(k + 1) = F(k+2) = 0$ to determine the three arbitrary constants that appear in the general solution of the linear third-order difference equation (15). In the case of Eq. (14), we use the conditions $F(k) = F(k + 1) = 0$ to determine the two arbitrary constants, and we set $F(k + 2) = 0$.*

Next, the above results can be generalized. Let us replace u_n^2 in the cost function $J(x)$ defined in Eq. (8) by $h(u_n) \geqslant 0$. Moreover, assume that the probability p is actually a function of u_n: $p = p(u_n)$. Then, writing $c_1 := h(1)+\lambda$, $d_1 := h(2)+\lambda$, $c_2 := 1 - p(1)$ and $d_2 := 1 - p(2)$, we find that $F(x)$ satisfies Eq. (1).

Finally, we have the following important result.

Proposition 3 *There is a unique value function associated with Eq. (1).*

Proof. First, the coefficient of $F^2(n)$ in Eq. (1) gives us the value of $p(1)$. Then, we deduce from the coefficient of $F(n)F(n + 1)$ the value of d_2, which yields $p(2)$. Next, we obtain the constants c_1 and d_1 from the coefficient of $F(n+3)$ and that of $F(n+2)$, respectively. Finally, notice that the value function $F(x)$ depends only on $h(i) + \lambda$, for $i = 1, 2$, and not on the function $h(\cdot)$ and the constant λ separately. □

In the next section, a particular problem will be solve explicitly.

3. An example

We can find the general solution of both difference equations. First, Eq. (14) is a second-order linear difference equation with constant coefficients:

$$F_1(x + 2) - F_1(x) + 2(1 + \lambda) = 0. \qquad (20)$$

Its general solution can be written as follows:

$$F_1(x) = r_1(-1)^x + r_2 - (1 + \lambda)x, \qquad (21)$$

where r_1 and r_2 are arbitrary constants. To determine the values of r_1 and r_2, we impose the conditions $F_1(k) = F_1(k + 1) = 0$. Moreover, we set $F_1(k + 2) = 0$.

Next, Eq. (15) is a third-order linear difference equation with constant coefficients:

$$F_2(x+3) + F_2(x+1) - 2F_2(x) + 2(4+\lambda) = 0. \tag{22}$$

We find that

$$F_2(x) = s_1 + s_2\left(-\frac{1}{2} + \frac{\sqrt{7}i}{2}\right)^x + s_3\left(-\frac{1}{2} - \frac{\sqrt{7}i}{2}\right)^x - \frac{4+\lambda}{2}x, \tag{23}$$

where s_1, s_2 and s_3 are constants that are determined from the boundary conditions $F_2(k) = F_2(k+1) = F_2(k+2) = 0$.

Remark 2

(i) Even though the expression for the function $F_2(x)$ contains complex terms, it is actually real for any integer $x \in \{0, 1, \ldots, k-1\}$.

(ii) The function $F_i(x)$ corresponds to the expected cost if we choose $u_0(x) \equiv i$, for $i = 1, 2$.

Let

$$G(x) := 1 + \lambda + \frac{1}{2}[\min\{F_1(x), F_2(x)\} + \min\{F_1(x+2), F_2(x+2)\}] \tag{24}$$

and

$$H(x) := 4 + \lambda + \frac{1}{2}[\min\{F_1(x+1), F_2(x+1)\} + \min\{F_1(x+3), F_2(x+3)\}], \tag{25}$$

so that

$$F(x) = \min\{G(x), H(x)\}. \tag{26}$$

To determine the optimal control $u_0^*(x)$ for any x in $\{0, 1, \ldots, k-1\}$, we can compare the value of $G(x)$ with that of $H(x)$.

3.1. A particular problem

Assume that $k = 4$. We find that

$$F_1(x) = -\frac{1}{2}(1+\lambda)(-1)^x + \frac{9}{2}(1+\lambda) - (1+\lambda)x \tag{27}$$

and that the constants s_1, s_2 and s_3 in Eq. (23) are given by

$$s_1 = \frac{19}{8}(4+\lambda), \quad s_2 = (4+\lambda)\frac{(3i-\sqrt{7})\sqrt{7}}{56i\sqrt{7}+168} \quad \text{and} \quad s_3 = -(4+\lambda)\frac{i\sqrt{7}}{56}. \tag{28}$$

Tables 1–4 give the value function $F(x)$, $F_1(x)$, $F_2(x)$, $G(x)$, $H(x)$ and the optimal control $u_0^*(x)$, for $x = 0, 1, 2, 3$, for various values of the parameter λ.

Table 1: Functions $F(x)$, $F_1(x)$, $F_2(x)$, $G(x)$ and $H(x)$, and optimal control $u_0^*(x)$, for $x = 0, 1, 2, 3$, when $\lambda = 1$

x	$F(x)$	$F_1(x)$	$F_2(x)$	$G(x)$	$H(x)$	$u_0^*(x)$
0	8	8	11.875	8	11	1
1	7	8	8.75	8	7	2
2	4	4	7.5	4	7	1
3	4	4	5	4	5	1

Table 2: Functions $F(x)$, $F_1(x)$, $F_2(x)$, $G(x)$ and $H(x)$, and optimal control $u_0^*(x)$, for $x = 0, 1, 2, 3$, when $\lambda = 2$

x	$F(x)$	$F_1(x)$	$F_2(x)$	$G(x)$	$H(x)$	$u_0^*(x)$
0	12	12	14.25	12	14.25	1
1	9	12	10.5	11.25	9	2
2	6	6	9	6	9	1
3	6	6	6	6	6	1 or 2

Table 3: Functions $F(x)$, $F_1(x)$, $F_2(x)$, $G(x)$ and $H(x)$, and optimal control $u_0^*(x)$, for $x = 0, 1, 2, 3$, when $\lambda = 5$

x	$F(x)$	$F_1(x)$	$F_2(x)$	$G(x)$	$H(x)$	$u_0^*(x)$
0	21.375	24	21.375	22.6875	21.375	2
1	15	24	15.75	18.375	15	2
2	12	12	13.5	12	13.5	1
3	9	12	9	10.5	9	2

Table 4: Functions $F(x)$, $F_1(x)$, $F_2(x)$, $G(x)$ and $H(x)$, and optimal control $u_0^*(x)$, for $x = 0, 1, 2, 3$, when $\lambda = 10$

x	$F(x)$	$F_1(x)$	$F_2(x)$	$G(x)$	$H(x)$	$u_0^*(x)$
0	33.25	44	33.25	38.125	33.25	2
1	24.5	44	24.5	30.25	24.5	2
2	21	22	21	21.5	21	2
3	14	22	14	18	14	2

Notice that, as expected, when λ is large, the optimal control is most often $u_0^*(x) = 2$.

To conclude this section, we will check that the values of the function $F(x)$ given in Table 1 (and using the fact that $F(x) = 0$ for $x \geqslant 4$) are such that Eq. (17) with $\lambda = 1$ is indeed satisfied, for $x = 0,\ 1,\ 2,\ 3$. First, when $x = 0$, we have

$$\begin{aligned} 0 = {} & 2 \times 8^2 - 8\,(7 + 2 \times 4 + 4 + 12 + 6) + 10 \times 4 \\ & + \ 4\,(7 + 4) + 7 \times 4 + 4 \times 4 + 40. \end{aligned} \tag{29}$$

Similarly, for $x = 1$, $x = 2$ and $x = 3$ we have respectively

$$\begin{aligned} 0 = {} & 2 \times 7^2 - 7\,(4 + 2 \times 4 + 0 + 12 + 6) + 10 \times 4 \\ & + \ 4\,(4 + 0) + 4 \times 4 + 4 \times 0 + 40, \end{aligned} \tag{30}$$

$$\begin{aligned} 0 = {} & 2 \times 4^2 - 4\,(4 + 2 \times 0 + 0 + 12 + 6) + 10 \times 0 \\ & + \ 4\,(4 + 0) + 4 \times 0 + 0 \times 0 + 40 \end{aligned} \tag{31}$$

and

$$\begin{aligned} 0 = {} & 2 \times 4^2 - 4\,(0 + 2 \times 0 + 0 + 12 + 6) + 10 \times 0 \\ & + \ 4\,(0 + 0) + 0 \times 0 + 0 \times 0 + 40. \end{aligned} \tag{32}$$

4. Conclusion

In this note, we presented a technique that enables us to obtain an expression for the solution of a certain boundary value problem for a non-linear difference equation of order three. We used the technique to solve explicitly a particular problem.

We can generalize the results obtained in Section 2 by assuming that the control variable u_n takes its values in a set $L := \{l_1, l_2\}$, where $l_i \in \mathbb{Z}$ for $i = 1,\ 2$. Moreover, L can contain more than two values: $L = \{l_1, \ldots, l_m\}$. Of course, if m is large, obtaining an explicit expression for the value function in the associated optimal control problem is rather tedious, and the corresponding non-linear difference equation will be quite involved.

References

[1] M. Kounta and N.J. Dawson: Linear quadratic Gaussian homing for Markov processes with regime switching and applications to controlled population growth/decay. *Methodology and Computing in Applied Probability*, **23** (2021), 1155–1172. DOI: 10.1007/s11009-020-09800-2.

[2] J. Kuhn: The risk-sensitive homing problem. *Journal of Applied Probability*, **22** (1985), 796–803. DOI: 10.2307/3213947.

[3] M. Lefebvre: Minimizing or maximizing the first-passage time to a time-dependent boundary. *Optimization*, **71**(2), (2022), 387–401. DOI: 10.1080/02331934.2021.1914039.

[4] M. Lefebvre: The homing problem for autoregressive processes. *IMA Journal of Mathematical Control and Information*, **39**(1), (2022), 322–344. DOI: 10.1093/imamci/dnab047.

[5] M. Lefebvre and M. Kounta: Discrete homing problems. *Archives of Control Sciences*, **23**(1), (2013), 5–18. DOI: 10.2478/v10170-011-0039-6.

[6] C. Makasu: Risk-sensitive control for a class of homing problems. *Automatica*, **45**(10), (2009), 2454–2455. DOI: 10.1016/j.automatica.2009.06.015.

[7] S. Stević, M.A. Alghamdi, A. Alotaibi and E.M. Elsayed: On a class of solvable higher-order difference equations. *Filomat*, **31**(2), (2017), 461–477. DOI: 10.2298/FIL1702461S.

[8] Z. Wang and Z. Zhou: Boundary value problem for a second-order difference equation with resonance. *Complexity*, **2020** (2020). DOI: 10.1155/2020/7527030.

[9] P. Whittle: *Optimization over Time*, **I**. Wiley, Chichester, 1982.

[10] P. Whittle: *Risk-sensitive Optimal Control*. Wiley, Chichester, 1990.